

On a Wavefunction Picture for the Direct Particle Theory of Gravitation and Electromagnetism
Author(s): F. Hoyle and J. V. Narlikar
Reviewed work(s):
Source: *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, Vol. 299, No. 1457 (Jun. 14, 1967), pp. 188-211
Published by: The Royal Society
Stable URL: http://www.jstor.org/stable/2415723
Accessed: 03/12/2011 04:38

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at http://www.jstor.org/page/info/about/policies/terms.jsp

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.



The Royal Society is collaborating with JSTOR to digitize, preserve and extend access to *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences.*

http://www.jstor.org

On a wavefunction picture for the direct particle theory of gravitation and electromagnetism

By F. Hoyle, F.R.S. and J. V. Narlikar

Institute of Theoretical Astronomy, University of Cambridge

(*Received* 5 *December* 1966)

The paper is in three parts. The direct particle theory developed in previous papers is reviewed in the introductory first part. A new result is obtained, that while long-range gravitational forces are necessarily attractive very short-range forces must be repulsive. The theory for strong gravitational fields differs radically therefore from general relativity. In general relativity a finite particle collapses into a singularity. Here a singularity explodes into a finite particle.

The concept of a finite particle is explored in the second and third parts. In the second part, the particle is treated as a classical world tube. A further new result emerges, that the inertial mass of such a particle must be very small. The gravitational mass is the same as in the line singularity case, but it turns out that a particle represented by a world tube shields its inertial mass through its own influence on the metric tensor. It is suggested that the very small ratio of gravitational to electrical forces may be due, not to the weakness of gravitation but to the smallness of the inertial mass. The advantage of this suggestion is that no very large dimensionless number need appear in the action formula.

In the third part we pass to a wavefunction picture for the particles. An action principle is used that leads to the Dirac equation for the particles. The field equations are the same as in the line singularity case, except that the energy-momentum tensor associated with matter is changed from the form used in general relativity to the form that is usual for a Dirac 'field'. The wave-function picture, as developed here, contains the simpler ideas of the quantum theory—the superposition principle, stationary states, the quasi-classical theory of radiative transitions. No attempt is made in this paper to extend the theory to include effects that are particular to quantum electrodynamics.

Introduction

In previous papers (1963, 1964*a*, 1966) we have developed a direct particle approach to the classical theory of gravitation and electromagnetism. The action takes the form

$$J = -\sum_{a<b}\sum \iint \tilde{G}(A,B)\,\mathrm{d}a\,\mathrm{d}b + \sum_{a<b}\sum 4\pi e_a e_b \iint \bar{G}_{i_A i_B}\,\mathrm{d}a^{i_A}\,\mathrm{d}b^{i_B}, \tag{1}$$

in which $\tilde{G}(A,B)$, $\bar{G}_{i_A i_B}$ are two-point symmetrical propagators satisfying a scalar and vector wave-equation respectively:

$$g^{i_x k_x}\tilde{G}(x,A)_{;i_x k_x} + \tfrac{1}{6}R(x)\,\tilde{G}(x,A) = (-\bar{g})^{-\frac{1}{2}}\delta^{(4)}(x,A), \tag{2}$$

$$g^{i_x k_x}\bar{G}(x,A)_{i_x i_A;j_x k_x} + R^{j_x}_{i_x}\bar{G}(x,A)_{j_x i_A} = (-\bar{g})^{-\frac{1}{2}}\bar{g}_{i_x i_A}\delta^4(x,A). \tag{3}$$

The signature of the metric is here taken as $(---+)$. Comparison with previous papers shows that the signs of both terms in (1) have been reversed. This is because a negative sign was formerly used on the right-hand sides of (2) and (3).

A mass field $m^{(b)}(x)$ and a potential field $A_i^{(b)}(x)$ are defined for particle b by

$$m^{(b)}(x) = \int \tilde{G}(x,B)\,\mathrm{d}b, \quad A_i^{(b)}(x) = 4\pi e_b \int \bar{G}_{i_x i_B}(x,B)\,\mathrm{d}b^{i_B}. \tag{4}$$

Once again we have inverted the sign in the definition of $m^{(b)}(x)$ in order to allow for the change in the wave equation for $\tilde{G}$. We have kept to the same sign as we used formerly in the definition of $A_i^{(b)}$, however. This means that our present $A_i^{(b)}$ is minus our former $A_i^{(b)}$. It can readily be verified that the present definition is in accord with the usual convention for the sign of the electromagnetic potential. We have made the change because the vector $A_i^{(b)}$ will appear explicitly in the wave equations obtained later in the paper. In classical theory, A_i does not appear in equations of physical relevance so the sign of the definition is immaterial. That is to say, A_i appears in the Dirac equation but not in Maxwell's equations. For comparison with other authors it is preferable to use the conventional sign for A_i.

Quantities m_a and A_a^i are defined by

$$m_a(x) = \sum_{b \neq a} m^{(b)}(x), \tag{5}$$

$$A_a^i(x) = g^{ik} \sum_{b \neq a} A_k^{(b)}(x). \tag{6}$$

Using (4) to (6), we can write the action in the form

$$J = \tfrac{1}{2} \sum_a J_a = \tfrac{1}{2} \sum_a \left[-\int m_a \, \mathrm{d}a + e_a \int A_a^i \, \mathrm{d}a_i \right], \tag{7}$$

which is similar in appearance to the action of the usual theory. There are several differences of principle, however. The electromagnetic field does not have independent degrees of freedom. The masses m_a arise from other particles—m_a would be zero if there were no particles other than a. If there were no particles at all there would be no theory. These features are evidently in agreement with the ideas of Mach.

The theory is invariant with respect to conformal geometrical transformations,

$$g_{ik}^* = \Omega^2 g_{ik}, \tag{8}$$

where Ω is any well-defined scalar function of position. In the usual theory, Maxwell's equations are conformally invariant but Einstein's equations are not. The incorporation of conformal invariance into the gravitational part of the present theory yields two interesting results: (1) long-range gravitational forces must be attractive, (2) there can be no cosmical term of the form λg_{ik} in the field equations.

The two terms on the right-hand side of (1) are strikingly similar in form, one built out of the biscalar $\tilde{G}(A, B)$ the other out of the bivector $\bar{G}_{i_A i_B}$. The factors $4\pi e_a e_b$ are numbers that determine the relative strength of electrical and gravitational forces. We have already remarked (1964*a*) on the circumstance that $4\pi |e_a e_b|$ must apparently be $\approx 10^{39}$ in order that the theory agree with observation. In this respect the present theory seemed no different from the usual theory. However, in the second part of this paper we obtain results that shed some new light on this point, which indeed suggest that no very large dimensionless numbers need be introduced into the action.

A further difference is that the present theory is symmetric between past and future, because $\tilde{G}(A, B)$, $\bar{G}_{i_A i_B}$ are symmetric between points A and B. How such a theory can be consistent with observation—i.e. with observed retarded solutions—

has been discussed (1963) along the lines first proposed by Wheeler & Feynman (1945, 1949).

Einstein's equations appear in the present theory only in a smooth fluid approximation, and only in a particular conformal frame—the frame in which

$$\sum_a m^{(a)}(x) = \text{constant} = m_0, \text{ say.} \tag{9}$$

The numerical value of m_0 can be changed by a scale change of the coordinates. Any such scale change of the masses cancels in the equations, however, so that their physical content is independent of the numerical value of m_0. The smooth fluid picture involves the following approximations. For any particle a,

$$\sum_{b \neq a} m^{(b)} \simeq \sum_b m^{(b)} = m_0, \tag{10}$$

$$\sum_{a<b} \sum m^{(a)} m^{(b)} \simeq \tfrac{1}{2} \left(\sum_a m^{(a)}\right) \left(\sum_b m^{(b)}\right) = \tfrac{1}{2} m_0^2. \tag{11}$$

The approximation (10) breaks down near particle a, but is valid in regions well removed from a. Approximation (11) is valid when m_0 is made up of a large number of comparable terms in the summation $\sum_b m^{(b)}$. This is the case for the contribution of the universe at large, and it is also true for macroscopic aggregations of particles, such as the Sun and planets. Because both (10) and (11) are valid in the solar system we cannot expect to test the differences between the present theory and Einstein's theory through observations of the kind used in the well-known tests of general relativity. Only in problems involving proximity to particles, or in strong gravitational fields, can we expect significant differences to arise.

The solution for a single particle, analogous to the usual Schwarzschild solution, showed (1966) the differences that arise when the smooth-fluid approximation is invalid. Taking a spherically symmetric static metric, as in the Schwarzschild solution,

$$ds^2 = e^{\nu} dt^2 - e^{-\nu} dr^2 - r^2(d\theta^2 + \sin^2\theta \, d\phi^2), \tag{12}$$

we obtained the following result:

$$e^{\nu} = \left(1 - \frac{p}{r}\right)^2, \quad \mu = \frac{m_0 p}{r - p}, \quad 2\pi m_0 p = 1. \tag{13}$$

This solution applies to a particle at rest in a coordinate frame in which g_{ik} is reduced locally to the Minkowski form. The quantity μ is the mass field generated by the particle. If we are dealing with particle a, for example, $\mu = m^{(a)}$. The conformal frame is then the one in which $\mu + \sum_{b \neq a} m^{(b)} = m_0$.

If the coordinates are changed by a scale factor ξ, all lengths, including p, are changed by ξ, while m_0, being the reciprocal of a length, is changed by ξ^{-1}. The 'gravitational mass' is p and the inertial mass is $m_0 - \mu(r = 0) = 2m_0$, so that the ratio of gravitational and inertial masses is $p/2m_0 = 1/(4\pi m_0^2)$, the same dependence on m_0^{-2} as the 'gravitational constant'. This is necessary, otherwise gravitational orbits would be affected by ξ. The result $p/2m_0 = 1/(4\pi m_0^2) > 0$ requires long-range gravitational forces to be attractive.

Here we note that so far as mass and gravitation are concerned only one kind of particle has been included in the action (1). (By putting $e_a = \pm e$ in the second term on the right-hand side of (1) both signs for the electric charge are catered for.) Hence it is no surprise that for a particular reference frame—i.e. a particular m_0—the length p is fixed. If instead of the first term in (1) we had written

$$-\sum_{a<b}\sum\iint w_a w_b \tilde{G}(A,B)\,\mathrm{d}a\,\mathrm{d}b, \tag{14}$$

with w_a a numerical weight factor associated with particle a, the values both of p and of the inertial mass for particle a would be multiplied by w_a. In this way particles with different masses could readily be included in the theory. At the present stage, however, we have no occasion to consider particles of different masses, so for simplicity we prefer to work with (1).

Consider a test particle released from rest in the metric given by (13). The test particle satisfies equations formally similar to those of the Schwarzschild solution, namely

$$\frac{\mathrm{d}^2 r}{\mathrm{d}s^2} - \tfrac{1}{2}\nu'\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 + \tfrac{1}{2}\mathrm{e}^{2\nu}\nu'\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 = 0, \tag{15}$$

$$\mathrm{d}t/\mathrm{d}s = c\,\mathrm{e}^{-\nu}, \tag{16}$$

where $\nu' = \mathrm{d}\nu/\mathrm{d}r$, and c is a constant. Initially $\mathrm{d}r/\mathrm{d}s = 0$, so initially

$$\mathrm{d}^2 r/\mathrm{d}s^2 = -\tfrac{1}{2}c^2\nu'. \tag{17}$$

With $\mathrm{e}^{\nu} = (1-p/r)^2$ we have $\nu' > 0$ for $r > p$, but $\nu' < 0$ for $r < p$. Noticing that r is a space-like coordinate even for $r < p$ (unlike the Schwarzschild solution), we see that, while for large r the test body falls towards the particle, for $r < p$ the body is repelled outwards. Our earlier statements (1964*a*, 1966) that gravitational forces are necessarily attractive in a conformally invariant theory must therefore be restricted to interactions over distances $> p$. However, since the gravitational masses of individual particles are very small compared to the distances involved in macroscopic gravitational problems—e.g. 10^{-52} cm for the proton compared to a distance scale of $\sim 10^{13}$ cm in the solar system—no change is required in the usual kind of problem.

