ON THE BEHAVIOR OF RADIATION NEAR MASSIVE BODIES*

J. Faulkner, F. Hoyle, and J. V. Narlikar
University of Cambridge, Cambridge, England
and
California Institute of Technology, Pasadena, California
Received April 4, 1964

ABSTRACT

The problem of communication between an observer on the surface of a massive body and an external Schwarzschild observer is examined. It is shown that when the body is freely imploding a signal from the external observer never reaches the surface provided the Schwarzschild coordinate of the external observer exceeds a critical value depending on the mass and radius of the body. The signals that reach the surface are redshifted. Time reversal of the solution then shows that radiation from an exploding massive body will be blueshifted when it reaches an external observer. This effect may supply the explanation for the origin of high-energy particles and quanta.

I. THE IMPLODING PROBLEM

The implosion problem for an object assumed uniform, spherically symmetric, without rotation and initially at rest was solved by Datt (1938) and by Oppenheimer and Snyder (1939). The following is a brief résumé of the solution:

Initial density, ρ_0; Definition, $\alpha = (8\pi G/3)\rho_0$.

a) Line Element in Terms of Co-moving Coordinates and with c = *1*

$$ds^2 = dt^2 - S^2(t)\left[\frac{dr^2}{1-\alpha r^2} + r^2 d\Omega^2\right] \qquad \text{(inside body)}, \tag{1}$$

$$ds^2 = dt^2 - S^2\left(\frac{t r_b^{3/2}}{r^{3/2}}\right)\left[\frac{K^2 dr^2}{1-\alpha r_b^3/r} + r^2 d\Omega^2\right] \quad \text{(outside body)}, \tag{2}$$

where $r = r_b$ is the radial co-moving coordinate of the boundary of the object, $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\psi^2$ and K is defined by

$$K = \frac{1}{S}\frac{\partial}{\partial r}(rS). \tag{3}$$

b) Dynamical Equations and Their Solution

Define $\chi = t$ (inside body), $\chi = t r_b^{3/2}/r^{3/2}$ (outside body).
Einstein's equations give

$$\left(\frac{dS}{d\chi}\right)^2 = \alpha\,\frac{(1-S)}{S}, \tag{4}$$

the imploding solution of which is given by

$$\alpha^{1/2}\chi = \frac{\pi}{2} - \sin^{-1}S^{1/2} + S^{1/2}(1-S)^{1/2}, \tag{5}$$

the constant of integration being taken so that $t = 0$ corresponds to the initial state of rest.

The object implodes into a singularity in proper time $(3\pi/32G\rho_0)^{1/2}$.

* Supported in part by the Office of Naval Research and the National Aeronautics and Space Administration.

c) The Event Horizon

A transformation from r, t to "Schwarzschild" coordinates R, T can be found for the exterior, the line element outside the body taking the form

$$ds^2 = dT^2\left(1 - \frac{2GM}{R}\right) - \frac{dR^2}{1 - 2GM/R} - R^2 d\Omega^2 , \tag{6}$$

provided $R > a r_b^3$. The event horizon is defined by

$$R = a r_b^3 . \tag{7}$$

The relation between R, r, t is

$$R = rS . \tag{8}$$

The event horizon develops when $S = a r_b^2$, i.e., as soon as the sphere defined by equation (7) belongs to the exterior.

d) Time Reversibility

The distinction between an imploding and an exploding object appears when the square root of equation (4) is taken. Choice of the negative root corresponds to the imploding solution, the positive root to the exploding case. The exploding case can evidently be obtained from a time reversal of the imploding solution.

II. RADIAL LIGHT TRACKS FOR THE IMPLODING CASE

The coordinate R has the meaning that the two-dimensional sphere defined by points of given R at any time T has proper area $4\pi R^2$. Initially, $S = 1$ and $R = r$.

Consider the following problem: Let light pulses be emitted radially inward from a set of spheres $R_1, R_2 \ldots$, such that $r_b < R_1 < R_2 \ldots$, and let $t = 0$ be the moment of emission of the pulses. What is the largest value of R for which such a pulse can be received by an observer co-moving with the boundary of the object? At first sight it might be supposed that the pulse must reach the observer before the latter crosses into the interior of the event horizon, but this is not the case. Inward-moving light can cross the event horizon; it is outward-moving light which cannot cross the horizon. This can readily be seen from the formulae given in Section I.

