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Power transfer in non-linear gravitational clustering and asymptotic universality

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ABSTRACT

We study the non-linear gravitational clustering of collisionless particles in an expanding background using an integro-differential equation for the gravitational potential. In particular, we address the question of how the non-linear mode–mode coupling transfers power from one scale to another in the Fourier space if the initial power spectrum is sharply peaked at a given scale. We show that the dynamical equation allows self-similar evolution for the gravitational potential $\phi_{\boldsymbol{k}}(t)$ in Fourier space of the form $\phi_{\boldsymbol{k}}(t) = F(t)D(\boldsymbol{k})$ where the function $F(t)$ satisfies a second-order non-linear differential equation. We analyse the relevant solutions of this equation, thereby determining the asymptotic time evolution of the gravitational potential and density contrast. The analysis suggests that both $F(t)$ and $D(\boldsymbol{k})$ have well-defined asymptotic forms indicating that the power transfer leads to a universal power spectrum at late times. The analytic results are compared with numerical simulations, showing good agreement over the range at which we could test them.

Key words: gravitation – methods: N-body simulations – methods: numerical – cosmology: theory – dark matter – large-scale structure of Universe.

1 INTRODUCTION

If power is injected at some scale L into an ordinary viscous fluid, it cascades down to smaller scales because of the non-linear coupling between different modes. The resulting power spectrum, for a wide range of scales, is well approximated by the Kolmogorov spectrum which plays a key useful role in the study of fluid turbulence. It is possible to obtain the form of this spectrum from fairly simple and general considerations, although the actual equations of fluid turbulence are intractably complicated.

In this Letter, we address the corresponding question for non-linear gravitational clustering of collisionless particles in an expanding background: if power is injected at a given length-scale very early on, how does the dynamical evolution transfer power to other scales at late times? In particular, does the non-linear evolution lead to an analogue of the Kolmogorov spectrum with some level of universality, in the case of gravitational interactions?

We will argue that the answer is essentially in the affirmative. If power is injected at a given scale $L = 2\pi/k_0$ then the gravitational clustering transfers the power to both larger and smaller spatial scales. At large spatial scales the power spectrum goes as $P(k) \propto k^4$ as soon as non-linear coupling becomes important. [This result is known in the literature: see for example Peebles (1980) and Padmanabhan (2005)]. More interestingly, the cascading of power to smaller scales leads to a *universal pattern* at late times just as in the case of fluid turbulence. By studying the relevant equations we will argue that the gravitational potential has the form $\phi_{\boldsymbol{k}}(t) = F(t)D(\boldsymbol{k})$ where $F(t)$ satisfies a non-linear differential equation and $D(\boldsymbol{k})$ satisfies an integral equation. We analyse the relevant equations analytically as well as verify the conclusions by numerical simulations. The study suggests that non-linear gravitational clustering does lead to a universal power spectrum at late times if the power is injected at a given scale initially.

While the gravitational evolution of collisionless particles in an expanding background is of considerable importance in cosmology, it is probably fair to say that our analytic understanding of this problem is rather patchy [for a general review of statistical mechanics of gravitating systems, see Padmanabhan (1990, 1989); previous work on analytic approximations includes the Zeldovich-like approximation (Zeldovich 1970; Gurbatov, Saichev & Shandarin 1989; Matarrese et al. 1992; Brainerd, Scherrer & Villumsen 1993; Bagla & Padmanabhan 1994; Padmanabhan & Engineer 1998; Engineer, Kanekar & Padmanabhan 2000; Tatekawa 2005), perturbative techniques (Buchert 1994; Valageas 2001, 2002) and non-linear scaling relations (Hamilton et al. 1991; Padmanabhan et al. 1996; Munshi & Padmanabhan 1997; Bagla, Engineer & Padmanabhan 1998; Kanekar & Padmanabhan 2001; Nityananda & Padmanabhan 1994; Padmanabhan 1996) among many other approaches]. In cosmology there is very little motivation to study the transfer of power by itself, and most of the numerical simulations in the past concentrated on evolving the broad-band initial power spectrum.

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The Letter is organized as follows. We describe our analytical model in Section 2. We run a series of numerical experiments to test our model; these are described in Section 3. We conclude by discussing the implications of our study in Section 4.