The situation for small interaction distances is of great theoretical interest, however. In general relativity the action is taken as the sum of two expressions, not a single expression as in (14). As we have previously remarked, this necessarily implies an ambiguity of sign for the ratio of the constants that multiply the two expressions used in general relativity. The 'gravitational constant' which appears in this ratio could either be positive or negative, corresponding to gravitational forces that are always attractive or to forces that are always repulsive. Observation at long range demands a choice of the attractive case. This inevitably leads to the implosion of sufficiently compact bodies into singularities, since in general relativity there is no possibility of inverting the sign of the gravitational constant at short range.

In the present theory, on the other hand, we have the remarkable result that long-range gravitational forces are necessarily attractive, sufficiently short range forces are necessarily repulsive. The effect is similar to what it would be in general relativity if at the critical Schwarzschild radius, $r = 2Gm$, the gravitational constant

G could be made to change sign. Our point of view is that gravitation is given by a single expression (14) in the action, not by two quite disparate terms. Separation of (14) into two expressions of the form used in general relativity happens to be a good approximation when we are dealing with the long-range interactions of many particles, but it is no approximation at all for very strong fields.

The real world is a quantized world. Manifestly then, the present theory can be of no ultimate value unless it is expressed in a quantized form. Yet we have hitherto been reluctant to move towards a quantized theory, because we have felt something more than a pragmatic view was needed. When a theoretical situation is obscure, a pragmatic attitude is necessary, but when a sophisticated theory already exists—as it does in the quantum theory—it is permissible to follow a less urgent line of development. We have avoided abandoning the classical theory until we arrived at a position that forces us to do so. It has seemed permissible to ask the question: Why is the universe not classical? Why from a theoretical point of view can the universe not be classical? Certainly there is no cause to abandon classical theory because of self-energy difficulties—these are overcome by the absorber theory. Indeed the classical theory is at least as well placed as the quantum theory in regard to self-energy problems, perhaps better placed. We have felt that a theoretical self-contradiction must eventually emerge. Now this would seem to have happened. If instead of our particles being strict line singularities we regard them as having radius $r = \epsilon \ll p$ we see from (17) that the surface of a particle experiences an outward acceleration $\mathrm{d}^2r/\mathrm{d}s^2 = -\frac{1}{2}c^2\nu'$. From the line element (12), $\mathrm{d}t/\mathrm{d}s = \mathrm{e}^{-\frac{1}{2}\nu}$ when $\mathrm{d}r = \mathrm{d}\theta = \mathrm{d}\phi = 0$. Hence the outward acceleration is $-\frac{1}{2}\nu'\mathrm{e}^{\nu}$ evaluated at $r = \epsilon$. For $\mathrm{e}^{\nu} = (1-p/r)^2$, the acceleration is p^2/ϵ^3. Unless the particle is 'glued' together by an infinite stress of unknown origin it must explode from a line singularity into a finite distribution. This is exactly opposite to the situation in general relativity. In general relativity, a finite distribution implodes into a singularity. The present result seems to us to justify our point of view in following the classical theory to its limit, for if we had not done so this short-range repulsion of gravitational forces would very likely have escaped notice.

Now we have a choice of procedures. We can continue using classical ideas but with particles represented by world tubes of finite radius, or we can represent particles by wave-functions in the sense of the quantum theory. The merit of the first alternative is that it may again lead to results that would be less easily obtained by the second method. The merit of the second alternative is that this is the direction in which the theory must eventually develop. In the following two parts of this paper we consider both alternatives.

Particles represented by classical world tubes

It will be useful in relation to later work to write out explicitly the equations given by the gravitational theory of the previous section. Omitting the electromagnetic term in (1), the field equations are (1964*a*, 1966):

$$\sum_{a<b}\sum [m^{(a)}m^{(b)}(R_{ik}-\tfrac{1}{2}g_{ik}R)+m^{(a)}(m^{(b)}_{;ik}-g_{ik}g^{pq}m^{(b)}_{;pq})+m^{(b)}(m^{(a)}_{;ik}-g_{ik}g^{pq}m^{(a)}_{;pq}) - 2(m^{(a)}_{;i}m^{(b)}_{;k}+m^{(a)}_{;k}m^{(b)}_{;i}-\tfrac{1}{2}g_{ik}m^{(a);l}m^{(b)}_{;l})]+3T_{ik}=0, \quad (18)$$

where
$$T^{ik}(x) = \sum_a \int \delta^{(4)}(x, A)\,[-\bar{g}(x, A)]^{-\frac{1}{2}} m_a \frac{\mathrm{d}a^i}{\mathrm{d}a}\frac{\mathrm{d}a^k}{\mathrm{d}a}\,\mathrm{d}a. \tag{19}$$

The functions $m^{(a)}$ satisfy
$$g^{pq} m^{(a)}_{;pq} + \tfrac{1}{6} R m^{(a)} = N^{(a)}(x), \tag{20}$$

in which
$$N^{(a)}(x) = \int \delta^{(4)}(x, A)\,[-\bar{g}(x, A)]^{-\frac{1}{2}}\,\mathrm{d}a. \tag{21}$$

In the absence of an electromagnetic field the equation of motion for particle a is
$$\frac{\mathrm{d}}{\mathrm{d}a}\left(m_a \frac{\mathrm{d}a^i}{\mathrm{d}a}\right) + m_a \Gamma^i_{kl}\frac{\mathrm{d}a^k}{\mathrm{d}a}\frac{\mathrm{d}a^l}{\mathrm{d}a} - g^{ik}\frac{\partial m_a}{\partial x^k} = 0. \tag{22}$$

In equations (18) to (21) we are still regarding the particles as line singularities. Suppose now we replace the world line of particle a by a bundle of parallel world lines, denoted by $a_1, a_2, \ldots$. Define
$$\left.\begin{aligned} m^{(a)} &= m^{(a_1)} + m^{(a_2)} + \ldots, \quad m_a = \sum_{b \neq a_1, a_2, \ldots} m^{(b)}, \\ N^{(a)} &= N^{(a_1)} + N^{(a_2)} + \ldots. \end{aligned}\right\} \tag{23}$$

Equation (20) continues to hold, but what of the field equations? If we admit interactions between $a_1, a_2, \ldots$ of the same kind as the interactions between $a, b, \ldots$ we have simply replaced particle a by a set of particles $a_1, a_2, \ldots$, all with the same status as the original particle. The situation is changed, however, if there are no interactions between $a_1, a_2, \ldots$ Then there are no quadratic terms of the form $m^{(a_1)} m^{(a_2)}$ in the field equations. The terms $m^{(a_1)}, m^{(a_2)} \ldots$ appear only linearly and can be summed to give $m^{(a)}$. Evidently, the field equations retain the form (18) with $m^{(a)}$ defined by (23). Moreover, T^{ik} is still given by (19), so that (18), (19) and (20) continue to hold. The only change is in the definition of $N^{(a)}$. Here a normalization factor is still required, because $N^{(a)}$ given by (23) weights each member of $a_1, a_2, \ldots$ equally with $b, c, \ldots$. If there are n members in the series $a_1, a_2, \ldots$ we take
$$nN^{(a)} = N^{(a_1)} + N^{(a_2)} + \ldots, \tag{23'}$$

in place of (23), where $N^{(a_1)}, \ldots$ each have the form (21). That is to say,
$$nN^{(a)}(x) = \int \delta^4(x, A_1)\,[-\bar{g}(x, A_1)]^{-\frac{1}{2}}\,\mathrm{d}a_1 + \ldots. \tag{24}$$

We can now replace the δ-functions in (24) to any required degree of accuracy by well-behaved functions, simply by using a member of any sequence of such functions that serve to define the delta function. The function $N^{(a)}(x)$ is then well-behaved, so that we have replaced the line singularity of particle a by a world tube. The choice of $a_1, a_2, \ldots$ permits us to weight the distribution of matter within the world tube in accordance with the field equations. The essential point is that equations (18), (19) and (20) continue to hold but with $N^{(a)}(x)$ a well-behaved function that defines the distribution of matter within the tube. We do not have *a priori* knowledge of $N^{(a)}(x)$, so this function must be determined by applying the field equations to an interior solution within the world tube. We proceed to consider this problem.

It will be useful to keep to the notation and method used previously for a line singularity. We consider particle a and write $\mu = m^{(a)}$. We again take the conformal frame in which $\mu + \sum_{b \neq a} m^{(b)} = \mu + m_a = m_0$, so that the inertial mass m_a is $m_0 - \mu$. In the neighbourhood of particle a we use the smooth fluid approximation for all particles other than a. The field equations then reduce to (1966):

$$\tfrac{1}{2}(m_0^2 - \mu^2)(R_{ik} - \tfrac{1}{2}g_{ik}R) = -3T_{ik} + \mu(\mu_{;ik} - g_{ik}g^{pq}\mu_{;pq}) - 2(\mu_{;i}\mu_{;k} - \tfrac{1}{4}g_{ik}\mu^{;l}\mu_{;l}). \quad (25)$$

Next we take a coordinate frame that is locally of the Minkowski form, except close to the particle. By a Lorentz transformation we reduce the particle locally to rest and we take the metric exterior to the particle to be of the form (12). The previous solution then holds in the exterior,

$$\mu = \frac{m_0 p}{r - p}, \quad \mathrm{e}^{\nu} = \left(1 - \frac{p}{r}\right)^2, \quad (26)$$

in which p is a constant of integration. This exterior solution will have to be matched to the interior solution.

We look for a static solution for the interior, arguing that because gravitational forces are attractive at long range and repulsive at short range an intermediate situation should exist in which the world tube of a particle is balanced between implosion and explosion.

In the line singularity case, (19) and (21) give

$$T^{ik}(x) = \sum_a m_a \frac{\mathrm{d}a^i}{\mathrm{d}a} \frac{\mathrm{d}a^k}{\mathrm{d}a} N^{(a)}(x). \quad (27)$$

When particle a is represented by the bundle $a_1, a_2, \ldots$ its contribution to T^{ik} becomes

$$m_a \left[\frac{\mathrm{d}a_1^i}{\mathrm{d}a_1} \frac{\mathrm{d}a_1^k}{\mathrm{d}a_1} N^{(a_1)}(x) + \frac{\mathrm{d}a_2^i}{\mathrm{d}a_2} \frac{\mathrm{d}a_2^k}{\mathrm{d}a_2} N^{(a_2)}(x) + \ldots \right]. \quad (28)$$

Since $a_1, a_2, \ldots$ are parallel world lines, (28) simplifies to

$$m_a \frac{\mathrm{d}a^i}{\mathrm{d}a} \frac{\mathrm{d}a^k}{\mathrm{d}a} [N^{(a_1)}(x) + N^{(a_2)}(x) + \ldots] = m_a N^{(a)}(x) \frac{\mathrm{d}a^i}{\mathrm{d}a} \frac{\mathrm{d}a^k}{\mathrm{d}a}, \quad (29)$$

because $\mathrm{d}a^i/\mathrm{d}a$ is the same for all members of the bundle. The right-hand side also gives the contributions to T^{ik} when $N^{(a)}(x)$ becomes a well-behaved function of x. For a general world tube the vector $\mathrm{d}a^i/\mathrm{d}a$ is also a function of x, but in the present static spherically symmetric case we can write

$$\mathrm{d}a^i/\mathrm{d}a = \mathrm{e}^{-\frac{1}{2}\nu} g_4^i, \quad (30)$$

the metric for the interior being

$$\mathrm{d}s^2 = \mathrm{e}^{\nu}\mathrm{d}t^2 - \mathrm{e}^{\lambda}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \quad (31)$$

Since other particles $b, c, \ldots$ make no contribution to T^{ik} in the interior of a, we have

$$T^{ik}(x) = [m_0 - \mu(x)]\,\mathrm{e}^{-\nu} g_4^i g_4^k N^{(a)}(x) \quad (32)$$

within the world tube of particle a.