Working in co-moving coordinates, and setting $ds = 0$, $d\Omega = 0$ in equation (2) we have

$$\left(1 - \frac{a r_b^3}{r}\right)^{1/2} dt = \pm S(t r_b^{3/2}/r^{3/2}) K dr \tag{9}$$

for radial light tracks in the exterior, with the plus sign for outward-moving light, the minus sign for inward-moving light. From the relation $R = rS$,

$$dR = r\frac{\partial S}{\partial t} dt + \frac{\partial}{\partial r}(rS) dr = \frac{r_b^{3/2}}{r^{1/2}} \frac{dS}{d\chi} dt + SK dr . \tag{10}$$

Substituting for dr from equation (9) and using the square root of equation (4) with minus sign (i.e., for the imploding case),

$$dR = dt\left[-\frac{a^{1/2} r_b^{3/2}}{r^{1/2}} \frac{(1-S)^{1/2}}{S^{1/2}} \pm \left(1 - \frac{a r_b^3}{r}\right)^{1/2}\right] . \tag{11}$$

Cancellation of the terms within the bracket is only possible in the case of outward-moving light. The condition for such a cancellation is $R = rS = a r_b^3$; outward-moving light stands still on the sphere with Schwarzschild coordinate $a r_b^3$, but not inward-moving light. The situation evidently is reversed in an exploding solution, since the positive square root of equation (4) must then be used.

Returning to our problem, we see that the observer co-moving with the body can receive signals from the exterior until the singularity develops. The largest R is that whose pulse reaches the observer exactly at the singularity. The actual determination of the largest R, given ρ_0, r_b, requires numerical integration of equation (9). The principle of the calculation is clear from equation (5) which determines S as a function of r at any specified t. The track of a light pulse is determined by some function $r(t)$. Using the value of r appropriate to a given t, determine S and K for this value of r and hence use equation (9) to integrate from r, t to $r + dr$, $t + dt$. The calculation begins with some specified $r = R_{\text{source}}$ ($> r_b$) at $t = 0$ and ends either when r decreases to r_b—i.e., the pulse reaches the co-moving observer—or when t increases to $(3\pi/32G\rho_0)^{1/2}$. In the latter case the observer falls into the singularity before the pulse can reach him. In Table 1 we give

TABLE 1

COMMUNICATION WITH IMPLODING BODY

ln S at Reception of Pulse	R_{source}/R_i								
	R_c/R_i =0 5	0 6	0.7	0.8	0.9	0 95	0 99	0 999	0 9999
− 1	2 3855	2 1655	1 9757	1.7987	1 6135	1 5011	1 3652	1 2927	1 2679
− 2	2 4666	2 2320	2 0303	1 8428	1 6474	1 5294	1 3874	1 3126	1 2884
− 3	2 4774	2 2406	2 0372	1 8482	1 6514	1 5327	1 3899	1 3148	1 2906
− 4	2 4789	2 2418	2 0381	1 8489	1 6520	1 5332	1 3903	1.3151	1 2909
− 6	2 4791	2 2420	2 0382	1 8491	1 6521				
− 8	2 4791								
−10	2 4791								
−12	2 4791								
−14	2 4792								
−16	2 4792								
−18	2 4792								

results for a number of cases in which the light pulses manage to reach the observer. Before consulting the table it is important to notice that, although there is a double infinity of cases of imploding bodies, depending on the choices for ρ_0, r_b, the behavior of the light tracks depends only on a single infinity—the behavior depends only on the ratio of the initial radius $R_i = r_b$ to the critical radius R_c defined in equation (7), αr_b^3. Thus $R_i/R_c = (\alpha r_b^2)^{-1}$ and choices for αr_b^2 give the single infinity of cases. It will be useful to imagine four concentric spheres:

i) An outermost sphere $R = R_{\text{source}}$, from which the light pulse is emitted at $t = 0$
ii) The initial boundary $R = R_i = r_b$ of the body
iii) The event horizon, defining the critical radius $R = R_c = \alpha r_b^3$
iv) The boundary of the object at the moment of reception of the pulse