2 THE MATHEMATICAL FORMALISM

In this section, we provide the mathematical details of our formalism. If $\boldsymbol{x}_i(t)$ is the trajectory of the ith particle, then equations for gravitational clustering in an expanding universe in the Newtonian limit can be summarized as

$$\ddot{\boldsymbol{x}}_i + \frac{2\dot{a}}{a}\dot{\boldsymbol{x}}_i = -\frac{1}{a^2}\nabla_x\phi; \quad \nabla_x^2\phi = 4\pi G a^2 \rho_\mathrm{b}\delta, \tag{1}$$

where $\rho_\mathrm{b}(t)$ is the smooth background density of matter. Usually one is interested in the evolution of the density contrast $\delta(t, \boldsymbol{x}) \equiv [\rho(t, \boldsymbol{x}) - \rho_\mathrm{b}(t)]/\rho_\mathrm{b}(t)$ rather than in the trajectories. Since the density contrast in the Fourier space $\delta_{\boldsymbol{k}}(t)$ can be related to the trajectories of the particles, one can obtain an equation for $\delta_{\boldsymbol{k}}(t)$ from the above equation. This equation, in turn, will involve the velocities of the particles and hence will not be closed. It is, however, possible to provide a closure condition for this equation using the following two facts. (1) At any given time t, particles that are already part of a bound, virialized cluster do not contribute significantly to the non-linear terms (see, for example, Peebles 1980, section 28.C). (2) The velocities of the remaining particles can be very well approximated by the Zeldovich ansatz which takes the velocities to be proportional to the gradient of the gravitational potential. Given this ansatz, it is possible to write down a *closed* integro-differential equation for the evolution of gravitational potential $\phi_{\boldsymbol{k}}(t)$ in the Fourier space [the details of this derivation can be found in the companion paper (Padmanabhan 2005) and will not be repeated here]. The final equation is

$$\ddot{\phi}_{\boldsymbol{k}} + 4\frac{\dot{a}}{a}\dot{\phi}_{\boldsymbol{k}} = -\frac{1}{3a^2}\int\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\phi_{\frac{1}{2}\boldsymbol{k}+\boldsymbol{p}}\phi_{\frac{1}{2}\boldsymbol{k}-\boldsymbol{p}}\mathcal{G}(\boldsymbol{k}, \boldsymbol{p}),$$
$$\mathcal{G}(\boldsymbol{k}, \boldsymbol{p}) = \frac{7}{8}k^2 + \frac{3}{2}p^2 - 5\left(\frac{\boldsymbol{k}\cdot\boldsymbol{p}}{k}\right)^2. \tag{2}$$

This equation governs the dynamical evolution of the system, but of course it is intractably complicated. In addition to the obvious difficulty of solving an integro-differential equation, we also need to tackle the issue of incorporating appropriate initial conditions, which will influence the evolution at the early stages. [Some results based on perturbation solutions to this equation are discussed in Padmanabhan (2005).] Our aim is to look for *late-time* scale-free evolution of the system, exploiting the fact that the above equation allows self-similar solutions of the form $\phi_{\boldsymbol{k}}(t) = F(t)D(\boldsymbol{k})$. Substituting this ansatz into equation (2), we obtain two separate equations for $F(t)$ and $D(\boldsymbol{k})$. It is also convenient at this stage to use the expansion factor $a(t) = (t/t_0)^{2/3}$ of the matter-dominated universe as the independent variable rather than the cosmic time t. Then simple algebra shows that the governing equations are

$$a\frac{\mathrm{d}^2F}{\mathrm{d}a^2} + \frac{7}{2}\frac{\mathrm{d}F}{\mathrm{d}a} = -F^2 \tag{3}$$

and

$$H_0^2 D_{\boldsymbol{k}} = \frac{1}{3}\int\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}D_{\frac{1}{2}\boldsymbol{k}+\boldsymbol{p}}D_{\frac{1}{2}\boldsymbol{k}-\boldsymbol{p}}\mathcal{G}(\boldsymbol{k}, \boldsymbol{p}). \tag{4}$$

Equation (3) governs the time evolution, while equation (4) governs the shape of the power spectrum. [The separation ansatz of course has the scaling freedom $F \to \mu F$, $D \to (1/\mu)D$ which will change the right-hand side of equation (3) to $-\mu F^2$ and the left-hand side of equation (4) to $\mu H_0^2 D_{\boldsymbol{k}}$. As is to be expected, however, our results will be independent of μ; so we have set it to unity.] Our interest lies in analysing the solutions of equation (3) subject to the initial conditions $F(a_i) =$ constant, $(\mathrm{d}F/\mathrm{d}a)_i = 0$ at some small enough $a = a_i$.