Replacing T^{ik} in (25) by the right-hand side of (32) we have the following situation. In the static spherically symmetric problem the field equations yield three independent equations, which together with

$$g^{pq}\mu_{;pq}+\tfrac{1}{6}R\mu = N^{(a)}, \tag{20'}$$

determine the four functions ν, λ, μ, $N^{(a)}$. Rather than tackle these four equations directly we change the grouping, beginning by contracting the field equations

$$R = \frac{2}{r^2}-e^{-\lambda}\left[\nu''+\tfrac{1}{2}\nu'^2-\tfrac{1}{2}\nu'\lambda'+\frac{2}{r}(\nu'-\lambda')+\frac{2}{r^2}\right] = \frac{6N}{m_0}. \tag{33}$$

Equation (20′) is used in establishing the second equality in (33). The (11) component of the field equations gives

$$\tfrac{1}{2}(m_0^2-\mu^2)\left[e^{-\lambda}\left(\frac{\nu'}{r}+\frac{1}{r^2}\right)-\frac{1}{r^2}\right] = -\mu\mu''e^{-\lambda}+\tfrac{1}{2}\mu\mu'\lambda'e^{-\lambda}+\tfrac{3}{2}\mu'^2e^{-\lambda}-\frac{N\mu}{m_0}(m_0-\mu), \tag{34}$$

while the (44) component gives

$$\tfrac{1}{2}(m_0^2-\mu^2)\left[e^{-\lambda}\left(\frac{1}{r^2}-\frac{\lambda'}{r}\right)-\frac{1}{r^2}\right] = -\tfrac{1}{2}\mu'^2e^{-\lambda}-\tfrac{1}{2}\mu\mu'\nu'e^{-\lambda}-\frac{N\mu}{m_0}(m_0-\mu)-3N(m_0-\mu). \tag{35}$$

Equations (20′), (33), (34) and (35) determine the problem. Once again, however, we avoid a direct attack on these equations. Often it happens that by obtaining an equation for $T^{ik}_{;k}$ the discussion is simplified, although of course this does not increase the number of independent equations. After some straightforward reductions, using (20′), it can be shown that

$$T^{ik}_{;k} = -\mu^i N. \tag{36}$$

With the form (32) for T^{ik}, only the r-component of (36) yields an equation. This equation can be taken in place of (20′), and is in fact easier to handle. Setting $i=1$ in (36) we have

$$T^{1k}_{;k} = \Gamma^1_{44}T^{44} = \tfrac{1}{2}\nu'e^{-\lambda}N(m_0-\mu) = e^{-\lambda}\mu'N, \tag{37}$$

which integrates to give

$$\mu = m_0(1-qe^{-\frac{1}{2}\nu}), \tag{38}$$

in which q is a constant of integration. Equations (33), (34), (35) and (38), together with the need to match μ, e^{ν}, e^{λ} to the exterior solution (26), determine the problem.

By a scale transformation of the t coordinate, and by choosing the appropriate square root of e^{ν}, it can be arranged that *provided* $q \neq 0$ we have $q=1$. This simplifies the substitution of (38) into (33), (34) and (35). After elimination of N two equations are left, which can be expressed after some reductions in the form

$$2\nu''-\tfrac{1}{2}\nu'^2-\nu'\lambda'+\frac{4\nu'+2\lambda'}{r}+\frac{2(e^{\lambda}-1)}{r^2} = 0, \tag{39}$$

$$\tfrac{1}{8}\nu'^2(2+e^{-\frac{1}{2}\nu})-\frac{\nu'}{2r}e^{-\frac{1}{2}\nu}+\frac{e^{\lambda}-1}{r^2}(1-\tfrac{1}{2}e^{-\frac{1}{2}\nu}) = 0. \tag{40}$$

For a world tube involving well-behaved functions the metric must not be singular at $r=0$; e^{ν}, e^{λ} must neither $\to 0$ nor $\to\infty$ as $r\to 0$. The solution for ν, λ near $r=0$ can be written $\nu = \sum_{s=0,1,\ldots}\nu_s r^s$, $\lambda = \sum_{s=0,1,\ldots}\lambda_s r^s$. An expansion in series that involved

a non-integral power of r would lead to singularities in the derivatives—i.e. after a sufficient number of differentiations $\nu^{(s)} \equiv \mathrm{d}^s\nu/\mathrm{d}r^s$ or $\lambda^{(s)}$ would $\to\infty$ as $r\to 0$. Suppose the expansion for λ begins with $\lambda_0 \neq 0$. Then $(\mathrm{e}^\lambda - 1)/r^2$ is singular. Suppose the expansion for λ begins with $\lambda_1 r$. Then (39) requires ν to begin with $\nu_0 - \lambda_1 r$, otherwise there would be $1/r$ singularity. The requirement that (40) also shall have no $1/r$ singularity leads immediately to $\lambda_1 = 0$. Suppose the expansion for λ begins with $\lambda_2 r^2$. To avoid a $1/r$ singularity ν_1 must be zero–i.e. $\nu = \nu_0 + \nu_2 r^2$. Substituting $\lambda = \lambda_2 r^2$, $\nu = \nu_0 + \nu_2 r^2$ in (39), (40) leads easily to $\lambda_2 = -2\nu_2$, $\nu_2 = \lambda_2(\mathrm{e}^{\frac{1}{2}\nu_0} - \frac{1}{2})$, requiring either $\lambda_2 = \nu_2 = 0$, or $\mathrm{e}^{\frac{1}{2}\nu_0} = 0$. The latter case is excluded since e^ν cannot be zero at $r = 0$. We now have $\nu'' = \lambda'' = \nu' = \lambda' = 0$ at $r = 0$. Repeated differentiations of (39), (40) then show that all derivatives of λ, ν are zero at $r = 0$. There is no solution that is non-singular at $r = 0$ when $q \neq 0$. The remaining possibility is the case $q = 0$, i.e. $\mu = m_0$.

Equations (34) and (35) are satisfied for $\mu = m_0$ while (33) determines N in terms of ν, λ. Since this interior solution must match (26) at the boundary of the world tube, the radius must be $2p$—just twice the radius $r = p$ at which gravitational forces reversed sign in the case of a line singularity.

Equation (33) is not alone sufficient to determine an explicit solution for ν, λ, even when N is given. A further condition is needed. Consider the requirement that the radial acceleration $\mathrm{d}^2r/\mathrm{d}s^2$ shall be zero for a test body at rest everywhere throughout the interior. From (17) this requires $\nu' = 0$ throughout the interior. Inserting $\nu' = \nu'' = 0$ in (33) we have

$$\frac{3N}{m_0} = \frac{1}{r^2} - \left(\frac{1}{r^2} - \frac{\lambda'}{r}\right)\mathrm{e}^{-\lambda}. \tag{41}$$

At $r = 0$ we again must have $\mathrm{e}^\lambda = 1$ to avoid a singularity. Integrating (41) therefore gives

$$\mathrm{e}^{-\lambda} = 1 - \frac{3}{m_0 r}\int_0^r N r^2\,\mathrm{d}r. \tag{42}$$

At $r = 2p$ the value given by (42) must match the value of $\mathrm{e}^{-\lambda}$ given by $\mathrm{e}^{-\lambda} = \mathrm{e}^\nu$ for the exterior solution. By (26) this is $\frac{1}{4}$, so that we must have

$$4\pi\int_0^{2p} N r^2\,\mathrm{d}r = 2\pi m_0 p. \tag{43}$$

In the line singularity solution the integral on the left-hand side of (43) is unity. Hence, if the world tube at present under consideration is given the same weight as the line singularity represented by (21), equation (44) becomes

$$2\pi m_0 p = 1, \tag{44}$$

as in (13) for the line singularity solution.

One might have expected the condition $\mathrm{d}^2r/\mathrm{d}s^2 = 0$ leading to $\nu' = 0$ to be necessary for the equilibrium of the world tube, and for it to have emerged from the field equations. The reason why this condition has to be imposed additionally to the field equations is that the inertial mass $m_0 - \mu$ is now zero throughout the interior. However, μ would not have turned out to be exactly equal to m_0 if we had included the small terms that were omitted in the smooth fluid approximation (for particles other

than a) or if electromagnetic terms had been included. It is therefore reasonable to use the additional condition imposed above, and hence to take m_0, p as being again related by (44).

So long as the inertial mass is not exactly zero we can write the equations of motion (22) in the form

$$\frac{1}{m_a}\frac{\mathrm{d}}{\mathrm{d}a}\left(m_a\frac{\mathrm{d}a^i}{\mathrm{d}a}\right)+\Gamma^i_{kl}\frac{\mathrm{d}a^k}{\mathrm{d}a}\frac{\mathrm{d}a^l}{\mathrm{d}a}-g^{ik}\frac{\partial \ln m_a}{\partial x^i}=0.$$

We see that m_a can be reduced by any constant factor we please without the gravitational orbit being affected. Hence the smallness of $m_a = m_0 - \mu$ is no embarrassment in a purely gravitational theory—here we note that it is the gravitational mass p that enters the Christoffel symbols, not the inertial mass. Because of $2\pi m_0 p = 1$ the length p is the same as in the line singularity solution.

What now of electromagnetic forces? Including electromagnetic forces, the equations of motion take the form

$$\frac{1}{m_a}\frac{\mathrm{d}}{\mathrm{d}a}\left(m_a\frac{\mathrm{d}a^i}{\mathrm{d}a}\right)+\Gamma^i_{kl}\frac{\mathrm{d}a^k}{\mathrm{d}a}\frac{\mathrm{d}a^l}{\mathrm{d}a}-g^{ik}\frac{\partial \ln m_a}{\partial x^i}=\frac{e_a}{m_a}\frac{\mathrm{d}a^k}{\mathrm{d}a}\sum_{b\neq a}F^{(b)\,i}{}_k, \tag{45}$$

in which $F^{(b)}_{ik}$ is the electromagnetic field due to particle b. Reducing m_a by a constant factor now changes the orbit, the Lorentz force becomes more dominant.

When $m_a = m_0 - \mu = 2m_0$, as it is in the line singularity solution, the orbits calculated from (45) agree with observed orbits only when $|4\pi e_a e_b|$ in the electromagnetic part of the action is taken to be a number of order 10^{39}. Yet the similarity of the structure of the two terms on the right-hand side of (1) makes $|4\pi e_a e_b| \approx 10^{39}$ seem very implausible. We would rather expect $|4\pi e_a e_b| \approx 1$, and this becomes possible in the present world tube solution, because m_a is now small. Indeed, if our solution $m_a = 0$ were taken literally, electromagnetic forces would become infinitely large.

The difference between infinity and 10^{39} is probably not significant in a classical model. We have already pointed out that the inclusion of small terms omitted above from the field equations would lead to m_a becoming non-zero. The simplest further step is to return to the strict gravitational equations, no longer using the smooth-fluid approximation for particles other than a. In place of (25) the field equations are

$$\begin{aligned}\tfrac{1}{2}\Big[(m_0^2-\mu^2)-\sum_{b\neq a}m^{(b)2}\Big](R_{ik}-\tfrac{1}{2}g_{ik}R) &= -3T_{ik}+\mu(\mu_{;ik}-g_{ik}g^{pq}\mu_{;pq})\\ &\quad -2(\mu_i\mu_k-\tfrac{1}{4}g_{ik}\mu^{;l}\mu_{;l})+\sum_{b\neq a}m^{(b)}(m^{(b)}_{;ik}-g_{ik}g^{pq}m^{(b)}_{;pq})\\ &\quad -2\sum_{b\neq a}(m^{(b)}_{;i}m^{(b)}_{;k}-\tfrac{1}{4}g_{ik}m^{(b);l}m^{(b)}_{;l}).\end{aligned} \tag{46}$$

Equations similar to (33), (34) and (35) can be written out in which quadratic terms in $m^{(b)}$ and its derivatives, $b \neq a$, now appear. To solve for $m_0 - \mu$ it is necessary to know the number n of terms that contribute significantly to sums like $\sum_{b\neq a} m^{(b)2}$. This is a problem of cosmology. If we assume that only particles within the observable universe make effective contributions then $n \simeq 10^{80}$. A solution of the form $m_0 - \mu \simeq n^{-\frac{1}{2}}m_0$ is needed. Unfortunately our investigation of (46) leads to $m_0 - \mu \simeq n^{-1}m_0$. This is for T_{ik} still given by (32). When other terms are included in

T_{ik}—i.e. other field interactions—it is possible to find arguments that lead to the required result. But we have not been able to convince ourselves of the cogency of these arguments, since when one knows the answer one is seeking, and when one is free to adjust a number of parameters, it is not usually difficult to arrive at a required result. Our view is that the present theory is probably too far removed from physical reality—i.e. the quantum theory—for the required result to emerge in an unequivocal form. To calculate a term less than the main term by $\sim 10^{39}$ demands a highly sophisticated theory, or so it would seem. The present theory has value, we believe, in giving the reason why the ratio of electrical forces to gravitational forces is observed to be very large; not because the electromagnetic coupling constant in the action is very large, but because the inertial masses are small. The deviation of the metric close to a particle from the Minkowski form enables the particle to shield its inertial mass almost completely. 'Bare' inertial masses exceed the shielded values by $\approx 10^{39}$, a factor which probably arises as the square root, $n^{\frac{1}{2}}$, of the number of particles of cosmological significance in the determination of $\sum_{b \neq a} m^{(b)}$.