We are considering cases where $R_c < R_i < R_{\text{source}}$, and for the results shown in Table 1 the boundary of the object lies inside the critical radius at reception. Although the latter is not a necessary condition, the cases of interest are those in which the object approaches the singularity and hence lies inside the event horizon. Because the transformation from co-moving coordinates to the Schwarzschild coordinates R, T cannot be made inside the event horizon, we cannot specify the boundary of the body at reception of a light pulse by the coordinate R used for (i), (ii), and (iii). The simplest specification is in terms of the scale factor S for the interior solution. The proper area of the boundary of the body at any moment is $4\pi r_b^2 S^2$, so that S^2 is the factor by which the proper area has decreased since $t = 0$.

The results tabulated for the case $R_i = 2R_c$ show that only a slight increase of R_{source} from 2.4791 R_i to 2.4792 R_i greatly affects the value of S at reception, ln S being -6 for $R_{\text{source}} = 2.4791\ R_i$ and ln $S = -18$ for $R_{\text{source}} = 2.4792\ R_i$. Evidently a value of R_{source} very slightly greater than 2.4792 R_i gives the answer to our problem of determining the outermost sphere from which light can be emitted at $t = 0$ and still reach the co-moving observer before the singularity. A similar situation occurs for the other values of R_c/R_i. Because the numbers change so little they have not been displayed in the table.

We turn now to a second problem. Suppose the emitted light is monochromatic with frequency ν_{source} as measured by an observer on the sphere $R = R_{\text{source}}$, but with frequency $\nu_{\text{reception}}$ as measured by the co-moving observer at reception. What is the ratio $\nu_{\text{reception}}/\nu_{\text{source}}$? Results are given in Table 2 for exactly the same cases as those shown

TABLE 2

THE RATIO ($\nu_{\text{reception}}/\nu_{\text{source}}$) FOR THE CASES INCLUDED IN TABLE 1

ln S AT RECEPTION	ln ($\nu_{\text{reception}}/\nu_{\text{source}}$)								
	R_c/R_i =0.5	0 6	0 7	0 8	0 9	0 95	0 99	0 999	0 9999
− 1	−0 6043	−0 6574	−0 7124	−0 7737	−0 8520	−0 9087	−0 9897	−1 0395	−1 0575
− 2	−1 021	−1 102	−1.185	−1 277	−1 392	−1 475	−1 591	−1 661	−1 685
− 3	−1 439	−1 539	−1 640	−1 750	−1 887	−1 984	−2 119	−2 200	−2 227
− 4	−1 876	−1 989	−2 101	−2 223	−2 374	−2 479	−2 626	−2 713	−2 742
− 6	−2 804	−2 930	−3 055	−3 189	−3 353	−3 468	−3 625	−3 718	−3 750
− 8	−3 774	−3 905	−4 036	−4 174	−4 344	−4 461	−4 623	−4 718	−4 750
−10	−4 762	−4 896	−5 028	−5 169	−5 340	−5 459	−5 622	−5 718	−5 750
−12	−5 758	−5 893	−6 026	−6 167	−6 339	−6 458	−6 622	−6 718	−6 750
−14	−6 757	−6 892	−7 025	−7 166	−7 338	−7 458	−7 622	−7 718	−7 750
−16	−7 756	−7 891	−8 024	−8 166	−8 338	−8 458	−8 622	−8 718	−8 750
−18	−8 756	−8 891	−9 024	−9 166	−9 338	−9 458	−9 622	−9 718	−9 750

in Table 1. The light is redshifted in all cases, and the redshifts become very large when reception takes place deep inside the critical radius.