Inspection shows that equation (3) has the exact solution $F(a) = (3/2)\,a^{-1}$. This, of course, is a special solution and will not satisfy the relevant initial conditions. However, equation (3) fortunately belongs to a class of non-linear equations which can be mapped to a homologous system. In such cases, the special power-law solutions will arise as the asymptotic limit. [The example well-known to astronomers is that of the isothermal sphere (Chandrasekhar 1939). Our analysis below has a close parallel]. To find the general behaviour of the solutions to equation (3), we will make the substitution $F(a) = (3/2)a^{-1}\,g(a)$ and change the independent variable from a to $q = \log a$. Then equation (3) reduces to the form

$$\frac{\mathrm{d}^2g}{\mathrm{d}q^2} + \frac{1}{2}\frac{\mathrm{d}g}{\mathrm{d}q} + \frac{3}{2}g(g-1) = 0. \tag{5}$$

This represents a particle moving in a potential $V(g) = (1/2)\,g^3 - (3/4)g^2$ under friction. For our initial conditions, the motion will lead the 'particle' to come asymptotically to rest at the stable minimum at $g = 1$ with damped oscillations. In other words, $F(a) \to (3/2)a^{-1}$ for large a, showing that this is indeed the asymptotic solution. From the Poisson equation in equation (1), it follows that $k^2\phi_{\boldsymbol{k}} \propto (\delta_{\boldsymbol{k}}/a)$ so that $\delta_{\boldsymbol{k}}(a) \propto g(a)k^2D(\boldsymbol{k})$, giving a direct physical meaning to the function $g(a)$. The asymptotic limit corresponds to a rather trivial case of $\delta_{\boldsymbol{k}}$ becoming independent of time. What will be more interesting – and accessible in simulations – will be the approach to this asymptotic solution. To obtain this, we introduce the variable $v = 2(\mathrm{d}g/\mathrm{d}q)$ so that our system reduces to the 'phase space' equations

$$\dot{v} = -\frac{1}{2}v - 3g(g-1); \quad \dot{g} = \frac{v}{2}, \tag{6}$$

where the dot denotes differentiation with respect to q. Dividing the first equation by the second and changing variables to $u = (v + g)$, we get the first-order form of the autonomous system to be

$$\frac{\mathrm{d}u}{\mathrm{d}g} = -\frac{6g(g-1)}{u-g}. \tag{7}$$

The critical points of the system are at (0, 0) and (1, 1). Standard analysis shows that: (i) the first one is an unstable critical point and the second one is the stable critical point; (ii) for our initial conditions the solution spirals around the stable critical point.

Fig. 1 describes the solution in the a–g and g–u planes. The $g(a)$ curves clearly approach the asymptotic value of $g \approx 1$ with superposed oscillations. The oscillations are rather dramatic in this figure, but from the physical point of view what is important is the existence of a well-defined scaling function rather than the oscillations. In fact, we will see below that we do not have sufficient resolution in the simulations to verify the oscillations themselves. The different curves in the top panel of Fig. 1 are for different initial values which arise from the scaling freedom mentioned earlier (the thick red line corresponds to the initial conditions used in the simulations described below). The bottom panel of Fig. 1 shows this *family* of solutions in the g–u plane. As usual, in going from the second-order equation for g to the first-order equation in the g–u plane, we map a family of solutions to a single curve (Chandrasekhar 1939). The stable critical point acts as an attractor. The solution $g(a)$ describes the time evolution and solves the problem of determining the asymptotic time evolution.