The present theory also has a bearing on the singularity problem in cosmology. Solutions of many-particle problems must show repulsive gravitational effects when the particles are sufficiently closely spaced. Replacement of the usual gravitational theory by the present theory yields cosmologies without singularities, but which are otherwise similar to the Friedmann models when the particles are not closely spaced. Cosmologies that 'bounce' are obtained. It is also of interest that if creation of matter is postulated it is possible to obtain the steady-state cosmology *without* introducing the C-field used in previous papers. It turns out that the mass fields can play an almost identical role to the C-field. We shall discuss these problems in a separate paper. They are not conveniently dealt with here because it is best to treat them in a different conformal frame from the one used throughout the above work. We simply note that cosmologies behave in many ways like single particles. For example, the interior solution of the Oppenheimer–Snyder problem of a collapsing discrete object has the same mathematical structure as a cosmological singularity.

Particles represented by wave functions

In the previous section we were able to discuss the theory in a simple classical way, using a world-tube in place of a line singularity, because we confined the explicit mathematical work to gravitation and inertia. In place of the line integral $\int \mathrm{d}a$ we introduced the scalar function $N^{(a)}(x)$. To extend the discussion to electromagnetism, a vector function of position is also needed—i.e. to replace $\int A_a^i \, \mathrm{d}a_i$ a vector is needed in place of $\mathrm{d}a_i$. If we had no *a priori* knowledge of the quantum theory we should probably seek to extend the theory by introducing a vector, $M_i^{(a)}(x)$ say, so that the world-tube of particle a would be represented by five functions $N^{(a)}$, $M_i^{(a)}$. We would probably argue by analogy with the line singularity case and proceed to set $N^{(a)}$ equal to the length of the vector $M_i^{(a)}$. Just as the classical line singularity theory has now been found unsatisfactory, we should probably find self-contradictions in the classical world-tube picture, and we should be led step by step in

a long argument to the representation of the vector and scalar properties of a particle in terms of wave functions. It accordingly seems preferable to proceed directly to a wave-function picture as an essential preliminary to a quantized form of the direct particle theory. Something is lost by this immediate procedure. One obtains the answer demanded by *a priori* knowledge without seeing exactly why this is the only possible answer.

The essential step is to introduce two spinors $u_\alpha^{(a)}$, $v_\alpha^{(a)}$ for each particle in place of $N^{(a)}$, $M_i^{(a)}$. We obtain a scalar immediately

$$\overline{\psi}\psi = u_\alpha v^\alpha + u_{\dot{\alpha}} v^{\dot{\alpha}}, \tag{47}$$

where ψ is the column matrix $\begin{pmatrix} u_\alpha \\ v^{\dot{\alpha}} \end{pmatrix}$ and $\overline{\psi}$ is the conjugate imaginary row matrix $(v^\alpha, u_{\dot{\alpha}})$. To obtain a vector from u_α, v_α it is necessary either to introduce differentiation or to use quantities $g_{i\alpha\dot{\beta}}$ satisfying

$$g_{i\alpha\dot{\beta}}\, g_k^{\alpha\dot{\beta}} = 2g_{ik}. \tag{48}$$

These quantities are introduced in the following way:

The matrices $\sigma^{i\alpha\dot{\beta}}$ are defined by

$$\left.\begin{aligned} \sigma^{1\alpha\dot{\beta}} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^{2\alpha\dot{\beta}} = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \\ \sigma^{3\alpha\dot{\beta}} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma^{4\alpha\dot{\beta}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}\right\} \tag{49}$$

Here we adopt the convention that α refers to rows and $\dot{\beta}$ to columns. If we use the usual spinor rules for lowering suffices, then

$$\left.\begin{aligned} \sigma^1_{\alpha\dot{\beta}} &= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \qquad \sigma^2_{\alpha\dot{\beta}} = \begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \\ \sigma^3_{\alpha\dot{\beta}} &= \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma^4_{\alpha\dot{\beta}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned}\right\} \tag{50}$$

in which α again refers to rows and $\dot{\beta}$ to columns. The definitions (49) and (50) lead to

$$\sigma^{i\alpha\dot{\beta}}\sigma^k_{\alpha\dot{\beta}} = 2\eta^{ik}, \tag{51}$$

in which η^{ik} is the metric tensor, diag $(-1, -1, -1, +1)$. We also define

$$\sigma_i^{\alpha\dot{\beta}} = \eta_{ik}\sigma^{k\alpha\dot{\beta}}. \tag{52}$$

It then follows that $$\eta_{ik}\sigma^{i\alpha\dot{\beta}}\sigma^k_{\gamma\dot{\rho}} = \sigma_k^{\alpha\dot{\beta}}\sigma^k_{\gamma\dot{\rho}} = 2\delta^\alpha{}_\gamma\,\delta^{\dot{\beta}}{}_{\dot{\rho}}. \tag{53}$$

Let $g_{ik}(x)$ be transformed to η_{ik} by

$$x'^i = x'^i(x), \quad g_{ik}(x) = \frac{\partial x'^l}{\partial x^i}\frac{\partial x'^m}{\partial x^k}\eta_{lm}.$$

Now define $g_i^{\alpha\dot{\beta}} = \sigma_i^{\alpha\dot{\beta}}$ in the x' frame, and let $g_i^{\alpha\dot{\beta}}$ transform like a covariant vector under any coordinate transformation, so that

$$g_i^{\alpha\dot{\beta}}(x) = \frac{\partial x'^l}{\partial x^i}\,\sigma_l^{\alpha\dot{\beta}}. \tag{54}$$

For the contravariant form we have

$$g^{i\alpha\dot{\beta}}(x) = g^{ik}g_k^{\alpha\dot{\beta}} = \frac{\partial x^i}{\partial x'^m}\frac{\partial x^k}{\partial x'^n}\eta^{mn}\frac{\partial x'^l}{\partial x^k}\sigma_l^{\alpha\dot{\beta}} = \frac{\partial x^i}{\partial x'^m}\sigma^{m\alpha\dot{\beta}}.$$

Using (51) we obtain

$$g_i^{\alpha\dot{\beta}}g_{k\alpha\dot{\beta}} = \left(\frac{\partial x'^l}{\partial x^i}\sigma_l^{\alpha\dot{\beta}}\right)\left(\frac{\partial x'^m}{\partial x^k}\sigma_{m\alpha\dot{\beta}}\right) = 2\frac{\partial x'^l}{\partial x^i}\frac{\partial x'^m}{\partial x^k}\eta_{lm} = 2g_{ik},$$

so that (48) is satisfied. We also have

$$g_{k\alpha\dot{\beta}}g^{k\gamma\dot{\kappa}} = \frac{\partial x'^l}{\partial x^k}\frac{\partial x^k}{\partial x'^m}\sigma_{l\alpha\dot{\beta}}\sigma^{m\gamma\dot{\kappa}} = \sigma_{m\alpha\dot{\beta}}\sigma^{m\gamma\dot{\kappa}} = 2\delta^{\gamma}{}_{\alpha}\delta^{\dot{\kappa}}{}_{\dot{\beta}}. \tag{55}$$

The transformation $x'^i = x'^i(x)$ is not unique. The numbers g_{ik} for any particular space-time point can be taken as the coefficients of a four-dimensional quadric. In general, the quadric has unequal principal axes and the transformation to principal axes is unique. So are the scale factors by which the coordinates must be changed to reduce further to the unit sphere. Any rotation of the unit sphere can now be made, however. This freedom of choice in the x' coordinates is of course expressed by the Lorentz group of transformation. But the freedom with respect to a Lorentz transformation is removed if we are given the $g_i^{\alpha\dot{\beta}}$ instead of g_{ik}, in the sense that there is now a unique x' frame in which $g_i^{\alpha\dot{\beta}} = \sigma_i^{\alpha\dot{\beta}}$. The sixteen functions $g_i^{\alpha\dot{\beta}}$ single out a special Lorentz frame, thereby removing a 'degeneracy' in the metric tensor g_{ik}. A knowledge of $g_i^{\alpha\dot{\beta}}$, for all i, α, $\dot{\beta}$, not only determines g_{ik}, it determines a unique Lorentz frame.

So far, the suffixes α, $\dot{\beta}$ are labels for the elements of the matrices in (49) and (50). We can write $\alpha = 1, 2$; $\dot{\beta} = \dot{1}, \dot{2}$. The connexion between (49) and (50) is that α, $\dot{\beta}$ obey the usual rules $\chi_1 = -\chi^2$, $\chi_2 = \chi^1$, $\xi_{\dot{1}} = -\xi^{\dot{2}}$, $\xi_{\dot{2}} = \xi^{\dot{1}}$ that govern the raising and lowering of spinor suffices. This again is a matter of definition. However, we now depart from definitions, by introducing a critical mathematical property. For any Lorentz transformation determined by coefficients a_{ik} a complex unimodular matrix

$$c_\gamma^\alpha \equiv \begin{bmatrix} c_1^1 & c_2^1 \\ c_1^2 & c_2^2 \end{bmatrix}$$

can be found that satisfies the equation

$$c_\gamma^\alpha \bar{c}_{\dot{\kappa}}^{\dot{\beta}} g_i^{\gamma\dot{\kappa}} = a_{ik}g^{k\alpha\dot{\beta}}, \tag{56}$$

where $\bar{c}_{\dot{\kappa}}^{\dot{\beta}}$ is the complex conjugate of c_κ^β. This well-known fact lies at the root of the whole of the following discussion.

We have seen that the 'degeneracy' of g_{ik} is removed if we are given $g_i^{\alpha\dot{\beta}}$. But how is this to be done? Consider $g_i^{\alpha\dot{\beta}}u_\alpha u_{\dot{\beta}}$ where $u_{\dot{\beta}}$ is the complex conjugate of u_β. A 'spin transformation' defined by the unimodular matrix c_γ^α applied to both suffixes α, and by $\bar{c}_{\dot{\gamma}}^{\dot{\alpha}}$ applied to both suffixes $\dot{\beta}$, leaves $g_i^{\alpha\dot{\beta}}u_\alpha u_{\dot{\beta}}$ invariant,

$$g_i^{\alpha\dot{\beta}}u_\alpha u_{\dot{\beta}} = (c_\alpha^\gamma \bar{c}_{\dot{\beta}}^{\dot{\kappa}} g_i^{\alpha\dot{\beta}})(c_\gamma^\rho u_\rho \bar{c}_{\dot{\kappa}}^{\dot{\sigma}} u_{\dot{\sigma}})$$

because c_γ^α is unimodular. Thus since a spin transformation on the suffixes α, $\dot{\beta}$ of $g_i^{\alpha\dot{\beta}}$ is equivalent to a Lorentz transformation on the suffix i, it follows that $g_i^{\alpha\dot{\beta}}u_\alpha u_{\dot{\beta}}$

is invariant with respect to the combination of a Lorentz transformation on the suffix i and an appropriate spin transformation on u_α, $u_{\dot\beta}$, namely the spin transformation for which (56) is satisfied. This means that it is immaterial in which Lorentz frame we choose to set $g_i^{\alpha\dot\beta} = \sigma_i^{\alpha\dot\beta}$, provided u_α is appropriately chosen. We can change the frame in which $g_i^{\alpha\dot\beta} = \sigma_i^{\alpha\dot\beta}$ by subjecting u_α to a spin transformation. This is just the situation we are accustomed to in quantum theory, that $\psi = (u_\alpha, v^{\dot\alpha})$ must be subject to a spin transformation whenever a Lorentz transformation is made. This means that the choice $g_i^{\alpha\dot\beta} = \sigma_i^{\alpha\dot\beta}$ for a particular Lorentz frame is equivalent to a specification of the wave function. This is the normal situation in quantum theory—the wave function must be specified in relation to the choice of Lorentz frame.

It remains to be emphasized that in order for these considerations to serve for the specification of $g_i^{\alpha\dot\beta}$ it is necessary that $g_i^{\alpha\dot\beta}$ shall appear in the theory only in forms like $g_i^{\alpha\dot\beta} u_\alpha u_{\dot\beta}$—i.e. with the spin suffixes contracted against suffixes belonging to wave functions—or in the combinations given in (48) and (55).