We end the present section by describing how the results of Table 2 were obtained. Suppose two light pulses are emitted near $t = 0$ by the observer on the sphere $R = R_{\text{source}}$. The first pulse may be taken as having a track $t = f(r)$ subject to $f(r = R_{\text{source}}) = 0$, while the second pulse has track $t = f(r) + \delta t(r)$, subject to $\delta t(R_{\text{source}}) = \delta t_0$. Because the body is at rest at $t = 0$ there is no distinction at that moment between a co-moving observer and a coincident observer on a Schwarzschild sphere $R =$ constant, so that δt_0 is the proper time between the emission of the two pulses as measured by the observer on $R = R_{\text{source}}$. Suppose both pulses reach the observer co-moving with the body. The proper time between reception for this observer is $\delta t(r_b)$ and our problem is solved by

$$\nu_{\text{reception}}\delta t(r_b) = \nu_{\text{source}} \times \delta t_0 , \tag{12}$$

provided $\delta t(r_b)$ can be determined when δt_0 is given. Evidently this could be done in principle simply by repeating the calculation of the type shown in Table 1—i.e., by performing this calculation separately for the two pulses. In practice, however, it is easier to subtract the differential equation (9) for the tracks of the two pulses and then to integrate the differenced equation. Remembering that the minus sign must be chosen in equation (9), we find that the differenced equation is

$$\left(1 - \frac{a r_b^3}{r}\right)^{1/2} d(\delta t) = -\delta(SK)\,dr = -\frac{\partial}{\partial r}(r\,\delta S)\,dr , \tag{13}$$

using the definition (4) of K. Also

$$\delta S = \frac{dS}{d\chi}\,\delta\chi = \left(\frac{r_b}{r}\right)^{3/2} \frac{dS}{d\chi}\,\delta t\,,$$

so that equation (13) can be written in the form

$$\frac{d \ln \delta t}{dr} = -\frac{\partial}{\partial r}\left[r \left(\frac{r_b}{r}\right)^{3/2} \frac{dS}{d\chi}\right] \div \left(1 - \frac{a r_b^3}{r}\right)^{1/2}. \tag{14}$$

The right-hand side of equation (14) is a known function of r, t—i.e., it can be constructed explicitly when r, t are specified.

Equation (14) must be integrated together with equation (9), which with appropriate sign can be written

$$\frac{dt}{dr} = -\frac{1}{(1 - a r_b^3/r)^{1/2}} \frac{\partial}{\partial r}(rS)\,, \tag{15}$$

the right-hand side of equation (15) also being known when r, t are specified. The integration begins with $r = R_{\text{source}}$, $t = 0$. Equation (15) determines t as a function of r, while (14) determines δt as a function of r, leading to the required value of $\delta t(r_b)$, and hence with the aid of equation (12) to $\nu_{\text{reception}}/\nu_{\text{source}}$.

III. RADIAL LIGHT TRACKS IN THE TIME-REVERSED SOLUTION

The sign of the right-hand side of equation (15) must be changed in the time-reversed situation—i.e., for an object exploding out of a singularity. The integrations set out in Table 1 are applicable to a situation in which the light pulse is emitted from the surface of the object and received *outside* the critical radius R_c. In fact, the table simply needs rewriting with "ln S at *emission* of pulse" over the left-hand column, and with the numbers appearing in the table interpreted as R for the light pulse divided by R for the boundary of the object at the time when the object reaches its maximum extension. For example, the column with $R_c/R_i = 0.5$ corresponds to a case in which the explosion takes the body to a maximum extension equal to twice the critical radius, and at the moment when the maximum extension is reached a pulse that started when ln S was -18 has reached a Schwarzschild sphere of radius $2.4792 \times 2 = 4.9584$ times the critical radius; the pulse has essentially escaped from the object in spite of the fact that it was emitted deep inside the critical radius. It will continue to escape, and the frequency of the radiation will not be much changed by subsequent events—i.e., it has essentially reached "infinity" so far as gravitational effects are concerned.

Table 2 can also easily be reinterpreted. The sign of the right-hand side of equation (14) must also be changed. However, this change is formal, because in taking the square root of equation (4) to obtain $dS/d\chi$ the opposite sign must be chosen, so that the two changes cancel and the right-hand side of equation (15) is numerically unchanged. Therefore δt has the same dependence on r as before. Since frequencies were higher on the outside, i.e., on $R = R_{\text{source}}$, than on the body they will be the same in the time-reversed problem. This means that radiation arrives outside the critical radius with a higher frequency than it had when emitted from the surface of the body. The radiation is blueshifted, and the numbers in Table 2 give the *reciprocal* of the blueshift factor. The point emerges that the blueshift is stronger the deeper inside the critical radius the boundary of the body at the moment of emission, in spite of the large gravitational redshift involved.