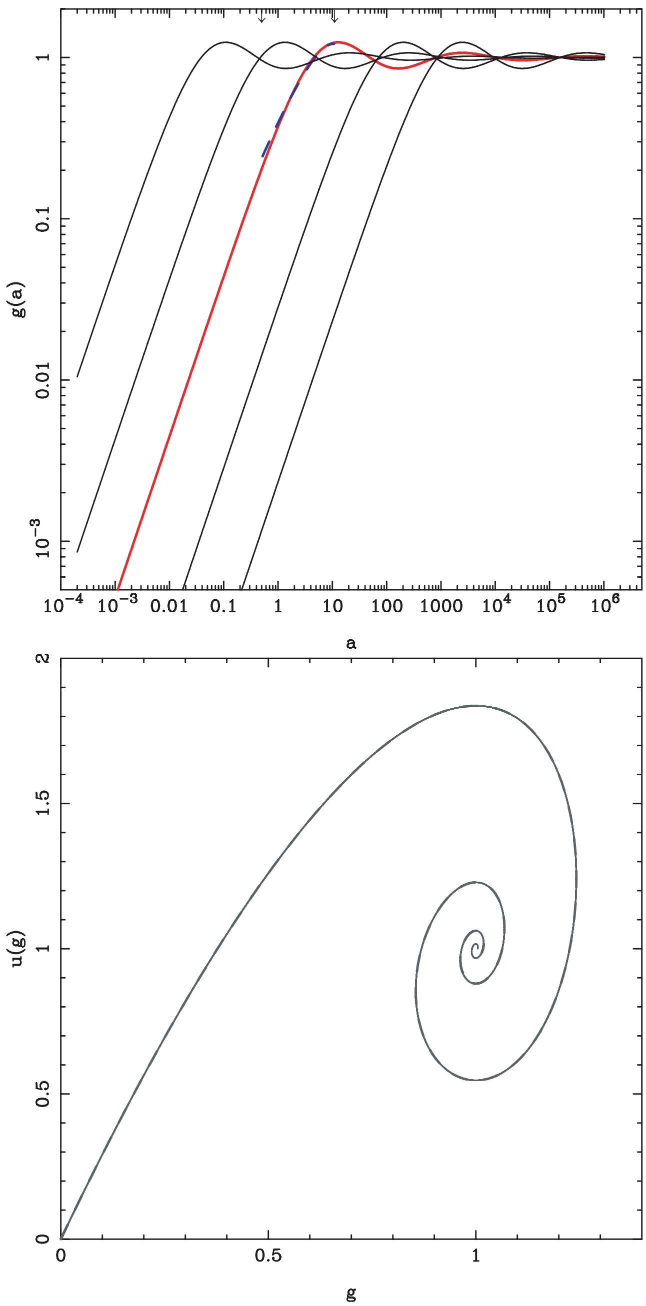

Figure 1. The solution to equation (5) is plotted in both the a–g plane and the g–u plane. The function $g(a)$ asymptotically approaches unity with oscillations which are represented by the spiral in the bottom panel. The different curves in the top panel correspond to the rescaling freedom in the initial conditions. The entire family of solutions is represented by a single curve in the g–u plane. One fiducial curve which was used to model the simulation is shown by the thick red line. The broken blue line overplotted over the fiducial curve represents a fitting function for $g(a)$ over $a = 0.5$–11.0 (refer to the text for details). Two small arrows on the x-axis at the top of the figure mark the approximate range of validity of the fitting function.

3 *N*-BODY SIMULATIONS

To test the correctness of our conclusions in the preceding section, we performed a high-resolution simulation using the TreePM method (Bagla 2002; Bagla & Ray 2003) and its parallel version (Ray & Bagla 2004) with 128^3 particles on a 128^3 grid. We used a cubic spline softened force with softening length $\epsilon = 0.4$ to ensure collisionless evolution in the simulations. We computed all relevant statistics at scales L larger than 2ϵ to avoid errors due to the force softening. Details about the code parameters can be found in Bagla & Ray (2003). The initial power spectrum $P(k)$ was chosen to be a Gaussian peaked at the scale of $k_{\rm p} = 2\pi/L_{\rm p}$ with $L_{\rm p} = 24$ grid lengths and with a standard deviation $\Delta k = 2\pi/L_{\rm box}$, where $L_{\rm box} = 128$ grid lengths is the size of one side of the simulation volume. The amplitude of the peak was taken such that $\Delta_{\rm lin}(k = k_{\rm p}, a = 0.25) = 1$. The underlying cosmological model was chosen to be an Einsten–de Sitter ($\Omega = 1$) universe.