We started the present discussion with the aim of constructing a vector from the spinors u_α, v_α. One such vector is

$$g^{i\alpha\dot\beta}(u_\alpha u_{\dot\beta} + v_\alpha v_{\dot\beta}), \tag{57}$$

in which $u_{\dot\beta} = \bar u_\beta$, $v_{\dot\beta} = \bar v_\beta$. This is the vector we shall use in the electromagnetic part of the action. It is customary to write (57) in the compact form $\bar\psi\gamma^i\psi$, where

$$\gamma^i = \begin{pmatrix} 0 & g^i_{\alpha\dot\beta} \\ g^{i\alpha\dot\beta} & 0 \end{pmatrix}, \quad \bar\psi = (v^\alpha, u_{\dot\beta}), \quad \psi = \begin{pmatrix} u_\alpha \\ v^{\dot\beta} \end{pmatrix}. \tag{58}$$

The action is expressed in terms of wave functions by

$$\begin{aligned} J = &-\sum_{a<b}\sum \iint (\bar\psi^{(a)}\psi^{(a)})_A \sqrt{(-g_A)}\,\mathrm{d}^4a\tilde G(A,B)\,(\bar\psi^{(b)}\psi^{(b)})_B \sqrt{(-g_B)}\,\mathrm{d}^4b \\ &+\sum_{a<b}\sum \iint 4\pi e_a e_b (\bar\psi^{(a)}\gamma^i\psi^{(a)})_A \sqrt{(-g_A)}\,\mathrm{d}^4a\bar G_{i_A i_B}(\bar\psi^{(b)}\gamma^i\psi^{(b)})_B \sqrt{(-g_B)}\,\mathrm{d}^4b \end{aligned} \tag{59}$$

in which $\psi^{(a)}, \psi^{(b)}, \ldots$ are the wave functions for particles $a, b, \ldots$ and $(\bar\psi^{(a)}\psi^{(a)})_A$ is the value of $\bar\psi^{(a)}\psi^{(a)}$ at the point A. The propagators $\tilde G(A,B)$, $\bar G_{i_A i_B}$ are the same as before. In writing J in the form (59) we have in mind the simple case in which $\psi^{(a)}, \psi^{(b)}, \ldots$ are non-overlapping. The discussion of the present paper is confined to this case.

A mass $m^{(b)}(x)$ and a potential field $A_i^{(b)}(x)$ are defined for particle b in analogy to (4)

$$m^{(b)}(x) = \int \tilde G(x,B)\,(\bar\psi^{(b)}\psi^{(b)})_B \sqrt{(-g_B)}\,\mathrm{d}^4b, \tag{60}$$

$$A^{(b)}_{ix}(x) = 4\pi e_b \int \bar G_{i_x i_B}(\bar\psi^{(a)}\gamma^i\psi^{(b)})_B \sqrt{(-g_B)}\,\mathrm{d}^4b, \tag{61}$$

and $m_a(x)$, $A^i_a(x)$ are again defined by $m_a = \sum_{b\neq a} m^{(b)}$, $A^i_a = \sum_{b\neq a} A^{(b)i}$. With these definitions the action can be written in the form

$$J = -\tfrac12\sum_a \int (m_a\bar\psi^{(a)}\psi^{(a)})_A \sqrt{(-g)}_A\,\mathrm{d}^4a + \tfrac12\sum_a e_a \int (A^i_a\bar\psi^{(a)}\gamma_i\psi^{(a)})_A \sqrt{(-g_A)}\,\mathrm{d}^4a, \tag{62}$$

by analogy to (7).

In our first attempt to develop a theory based on spinors we did not use the action (59), which is quadratic in the spinors both at A and B. It seemed more elegant to seek a theory involving only a single spinor at A and B. An attempt to find such a theory has indeed been made by Islam (1966). It leads to considerable complexity. Nevertheless, we feel such an approach to be an attractive one. However, in the present paper we prefer to develop the theory along lines that are more directly analogous to the usual theory.

The field equations are again to be given by a variation of geometry. Here a choice must be made, of whether the variation is to be with respect to g_{ik}, as before, or with respect to $g_i^{\alpha\dot{\beta}}$. We choose $g_i^{\alpha\dot{\beta}}$, giving a wider condition. All changes δg_{ik} can be represented by some $\delta g_i^{\alpha\dot{\beta}}$, whereas the reverse is not true—a Lorentz transformation changes $\sigma_i^{\alpha\dot{\beta}}$, but leaves η_{ik} unchanged. The significance of using $\delta g_i^{\alpha\dot{\beta}}$ instead of δg_{ik} will appear at a later stage—the extra 'width' of the condition turns out to require that the wave functions satisfy the Dirac equation.

The propagators $\tilde{G}, \bar{G}$ depend only on the metric tensor. Their variation, whether with respect to δg_{ik} or $\delta g_i^{\alpha\dot{\beta}}$, leads to terms that are directly analogous to the world line picture,

$$\frac{1}{6}\int [\sum_{a<b}\sum \{m^{(a)}m^{(b)}(R_{ik}-\tfrac{1}{2}g_{ik}R)+m^{(a)}(m^{(b)}_{;ik}-g_{ik}g^{pq}m^{(b)}_{;pq})+m^{(b)}(m^{(a)}_{;ik}-g_{ik}g^{pq}m^{(a)}_{;pq})$$
$$-2(m^{(a)}_{;i}m^{(b)}_{;k}+m^{(a)}_{;k}m^{(b)}_{;i}-\tfrac{1}{2}g_{ik}m^{(a);l}m^{(a)}_{;l})\}+3g_{ip}g_{kq}T^{pq}_{\text{el.}}]\sqrt{(-g)}\,\delta g^{ik}\,\mathrm{d}^4x. \quad (63)$$

Here $T^{pq}_{\text{el.}}$ is the electromagnetic energy-momentum tensor constructed from the $A_i^{(a)}$ in the manner discussed in detail in a previous paper (1964*b*). Equating the integrand of (63) to zero would give field equations analogous to those of the line singularity picture, set out above in (18), except that the energy-momentum tensor of the particles, given for the line singularity picture in (19), would be absent. The main problem in passing from the line singularity picture to the wave-function picture is to obtain $T^{ik}_{\text{mat.}}$ in a form appropriate to the wave-function picture, and to show that $T^{ik}_{\text{mat.}}$ takes its place in the field equations.

In the line-singularity picture $T^{ik}_{\text{mat.}}$ was obtained from

$$-\sum_a \int m_a \delta(\mathrm{d}a), \quad (64)$$

which appears in the variation of the first term on the right-hand side of (7). In analogy to (64) we might expect to obtain $T^{ik}_{\text{mat.}}$ from

$$-\sum_a \int m_a \delta(\bar{\psi}^{(a)}\psi^{(a)}\sqrt{(-g)})\,\mathrm{d}^4a. \quad (65)$$

Consideration of (65) forces us to make a decision as to whether a variation of geometry, $g^{i\alpha\dot{\beta}} \to g^{i\alpha\dot{\beta}} + \delta g^{i\alpha\dot{\beta}}$, implies a variation of the wave functions. This decision also affects the contribution of the second term on the right-hand side of (62). In any case we have still to include the variation of γ^i in the second term. Thus δJ contains, in addition to the sum of (63) and (65), the further term

$$\sum_a e_a \int A_a^i \delta(\bar{\psi}^{(a)}\gamma_i\psi^{(a)}\sqrt{(-g)})\,\mathrm{d}^4a. \quad (66)$$

What is missing so far from our argument is a conservation condition for the particles. Such a condition appears automatically in the line-singularity picture, because we take the world lines of particles to be given by fixed parametric equations—the particle is defined by the same functions $a^i(\tau)$ of the parameter τ both before and after the variation of geometry. In view of our *a priori* knowledge it would be natural to assume that (57) has zero divergence,

$$[g^{i\alpha\dot{\beta}}(u_\alpha u_{\dot{\beta}}+v_\alpha v_{\dot{\beta}})]_{;\,i}=0 \tag{67}$$

for each particle, since the condition (67) is the usual conservation law for the Dirac current. However, although (67) will eventually be found to hold good, we prefer to proceed otherwise, for the following reason. Eventually we wish to go to a fully quantized theory. Preceding second quantization it is necessary to write our equations in terms of wave functions involving all particles. This will be more awkward if our conservation condition is expressed as a separate equation for each particle. In contrast to this, (62) for J involves all particles and is in a form suitable for generalization in the required way.

The equations (67) direct attention to the concept of covariant differentiation of a spinor function. In order that a quantity such as $g^{i\alpha\dot{\beta}}u_{\dot{\beta}}u_{\alpha;\,i}$ shall have meaning it is necessary for $u_{\alpha;\,i}$ to transform under spin transformations in the same way as u_α. In terms of the affine connexion $L^\beta_{\alpha i}$ we take

$$u_{\alpha;\,i}=\frac{\partial u_\alpha}{\partial x^i}-L^\beta_{\alpha i}u_\beta, \tag{68}$$

for any spinor u_α. Writing the complex conjugate of $L^\beta_{\alpha i}$ as $L^{\dot{\beta}}_{\dot{\alpha} i}$, the equation conjugate to (68) is

$$u_{\dot{\alpha};\,i}=\frac{\partial u_{\dot{\alpha}}}{\partial x^i}-L^{\dot{\beta}}_{\dot{\alpha} i}u_{\dot{\beta}}. \tag{69}$$

Consider a spin transformation $u_\alpha\rightarrow u'_\alpha$, where $u'_\alpha=c^\beta_\alpha u_\beta$, $|c^\beta_\alpha|$ any unimodular matrix. Let $L'^\beta_{\alpha i}$ be the affine connexion giving $u'_{\alpha;\,i}$, namely

$$u'_{\alpha;\,i}=\frac{\partial u'_\alpha}{\partial x^i}-L'^\beta_{\alpha i}u'_\beta. \tag{70}$$

We require $L'^\beta_{\alpha i}$ to be such that $u'_{\alpha;\,i}=c^\beta_\alpha u_{\beta;\,i}$. This leads to

$$\frac{\partial u'_\alpha}{\partial x^i}-L'^\beta_{\alpha i}u'_\beta=\frac{\partial}{\partial x^i}(c^\beta_\alpha u_\beta)-L'^\beta_{\alpha i}c^\gamma_\beta u_\gamma=c^\beta_\alpha\left(\frac{\partial u_\beta}{\partial x^i}-L^\gamma_{\beta i}u_\gamma\right).$$

If this is to hold for an arbitrary spinor u_α we must have

$$L'^\beta_{\alpha i}c^\gamma_\beta=c^\beta_\alpha L^\gamma_{\beta i}+\frac{\partial c^\gamma_\alpha}{\partial x^i}. \tag{71}$$

It is convenient to choose $L^\beta_{\alpha i}$ so that

$$g^i_{\alpha\dot{\beta};\,k}\equiv\frac{\partial g^i_{\alpha\dot{\beta}}}{\partial x^k}-L^\gamma_{\alpha k}g^i_{\gamma\dot{\beta}}-L^{\dot{\gamma}}_{\dot{\beta}k}g^i_{\alpha\dot{\gamma}}=0 \tag{72}$$

for all i, k, α, $\dot{\beta}$. It can be shown, after some manipulative work, that (72) leads to

$$4L^\beta_{\alpha i}=g^{l\dot{\beta}\dot{\gamma}}g_{m\alpha\dot{\gamma}}\Gamma^m_{il}-g^{l\dot{\beta}\dot{\gamma}}\frac{\partial g_{i\alpha\dot{\gamma}}}{\partial x^i}, \tag{73}$$

in which Γ^m_{il} is the usual Christoffel symbol. It is not hard to show that the choice (73) for $L^{\beta}_{\alpha i}$ satisfies (71). It can also be shown without difficulty that spinor suffices can be raised or lowered within the differentiation sign.