IV. ANALOGY WITH GEOMETRICAL OPTICS

In the above treatment we have assumed that light signals propagate along null lines $ds = 0$. In practice, however, light signals from a source do not propagate sharply along the null cone from that source. There is a scattering of light inside the null cone. This curious effect arises from the curvature of space time and can be understood in the following way.

The electromagnetic 4-potential at a point x, due to a moving electric charge e with world line at a^{i_A} is given by

$$A_{i_x} = 4\pi e \int G_{i_x i_A}{}^{(\mathrm{Ret})} d a^{i_A} , \tag{16}$$

where $G_{i_x i_A}{}^{(\mathrm{Ret})}$ is a two-point vector Green's function. DeWitt and Brehme (1960) have shown that $G_{i_x i_A}{}^{(\mathrm{Ret})}$ can be written as twice the retarded part of

$$\bar{G}_{i_x i_A} = f\delta(\sigma) + g\theta(\sigma) , \tag{17}$$

where f, g are known functions of space-time geometry, 2σ the square of the "interval" between x, A; and δ, θ are the delta and Heaviside functions.

The electromagnetic field can be obtained from the curl of equation (16). The first term in equation (17)—the one involving the delta function—is responsible for sharp propagation along the null cone and is therefore analogous to light propagation in geometrical optics. The second term represents scattering inside the light cone ($\sigma > 0$) and has no analogue in geometrical optics. This term disappears only if the space time is conformally flat.

The line element (2) is not conformally flat, and scattering inside the light cone will take place. In the present paper, in considering light tracks as given by $ds = 0$, we have ignored this effect. This is justified in the present context since propagation along the light cone, represented by the first term is always "ahead" of the propagation inside the light cone, represented by the second term. We are interested only in the signal that travels the "fastest."

V. CONCLUSION

In a recent paper, two of us have discussed the effect of the C-field on imploding objects, and have shown that singularities do not develop (Hoyle and Narlikar 1964). Positive energy flows out of a body as the singularity is approached causing a repulsive gravitational effect to set in that ultimately halts the implosion. The present investigation suggests that interesting blueshift effects may arise in the re-expansion phase which follows the halting of the implosion. It must be emphasized, however, that the re-expansion is not as simple as the time-reversed situation considered above. Time reversal affects the line element inside and outside the body instantaneously. This is probably not a bad approximation for the interior, but for the exterior the reversal will not be everywhere instantaneous, but will be propagated ahead of the re-expanding body. The energy-momentum tensor is non-zero outside the body and care is needed in order not to be misled by static concepts. Our belief is that concepts based on the static Schwarzschild solution for a point mass are seriously misleading in the dynamic problem. Effects analogous to the Doppler shift arise, and indeed the results of Table 2 are in the main due to the "Doppler shift" which plays a stronger role than the "gravitational shift." The essential point is that the radiation is emitted by a surface which moves, in the sense that the proper area of the surface changes with time. The latter effect becomes stronger the deeper inside the critical radius the emitting surface happens to be.

The existence of strong blueshifts is of great interest in relation to the problem of the emission of high-energy quanta and particles from collapsed objects. It may well be significant that dynamical oscillatory effects, at appropriate phases of oscillation, appear capable of greatly lifting quantum energies. It was this effect which Hoyle, Fowler, Burbidge, and Burbidge (1964) had in mind in a recent general discussion of the problems of relativistic astrophysics.

REFERENCES

Datt, B. 1938, *Z. f Phys.*, **108**, 314.
DeWitt, B. S , and Brehme, R. W. 1960, *Ann. Phys* , **9**, 220.
Hoyle, F., Fowler, W. A., Burbidge, G. R., and Burbidge, E. M. 1964, *Ap. J.*, **139**, 909.
Hoyle, F., and Narlikar, J. V. 1964, *Proc. Roy. Soc. London*, *A*, **278**, 465.
Oppenheimer, J. R., and Snyder, H. 1939, *Phys. Rev.*, **56**, 455.