The late-time evolution of the power spectrum [in terms of $\Delta_k^2 \equiv k^3 P(k)/2\pi^2$ where $P = |\delta_k|^2$ is the power spectrum of density fluctuations] obtained from the simulation is shown in the top panel of Fig. 2. In the bottom panel of Fig. 2, we have rescaled the Δ_k, using the appropriate solution $g(a)$. The fact that the curves fall on top of each other shows that the late-time evolution indeed scales as $g(a)$ within the numerical accuracy. A reasonably accurate fit for $g(a)$ used in this figure at late times ($a = 0.5$–11.0) is given by

$$g(a) = 0.398a(1 - 0.3 \ln a). \tag{8}$$

[Of course, this fit does not capture the oscillations, but our simulations also do not proceed that far. In the top panel of Fig. 1, we have superimposed the fitting function on $g(a)$ by a thick broken blue line over the range of epochs where the fit is valid. This corresponds to the highest non-linear regime that we could explore in our simulation. Our simulation is not sufficiently evolved to test the prediction of asymptotic oscillatory behaviour, as demonstrated by the thick red line in the top panel of Fig. 1.]

In equation (8), the term $0.3 \ln a$ is about a 20 per cent correction, say, to a $g(a) \propto a$ behaviour during $0.5 \lesssim a \lesssim 2$ when the evolution is already non-linear, i.e. $\Delta(k)$ is no longer small compared with unity at the scale k that we are considering. [This was first noticed from somewhat lower resolution simulations in Bagla & Padmanabhan (1997).] On the other hand, since the evolution at *linear* scales is always $\delta \propto a$ in the $\Omega = 1$ universe that we are studying, this allows for an approximately form-invariant evolution (except for logarithmic corrections amounting to less than 20 per cent) of the power spectrum at both quasi-linear and linear scales for this range of a.

The overall behaviour is fairly generic, and we have performed a series of simulations with different initial conditions to test this claim. Fig. 3 shows Δ_k scaled by $g(a)$ for a simulation where we have initial power concentrated in two narrow windows in k-space. In addition to power around $L_{\rm p} = 24$ grid lengths as in the previous case, we added power at $k_1 = 2\pi/L_1 (L_1 = 8$ grid lengths) using a Gaussian with the same width as before. The amplitude at L_1 was chosen to be five times higher than that at $L_{\rm p}$, which means that $\Delta_{\rm lin}(k_1, a = 0.05) = 1$. The fact that the curves fall on top of each other in this case too shows that the late-time evolution indeed scales as $g(a)$, independent of initial conditions.

The form of the power spectrum in Fig. 2 is well approximated by

$$P(k) \propto \frac{k^{-0.4}}{1 + (k/k_0)^{2.6}}; \qquad \frac{2\pi}{k_0} \approx 4.5. \tag{9}$$

This fit is shown by the broken blue line in the figure which completely overlaps with the data and is barely visible. [Note that this fit is applicable only at $L < L_{\rm p}$ since the k^4 tail will dominate scales to the right of the initial peak; see the discussion in Bagla & Padmanabhan (1997).] At non-linear scales, $P(k) \propto k^{-3}$ making