With the choice (73) for $L^{\beta}_{\alpha i}$ the integral

$$\sum_a \int \overline{\psi}^{(a)}\gamma^i\psi^{(a)}_{;i}\sqrt{(-g)} = \sum_a \int g^{i\alpha\dot{\beta}}[u^{(a)}_{\beta}u^{(a)}_{\alpha;i} + v^{(a)}_{\alpha}v^{(a)}_{\beta;i}]\sqrt{(-g)}\,\mathrm{d}^4a \tag{74}$$

is equal to

$$-\sum_a \int g^{i\alpha\dot{\beta}}[u^{(a)}_{\alpha}u^{(a)}_{\beta;i} + v^{(a)}_{\beta}v^{(a)}_{\alpha;i}]\sqrt{(-g)}\,d^4a, \tag{75}$$

provided the flow of the total Dirac current through the boundary of the region of integration is zero. In such a case (74) is pure imaginary and we can choose a real number $\hbar$ such that

$$-\mathrm{i}\hbar\sum_a \int \overline{\psi}^{(a)}\gamma^i\psi^{(a)}_{;i}\sqrt{(-g)}\,\mathrm{d}^4a = J. \tag{76}$$

Even if the flow of the total Dirac current were not zero we could still write (76) with complex $\hbar$. We emphasize that (76) involves no physical assumption—the equation is simply a definition of $\hbar$.

The condition $\delta J = 0$ for $g^{i\alpha\dot{\beta}} \to g^{i\alpha\dot{\beta}} + \delta g^{i\alpha\dot{\beta}}$ has always seemed a mysterious and rather obscure postulate. What is the meaning of J and why should its stationary property with respect to variations of geometry be of such decisive importance? Equation (76) gives the beginning of an answer to the first of these questions. In quantum theory the momentum operator in a coordinate representation is equivalent to the operation of differentiation. Hence we can interpret $-i\hbar\overline{\psi}^{(a)}\gamma^i\psi^{(a)}_{;i}$ as a scalar 'momentum' for particle a, so that the left-hand side of (76) is the total scalar momentum of all the particles. If we now interpret $-J$ as a scalar momentum associated with particle couplings, equation (76) can be read as a statement that the total momentum of the whole universe is zero. Dividing (76) by $\hbar$, and using (62) for J, we have

$$-\mathrm{i}\int \overline{\psi}^{(a)}\gamma^i\psi^{(a)}_{;i}\sqrt{(-g)}\,\mathrm{d}^4a = -\tfrac{1}{2}\sum_a \int \frac{m_a}{\hbar}\overline{\psi}^{(a)}\psi^{(a)}\sqrt{(-g)}\,\mathrm{d}^4a + \sum_{a<b}\sum \frac{4\pi e_a e_b}{\hbar}$$
$$\times \iint \overline{\psi}^{(a)}\gamma^{i_A}\psi^{(a)}\sqrt{(-g_A)}\,\mathrm{d}^4a\bar{G}_{i_A i_B}\overline{\psi}^{(b)}\gamma^{i_B}\psi^{(b)}\sqrt{(-g_B)}\,\mathrm{d}^4b. \tag{77}$$

The interesting point emerges that the 'couplings' $m_a/\hbar, \ldots, |e_a e_b|/\hbar$ on the right-hand side have the forms of the inverse Compton wavelengths for particle $a, \ldots$, and of the fine structure constant. This suggests that if other particle interactions were included in J the multiplying numbers on the right-hand side would be just the usual coupling constants associated with physical 'fields'. In this connexion it is of interest that the beta decay coupling is known to have a similar form to our present gravitational and electromagnetic terms, in the sense that products appear involving four spinor wave functions. As we remarked before, one has the suspicion that the several terms appearing in J represent the break-up of a single master term into a number of fragments.

The question now arises as to whether our basic postulate is to be $\delta J = 0$ for $g^{i\alpha\dot{\beta}} \to g^{i\alpha\dot{\beta}} + \delta g^{i\alpha\dot{\beta}}$ or are we to take

$$\delta\left(J + \mathrm{i}\hbar \sum_a \int \overline{\psi}^{(a)} \gamma^i \psi^{(a)}_{;i} \sqrt{(-g)}\, \mathrm{d}^4 a\right) = 0\,? \tag{78}$$

If equation (76) has a deep-rooted physical meaning then evidently it is this equation that should be unchanged by $g^{i\alpha\dot{\beta}} \to g^{i\alpha\dot{\beta}} + \delta g^{i\alpha\dot{\beta}}$. The action J ceases to be of sole importance. In the following work we adopt (78).

We return now to the problem of the conservation condition for the particles. Our requirement is for (78) to hold for $g^{i\alpha\dot{\beta}} \to g^{i\alpha\dot{\beta}} + \delta g^{i\alpha\dot{\beta}}$ subject to some conservation condition. The action J is easily seen to be invariant with respect to phase changes of the wave functions, $\psi^{(a)} \to \psi^{(a)} \mathrm{e}^{\mathrm{i}\theta_a}$, where θ_a is any well-behaved function of position. Consider the effect of imposing invariance for (76) under such phase changes. We require

$$\sum_a \int g^{i\alpha\dot{\beta}} [u_{\dot{\beta}} u_{\alpha;i} + v_\alpha v_{\dot{\beta};i} - \mathrm{e}^{-\mathrm{i}\theta} u_{\dot{\beta}} (u_\alpha \mathrm{e}^{\mathrm{i}\theta})_{;i} - \mathrm{e}^{-\mathrm{i}\theta} v_\alpha (\mathrm{e}^{\mathrm{i}\theta} v_{\dot{\beta}})_{;i}]_a \sqrt{(-g_A)}\, \mathrm{d}^4 a = 0,$$

where $[\ ]_a$ is evaluated for particle a. This leads immediately to

$$\sum_a \int g^{i\alpha\dot{\beta}} [\theta (u_\alpha u_{\dot{\beta}} + v_\alpha v_{\dot{\beta}})_{;i}]_a \sqrt{(-g_A)}\, \mathrm{d}^4 a = 0, \tag{79}$$

provided the phase changes $\theta_a, \ldots$ are zero on the boundary of the region of integration. With $\theta_a, \ldots$ arbitrary functions within the region of integration, we must have

$$g^{i\alpha\dot{\beta}} (u_\alpha u_{\dot{\beta}} + v_\alpha v_{\dot{\beta}})_{;i} = 0 \tag{80}$$

for every particle. This is the expected conservation condition—the divergence of the Dirac current is zero for each particle.

The present derivation of (80) is much to be preferred to imposing these conditions from the outset, because it provides an example of the importance of equation (76). It adds further plausibility to the assumption of (78) as the basic postulate of the theory. Our statement now is that (78) holds both with respect to $g^{i\alpha\dot{\beta}} \to g^{i\alpha\dot{\beta}} + \delta g^{i\alpha\dot{\beta}}$ and with respect to arbitrary phase changes of the wave functions. This statement includes the required conservation condition. We now show that it leads to the Dirac equation for the wavefunction.

In the line singularity picture the equations of motion of the particles can be obtained in two ways. The more formal way is to begin by obtaining the field equations and by then taking their divergence. The more direct way is to vary the positions of the world lines keeping the geometry fixed. The reason why this second method succeeds is that it is possible to represent small displacements of the world lines at fixed geometry by the combination of a coordinate transformation and a geometrical variation. Let $a^i(\tau) \to a^i(\tau) + \delta a^i(\tau)$ be a small change in the parametric representation of world line a. Make an infinitesimal coordinate transformation $x'^i = x'^i(x)$ such that $a'^i(\tau) = a^i(\tau) - \delta a^i(\tau)$ for all values of τ. That is to say, in the x' coordinate frame the world line is represented by the same functions of τ as before. The coordinate transformation changes g_{ik} to $g_{ik} + \delta g_{ik}$, say. Now make a geometrical variation $g_{ik} \to g_{ik} - \delta g_{ik}$. The nett effect of coordinate transformation plus geometrical change is to leave the g_{ik} the same as before. We are back to the

original situation. Evidently $\delta J = 0$ for the coordinate change while $\delta J = 0$ also for the geometrical change. Hence $\delta J = 0$ for the combination of the two, which we have seen to be equivalent to the displacement of the world line at fixed geometry.

We now apply a similar method to the wavefunction picture. We saw above that it is irrelevant in which local Lorentz frame we set $g^{i\alpha\dot{\beta}} = \sigma^{i\alpha\dot{\beta}}$ provided the wavefunctions are appropriately matched to the Lorentz frame. Suppose at each point an infinitesimal change is made in the Lorentz frame in which $g^{i\alpha\dot{\beta}} = \sigma^{i\alpha\dot{\beta}}$, the six parameters determining the change being well-behaved functions of position. The change defines

$$c_\alpha^\beta = \begin{bmatrix} 1+\epsilon & \eta \\ \xi & 1-\epsilon \end{bmatrix}$$

in which ϵ, η, ξ are infinitesimal functions of position. The metric tensor g_{ik} is unaffected, so that the propagators $\tilde{G}, \bar{G}$ are unaffected. So are quantities like $g^{i\alpha\dot{\beta}} u_\alpha u_{\dot{\beta}}$ when the transformation c_α^β is applied to all spinor suffixes. Indeed all terms in (76) are unaffected, so (78) is trivially satisfied. But the transformation produces a change of the type $g^{i\alpha\dot{\beta}} \to g^{i\alpha\dot{\beta}} + \delta g^{i\alpha\dot{\beta}}$. Now make a change $g^{i\alpha\dot{\beta}} \to g^{i\alpha\dot{\beta}} - \delta g^{i\alpha\dot{\beta}}$. This second change is 'geometrical'—it does not affect u_α. By our basic postulate (78) is satisfied for this second change. Consider now the combination of the two changes. Evidently $g^{i\alpha\dot{\beta}}$ is unchanged, since the two changes compensate each other so far as $g^{i\alpha\dot{\beta}}$ is concerned. But the wave functions are altered because they were subject to variation in the spin transformation but not in the 'geometrical' variation. Hence (76) must be satisfied for an arbitrary infinitesimal spin transformation of the wave functions at fixed geometry. Equation (78) is also satisfied for arbitrary phase changes of the wave functions.

We note at this stage that, because of (80), the integral on the left-hand side of (76) is pure imaginary, so that $\hbar$ is real.

Consider a change $u_\alpha^{(a)} \to u_\alpha^{(a)} + \delta u_\alpha^{(a)}$, $v_\alpha^{(a)} \to v_\alpha^{(a)} + \delta v_\alpha^{(a)}$ for particle a alone. Equation (78) takes the form

$$0 = \mathrm{i}\hbar \int g^{i\alpha\dot{\beta}} \sqrt{(-g)}\, \delta(u_{\dot{\beta}} u_{\alpha;i} + v_\alpha v_{\dot{\beta};i})_a \, \mathrm{d}^4a + e_a \int A_a^i g_i^{\alpha\dot{\beta}} \sqrt{(-g)}\, \delta(u_\alpha u_{\dot{\beta}} + v_\alpha v_{\dot{\beta}}) \, \mathrm{d}^4a$$
$$- \int m_a \sqrt{(-g)}\, \delta(u_\alpha v^\alpha + u_{\dot{\beta}} v^{\dot{\beta}})_a \, \mathrm{d}^4a. \tag{81}$$

Putting $\delta u_\alpha = \delta v_\alpha = 0$ on the boundary of the region of integration, it can readily be shown that the first team on the right-hand side of (81) can be expressed in the form

$$\mathrm{i}\hbar \int g^{i\alpha\dot{\beta}} \sqrt{(-g)}\, (u_{\alpha;i}\, \delta u_{\dot{\beta}} + v_{\dot{\beta};i}\, \delta v_\alpha - u_{\dot{\beta};i}\, \delta u_\alpha - v_{\alpha;i}\, \delta v_{\dot{\beta}})_a \, \mathrm{d}^4a. \tag{82}$$

Collecting terms we can write (81) as

$$\int \sqrt{(-g)}\, \delta u_\alpha (-\mathrm{i}\hbar g^{i\alpha\dot{\beta}} u_{\dot{\beta};i} + e_a A_a^i g_i^{\alpha\dot{\beta}} u_{\dot{\beta}} - m_a v^\alpha) \, \mathrm{d}^4a$$
$$+ \int \sqrt{(-g)}\, \delta u_{\dot{\beta}} (\mathrm{i}\hbar g^{i\alpha\dot{\beta}} u_{\alpha;i} + e_a A_a^i g_i^{\alpha\dot{\beta}} u_\alpha - m_a v^{\dot{\beta}}) \, \mathrm{d}^4a$$
$$+ \int \sqrt{(-g)}\, \delta v_\alpha (\mathrm{i}\hbar g^{i\alpha\dot{\beta}} v_{\dot{\beta};i} + e_a A_a^i g_i^{\alpha\dot{\beta}} v_{\dot{\beta}} + m_a u^\alpha) \, \mathrm{d}^4a$$
$$+ \int \sqrt{(-g)}\, \delta v_{\dot{\beta}} (-\mathrm{i}\hbar g^{i\alpha\dot{\beta}} v_{\alpha;i} + e_a A_a^i g_i^{\alpha\dot{\beta}} v_\alpha + m_a u^{\dot{\beta}}) \, \mathrm{d}^4a. \tag{83}$$

The first two terms in (83) are complex conjugates, as also are the last two terms. Hence we require the real part of the following expression to vanish

$$\int \sqrt{(-g)}\,\delta u_\alpha(-\mathrm{i}\hbar g^{i\alpha\dot\beta}u_{\dot\beta;\,i}+e_a A^i_a g^{\alpha\dot\beta}_i u_{\dot\beta}-m_a v^\alpha)\,\mathrm{d}^4a$$
$$+\int \sqrt{(-g)}\,\delta v_\alpha(\mathrm{i}\hbar g^{i\alpha\dot\beta}v_{\dot\beta;\,i}+e_a A^i_a g^{\alpha\dot\beta}_i v_{\dot\beta}+m_a u^\alpha)\,\mathrm{d}^4a. \tag{84}$$

If $\delta u_\alpha, \delta v_\alpha$ are taken as entirely arbitrary variations we must have

$$\left.\begin{aligned}-\mathrm{i}\hbar g^{i\alpha\dot\beta}u_{\dot\beta;\,i}+e_a A^i_a g^{\alpha\dot\beta}_i u_{\dot\beta}-m_a v^\alpha &= 0,\\ \mathrm{i}\hbar g^{i\alpha\dot\beta}v_{\dot\beta;\,i}+e_a A^i_a g^{\alpha\dot\beta}_i v_{\dot\beta}+m_a u^\alpha &= 0,\end{aligned}\right\} \tag{85}$$

and by taking complex conjugates we have

$$\left.\begin{aligned}\mathrm{i}\hbar g^{i\alpha\beta}u_{\alpha;\,i}+e_a A^i_a g^{\alpha\dot\beta}_i u_\alpha-m_a v^{\dot\beta} &= 0,\\ -\mathrm{i}\hbar g^{i\alpha\beta}v_{\alpha;\,i}+e_a A^i_a g^{\alpha\beta}_i v_\alpha+m_a u^{\dot\beta} &= 0.\end{aligned}\right\} \tag{86}$$

Equations (85) and (86) are just the Dirac equation written out explicitly in terms of the two spinors u_α, v_α and of their complex conjugates. In compact form, (85) and (86) give

$$\mathrm{i}\hbar\gamma^i\psi^{(a)}_{;\,i}+e_a A^i_a\gamma_i\psi^{(a)}-m_a\psi^{(a)}=0, \tag{87}$$

where γ^i and ψ are defined in (58).

However, before accepting (87) it is necessary to verify that we are permitted sufficient freedom over δu_α, δv_α. The most general variations of δu_1, δu_2, δv_1, δv_2 involve 8 parameters since these are complex functions. The general infinitesimal unimodular transformation supplies 6 parameters. The phase change

$$\psi\to\psi\,\mathrm{e}^{\mathrm{i}\theta}=\psi+\mathrm{i}\theta\psi$$

when θ is small gives a further measure of freedom in the choice of δu_α, δv_α. So far, we have thought of θ as a real angle, in which case we have a total of 7 parameters. But (78) holds for complex θ if it holds for θ real. This is because (80), which follows from the case in which θ is real, permits (76) to be satisfied when θ is complex. The combination of a general infinitesimal unimodular transformation with the case in which θ is complex is equivalent to a transformation matrix

$$\begin{bmatrix}1+\epsilon & \eta\\ \xi & 1+\chi\end{bmatrix},$$

in which ϵ, η, ξ, χ are arbitrary complex functions of position. Provided u_α, v_α are different spinors, which we assume to be the case, this gives the most general δu_α, δv_α.

The emergence of the Dirac equation identifies $\hbar$ with Planck's constant.

It remains to derive the field equations. We now make the change $g^{i\alpha\dot\beta}\to g^{i\alpha\dot\beta}+\delta g^{i\alpha\dot\beta}$ keeping the wave functions constant. Provided the change is not equivalent to a pure spin transformation—i.e. provided $\delta g^{ik}\neq 0$, the propagators $\tilde G$, $\bar G$ are altered, and this leads in (78) to terms that have already been set out in (63). Denote these by δJ_1. Then (78) can be written

$$\delta J_1+\sum_a\int(\mathrm{i}\hbar\overline{\psi}\gamma^i\psi_{;\,i}+eA_i\overline{\psi}\gamma^i\psi-m\overline{\psi}\psi)_a\,\delta(\sqrt{(-g)}\,\mathrm{d}^4a$$
$$+\sum_a\int[\mathrm{i}\hbar\delta(\overline{\psi}\gamma^i\psi_{;\,i})+eA_i\,\delta(\overline{\psi}\gamma^i\psi)]_a\sqrt{(-g)}\,\mathrm{d}^4a=0. \tag{88}$$

The term in $\delta(\sqrt{(-g)})$ vanishes because of (87). Hence the last term on the left-hand side of (88) determines the energy-momentum tensor of matter, $T^{ik}_{\text{mat.}}$, the definition of $T^{ik}_{\text{mat.}}$ being

$$-\frac{1}{2}\int \delta g_{ik} T^{ik}_{\text{mat.}} \sqrt{(-g)}\, \mathrm{d}^4x = \sum_a \int [i\hbar\delta(\overline{\psi}\gamma^i\psi_{;i}) + eA_i\delta(\overline{\psi}\gamma^i\psi)]_a \sqrt{(-g)}\, \mathrm{d}^4x \quad (89)$$

provided $\delta g^{i\alpha\dot{\beta}}$ is a pure geometrical change—it has no spin transformation concealed in its components. On the right-hand side of (89) we have changed the elements of integration $\mathrm{d}^4a, \mathrm{d}^4b, \ldots$ all to d^4x, because the integrals are all over the same space time volume.

The manipulative treatment of (89) is somewhat awkward, so we defer details to an Appendix, simply stating the result

$$4g_{ip}g_{kq}T^{pq}_{\text{mat.}} = \sum_a [i\hbar(\overline{\psi}\gamma_i\psi_{;k} + \overline{\psi}\gamma_k\psi_{;i} - \overline{\psi}_{;i}\gamma_k\psi - \overline{\psi}_{;k}\gamma_i\psi) + 2e(A_i\overline{\psi}\gamma_k\psi + A_k\overline{\psi}\gamma_i\psi)]_a \quad (90)$$

in which γ^i, ψ, $\overline{\psi}$ have the forms given in (58), and the subscript a on the square bracket denotes quantities for particle a.

The field equations are

$$\sum_{a<b}\sum [(R_{ik} - \tfrac{1}{2}g_{ik}R)\, m^{(a)}m^{(b)} + m^{(a)}(m^{(b)}_{;ki} - g_{ik}g^{pq}m^{(b)}_{;pq}) + m^{(b)}(m^{(a)}_{;ik} - g_{ik}g^{pq}m^{(a)}_{;pq}) - 2(m^{(a)}_{;i}m^{(b)}_{;k} + m^{(a)}_{;k}m^{(b)}_{;i} - \tfrac{1}{2}g_{ik}m^{(a);l}m^{(b)}_{;l})] + 3g_{ip}g_{kq}(T^{pq}_{\text{el.}} + T^{pq}_{\text{mat.}}) = 0. \quad (91)$$

This completes the work of the present paper. It takes the theory to the stage where the usual ideas of quantum mechanics can be applied—the superposition principle, transformation theory, stationary states, perturbation methods, and the quasi-classical theory of radiative transitions. However, applications special to quantum electrodynamics—for example the Lamb–Rutherford shift—cannot be dealt with until the theory is further developed. We shall attempt such a development in a future paper.

Appendix

First, we notice that the most general choice for $\delta g^{i\alpha\dot{\beta}}$ contains an infinitesimal spin transformation as well as the changes required to give a variation of the metric tensor $g^{ik} \to g^{ik} + \delta g^{ik}$. This can be seen explicitly by considering the choice

$$\delta g^{i\alpha\dot{\beta}} = \tfrac{1}{2}g^{\alpha\dot{\beta}}_k\, \delta g^{ik} + c^\alpha_\gamma g^{i\gamma\dot{\beta}} + \bar{c}^{\dot{\beta}}_{\dot{\rho}} g^{i\alpha\dot{\rho}}, \quad (\text{A } 1)$$

where $c^\alpha_\gamma + \delta^\alpha_\gamma$ is any infinitesimal unimodular transformation—i.e. any infinitesimal transformation satisfying $c^\alpha_\alpha = 0$. The choice (A 1) satisfies the requirement that $\delta g^{i\alpha\dot{\beta}}$ and $\delta g^{i\beta\dot{\alpha}}$ are complex conjugates. From (48) of the main text we have

$$2\delta g^{ik} = \delta g^{i\alpha\dot{\beta}} g^k_{\alpha\dot{\beta}} + \delta g^{k\alpha\dot{\beta}} g^i_{\alpha\dot{\beta}}. \quad (\text{A } 2)$$

Substituting (A 1) in the right-hand side of (A 2) gives

$$\delta g^{ik} = \delta g^{ik} + g^{ik}(c^\alpha_\alpha + \bar{c}^{\dot{\alpha}}_{\dot{\alpha}}) = \delta g^{ik}$$

because of the unimodular condition. The spin part of (A 1), involving c^{α}_{γ}, has no effect on the metric tensor.

Our purpose is to calculate the change of

$$\sum_a \int (\mathrm{i}\hbar\overline{\psi}\gamma^i\psi_{;\,i} + eA_i\overline{\psi}\gamma^i\psi - m\overline{\psi}\psi)_a \sqrt{(-g)}\,\mathrm{d}^4a, \tag{A 3}$$

resulting from $g^{i\alpha\dot{\beta}} \to g^{i\alpha\dot{\beta}} + \delta g^{i\alpha\dot{\beta}}$. Before the change, (A 3) is zero because of (87). After the change (A 3) is not in general zero because (87) need not apply after the change. Manifestly, (A 3) is unchanged by any arbitrary spin transformation. Such a transformation alters both ψ and $g^{i\alpha\dot{\beta}}$. Choose an infinitesimal transformation $-c^{\alpha}_{\gamma}$. The nett effect of the choice (A 1) for $\delta g^{i\alpha\dot{\beta}}$, followed by $-c^{\alpha}_{\gamma}$, is to give

$$\delta g^{i\alpha\dot{\beta}} = \tfrac{1}{2} g^{\alpha\dot{\beta}}_{k}\,\delta g^{ik}. \tag{A 4}$$

However, (A 3) is now subject to the change of ψ resulting from $-c^{\alpha}_{\gamma}$, as well as to the change (A 4). But $\psi \to \psi + \delta\psi$ leaves (A 3) unaltered, as we saw in arriving at (87). Hence we can use (A 4) as the choice for $\delta g^{i\alpha\dot{\beta}}$. This choice carries full information concerning the change of the metric tensor. The wavefunction ψ is left unchanged.

All this can be seen by counting parameters. The most general choice for $\delta g^{i\alpha\dot{\beta}}$, taking into account the requirement that $\delta g^{i\alpha\dot{\beta}}$, $\delta g^{i\beta\dot{\alpha}}$ must be complex conjugates, contains 16 parameters, whereas the most general choice for δg^{ik} has only 10 parameters. The remaining six parameters are contained in the infinitesimal spin transformation $c^{\alpha}_{\gamma} + \delta^{\alpha}_{\gamma}$. The variation of $\sqrt{(-g)}$ leads to no change in (A 3), because of (87). As pointed out in the main text, we therefore require

$$\sum_a \int [\mathrm{i}\hbar\delta(\overline{\psi}\gamma^i\psi_{;\,i}) + eA_i\delta(\overline{\psi}\gamma^i\psi)]_a \sqrt{(-g)}\,\mathrm{d}^4x \tag{A 5}$$

calculated for $\delta g^{i\alpha\dot{\beta}}$ given by (A 4). The resulting change is equal to

$$-\frac{1}{2}\int \delta g_{ik} T^{ik}_{\mathrm{mat.}} \sqrt{(-g)}\,\mathrm{d}^4x,$$

this being the definition of $T^{ik}_{\mathrm{mat.}}$.

Our aim is to establish the result (90) for $T^{ik}_{\mathrm{mat.}}$. This can be done by a short 'slick' argument or by direct 'slogging'. Here we give the latter because, if it were not for the existence of the latter, one would be inclined to wonder whether the former might not conceal an error. (The cancellations of terms which occurs in the slogging method is an indication that the short method is very probably right.) A similar problem has been considered by Wheeler & Brill (1957) who describe the method of attack but do not give the details—these authors refer to an unpublished manuscript of B. S. DeWitt. Other authors appear always to have quoted the result, as we have done in the main text.

If for the moment we set $\delta\psi_{;\,i} = 0$ in (A 5) we have

$$\sum_a \int \delta g^{i\alpha\dot{\beta}} [\tfrac{1}{2}\mathrm{i}\hbar(u_{\dot{\beta}}u_{\alpha;\,i} - u_{\dot{\beta};\,i}u_{\alpha} - v_{\dot{\beta}}v_{\alpha;\,i} + v_{\dot{\beta};\,i}v_{\alpha}) + eA_i(u_{\alpha}u_{\dot{\beta}} + v_{\alpha}v_{\dot{\beta}})]_a \sqrt{(-g)}\,\mathrm{d}^4x \tag{A 6}$$

in terms of the individual spinors u_{α}, v_{α}. Here we have symmetrized with respect to

dotted and undotted spinors using (80). Substituting for $\delta g^{i\alpha\dot{\beta}}$ from (A 4) this becomes

$$\begin{aligned}
&\tfrac{1}{2}\sum_a \int \delta g^{ik} g_k^{\alpha\dot{\beta}} [\tfrac{1}{2}\mathrm{i}\hbar(u_\beta u_{\alpha;i} - u_{\dot{\beta};i} u_\alpha - v_\beta v_{\alpha;i} + v_{\beta;i} v_\alpha) + eA_i(u_\alpha u_{\dot{\beta}} + v_\alpha v_{\dot{\beta}})]_a \sqrt{(-g)}\, \mathrm{d}^4x \\
&\quad = \tfrac{1}{2}\sum_a \int \delta g^{ik} [\tfrac{1}{2}\mathrm{i}\hbar(\overline{\psi}\gamma_k \psi_{;i} - \overline{\psi}_{;i}\gamma_k \psi) + eA_i \overline{\psi}\gamma_k \psi]_a \sqrt{(-g)}\, \mathrm{d}^4x \\
&\quad = \tfrac{1}{2}\sum_a \int \delta g^{ik} [\tfrac{1}{4}\mathrm{i}\hbar(\overline{\psi}\gamma_k \psi_{;i} + \overline{\psi}\gamma_i \psi_{;k} - \overline{\psi}_{;i}\gamma_k \psi - \overline{\psi}_{;k}\gamma_i \psi) \\
&\qquad\qquad + \tfrac{1}{2}e(A_i \overline{\psi}\gamma_k \psi + A_k \overline{\psi}\gamma_i \psi)]_a \sqrt{(-g)}\, \mathrm{d}^4x. \qquad \text{(A 7)}
\end{aligned}$$

In view of the definition of $T^{ik}_{\text{mat.}}$, (A 7) gives the result set out in the main text, in (90). Hence we see that the essential point is to establish that the terms in (A 5) of the kind $\delta(\psi_{;i})$ vanish. Although the variation (A 5) is for constant ψ this does not immediately lead to $\delta(\psi_{;i}) = 0$ because the operation $;i$ is different after the variation $\delta g^{i\alpha\dot{\beta}}$ from what it was initially. It therefore remains to consider the differentiation terms in (A 5) and to show that, taken together, they vanish. The terms in question are

$$\sum_a \int (u_{\dot{\beta}} \delta u_{\alpha;i} - u_\alpha \delta u_{\dot{\beta};i} + v_\alpha \delta v_{\dot{\beta};i} - v_{\dot{\beta}} \delta v_{\alpha;i})_a g^{i\alpha\dot{\beta}} \sqrt{(-g)}\, \mathrm{d}^4x. \qquad \text{(A8)}$$

Consider

$$\int u_{\dot{\beta}} \delta u_{\alpha;i} g^{i\alpha\dot{\beta}} \sqrt{(-g)}\, \mathrm{d}^4x.$$

We have

$$\begin{aligned}
g^{i\alpha\dot{\beta}} u_{\dot{\beta}} \delta u_{\alpha;i} &= g^{i\alpha\dot{\beta}} u_{\dot{\beta}} \delta\left(\frac{\partial u_\alpha}{\partial x^i} - L^\beta_{\alpha i} u_\beta\right) \\
&= -g^{i\alpha\dot{\beta}} u_\beta u_{\dot{\beta}} \delta L^\beta_{\alpha i}, \qquad \text{(A 9)}
\end{aligned}$$

where $L^\beta_{\alpha i}$ is the affine connexion defined in (73). The final step in the proof will be to show that

$$\delta L^\beta_{\alpha i} = -\tfrac{1}{8} g^{n\beta\dot{\gamma}} g^k_{\alpha\dot{\gamma}} (\delta g_{ik;n} - \delta g_{in;k}). \qquad \text{(A 10)}$$

Accepting (A 10) for the moment, the above integral becomes

$$\frac{1}{8}\int g^{i\alpha\dot{\beta}} g^{n\beta\dot{\gamma}} g^k_{\alpha\dot{\gamma}} (\delta g_{ik;n} - \delta g_{in;k}) u_\beta u_{\dot{\beta}} \sqrt{(-g)}\, \mathrm{d}^4x. \qquad \text{(A 11)}$$

Integrating by parts, setting $\delta g_{ik} = 0$ on the boundary of the region of integration, and remembering that $g^{i\alpha\dot{\beta}}_{;k} = 0$ for all $i, k, \alpha, \dot{\beta}$, (A 11) is easily shown to be equal to

$$\frac{1}{8}\int \delta g_{ik} (u_\beta u_{\dot{\beta}})_{;n} g^{i\alpha\dot{\beta}} (g^{k\beta\dot{\gamma}} g^n_{\alpha\dot{\gamma}} - g^{n\beta\dot{\gamma}} g^k_{\alpha\dot{\gamma}}) \sqrt{(-g)}\, \mathrm{d}^4x. \qquad \text{(A 12)}$$

The next step in the argument is to show that the integrand of (A 12) is real. Then because the two terms in (A 8) involving the spinor u are complex conjugates subtracted from each other it follows that the contribution of these terms to (A 8) is zero. An exactly similar argument shows that the contribution of the spinor v is zero, so (A 8) vanishes, as required.

Two steps are needed to see that the integrand of (A 12) is real. The term $\delta g_{ik}(u_\beta u_{\dot{\beta}})_{;n} g^{i\alpha\dot{\beta}} g^{k\dot{\beta}\dot{\gamma}} g^n_{\alpha\dot{\gamma}}$ is easily seen to be equal to its complex conjugate—all that is needed is to take account of the symmetry of δg_{ik} and to rearrange the dummy

suffixes after taking the complex conjugate, remembering $g^{i\alpha\dot{\beta}}$ and $g^{i\dot{\beta}\alpha}$ are complex conjugates. For the term $-\delta g_{ik}(u_\beta u_{\dot{\beta}})_{;\,n}\, g^{i\alpha\dot{\beta}} g^{n\beta\dot{\gamma}} g^k_{\alpha\dot{\gamma}}$ we note a relation

$$g^{i\alpha\dot{\beta}} g^k_{\alpha\dot{\gamma}} = S^{ik\dot{\beta}}{}_{\dot{\gamma}} + g^{ik}\delta^{\dot{\beta}}_{\dot{\gamma}}, \tag{A 13}$$

in which $S^{ik\dot{\beta}}{}_{\dot{\gamma}}$ is antisymmetric in i, k. The original definition (54) in terms of $\sigma_i^{\alpha\dot{\beta}}$, together with the explicit forms of $\sigma_i^{\alpha\dot{\beta}}$, are needed to prove this antisymmetry. The proof is straightforward (cf. Corson 1953). Now because $S^{ik\dot{\beta}}{}_{\dot{\gamma}}$ is multiplied by the symmetrical δg_{ik} we are left with $-\delta g_{ik}(u_\beta u_{\dot{\beta}})_{;\,n}\, g^{n\beta\dot{\gamma}} g^{ik}\delta^{\dot{\beta}}_{\dot{\gamma}}$, i.e. with $-\delta g_{ik} g^{ik} g^{n\beta\dot{\beta}}(u_\beta u_{\dot{\beta}})_{;\,n}$ and this also is real.

It only remains to establish (A 10). In deriving a relation between quantities all taken at the same point in Riemannian geometry one often uses the device of choosing a coordinate system in which $\partial g_{ik}/\partial x^l = 0$ at the point in question. The algebraic manipulation is thereby much simplified. The result at the end is expressed in covariant form, which immediately permits a transfer back to the original coordinate system. Here the choice of a coordinate system giving $\partial g_{ik}/\partial x^l = 0$ is not sufficient to require $\partial g^{i\alpha\dot{\beta}}/\partial x^l = 0$. For this we must also make an appropriate spin transformation. Provided at the end of the calculation we write our result covariantly in both tensor and spin suffixes we can transfer immediately back to the original coordinate and spin frames.

From the definition (73) we have

$$\delta L^\beta_{\alpha i} = \tfrac{1}{4} g^{n\beta\dot{\gamma}}(\partial \delta g_{n\alpha\dot{\gamma}}/\partial x^i - g_{k\alpha\dot{\gamma}}\,\delta\Gamma^k_{in}). \tag{A 14}$$

The relation for $\delta g_{n\alpha\dot{\gamma}}$ corresponding to (A 4) is

$$\delta g_{n\alpha\dot{\gamma}} = \tfrac{1}{2}\delta g_{nk} g^k_{\alpha\dot{\gamma}}. \tag{A 15}$$

This is easily shown from the requirement $\delta(g^{i\alpha\dot{\gamma}} g_{n\alpha\dot{\gamma}}) = 0$. Hence, using the usual formula for the Christoffel symbol $\delta\Gamma^k_{in}$ we have

$$\delta L^\beta_{\alpha i} = \tfrac{1}{8} g^{n\beta\dot{\gamma}} \frac{\partial}{\partial x^i}(\delta g_{nk} g^k_{\alpha\dot{\gamma}}) - \tfrac{1}{8} g^{n\beta\dot{\gamma}} g^k_{\alpha\dot{\gamma}} \left(\frac{\partial \delta g_{ik}}{\partial x^n} + \frac{\partial \delta g_{nk}}{\partial x^i} - \frac{\partial \delta g_{in}}{\partial x^k} \right),$$

which leads without difficulty to

$$\begin{aligned} \delta L^\beta_{\alpha i} &= -\tfrac{1}{8} g^{n\beta\dot{\gamma}} g^k_{\alpha\dot{\gamma}} \left(\frac{\partial \delta g_{ik}}{\partial x^n} - \frac{\partial \delta g_{in}}{\partial x^k} \right) \\ &= -\tfrac{1}{8} g^{n\beta\dot{\gamma}} g^k_{\alpha\dot{\gamma}} (\delta g_{ik;\,n} - \delta g_{in;\,k}). \end{aligned} \tag{A 16}$$

The right-hand side of (A 16) has the appropriate covariant form in all suffixes.

References

Corson, E. M. 1953 *Introduction to tensors, spinors and relativistic wave equations.* Glasgow: Blackie & Son Ltd.

Hoyle, F. & Narlikar, J. V. 1963 *Proc. Roy. Soc.* A **277**, 1.

Hoyle, F. & Narlikar, J. V. 1964*a* *Proc. Roy. Soc.* A **282**, 191.

Hoyle, F. & Narlikar, J. V. 1964*b* *Proc. Roy. Soc.* A **282**, 184.

Hoyle, F. & Narlikar, J. V. 1966 *Proc. Roy. Soc.* A **294**, 138.

Islam, J. N. 1966 *Proc. Roy. Soc.* A **294**, 437.

Wheeler, J. A. & Feynman, R. P. 1945 *Rev. Mod. Phys.* **17**, 157.

Wheeler, J. A. & Feynman, R. P. 1949 *Rev. Mod. Phys.* **21**, 425.

Wheeler, J. A. & Brill, D. R. 1957 *Rev. Mod. Phys.* **29**, 477.