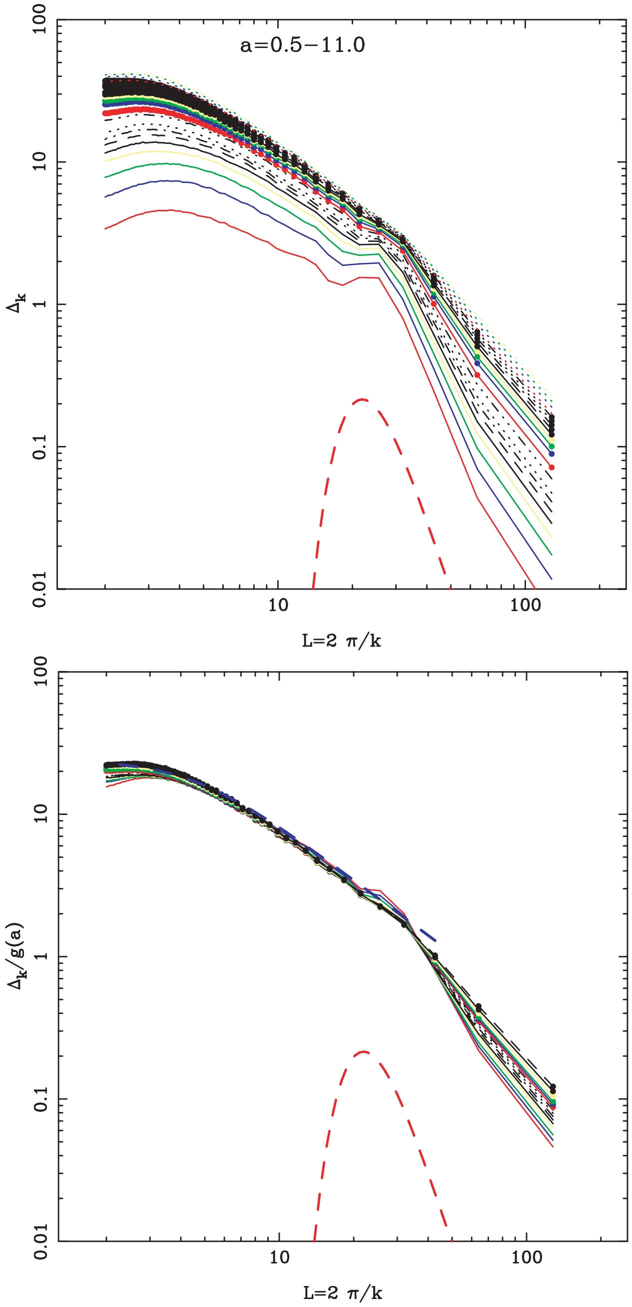

Figure 2. Top panel: the results of the numerical simulation with an initial power spectrum that is a Gaussian peaked at $L = 24$. The y-axis gives Δ_k where $\Delta_k^2 = k^3 P(k)/2\pi^2$ is the power per logarithmic band, and this quantity is plotted as a function of length-scale $L = 2\pi/k$ on the x-axis. The evolution generates a well-known k^4 tail at large scales (see for example Bagla & Padmanabhan 1997), and leads to cascading of power to small scales. Bottom panel: the simulation data are re-expressed by factoring out the time evolution function $g(a)$ obtained by integrating equation (5). The fact that the curves fall nearly on top of each other shows that the late-time evolution is scale-free and described by the ansatz discussed in the text. The rescaled spectrum is very well described by $P(k) \propto k^{-0.4}/[1 + (k/k_0)^{2.6}]$, which is shown by the completely overlapping, broken blue curve.

Δ_k flat, as seen in Fig. 2. (This is *not* a numerical artefact and we have sufficient dynamic range in the simulation to ascertain this.) At quasi-linear scales $P(k) \propto k^{-0.4}$. The power spectrum in Fig. 3 is also well-approximated by equation (9) except for the fact that

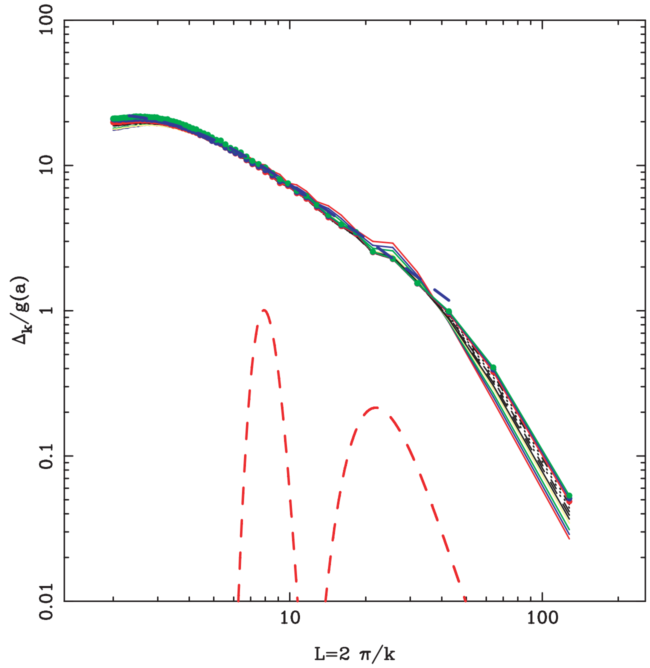

Figure 3. The results of a numerical simulation with an initial power spectrum that has two Gaussians peaked at $L = 24$ and 8. The figure has been plotted by factoring out the time evolution function $g(a)$ (equation 5) from Δ_k. The fact that the curves fall nearly on top of each other in this case as well shows that the late-time evolution is scale-free and is described by the ansatz discussed in the text, *independent of initial conditions*. The rescaled spectrum is once again described by $P(k) \propto k^{-0.4}/[1 + (k/k_0)^{2.6}]$ which is shown by the completely overlapping, broken blue curve.

the scale $2\,\pi/k_0 \approx 4.1$ in this case. A lower value for the scale is to be expected because of the additionally injected power at the lower scale L_1.

4 DISCUSSION

The analysis shows that non-linear evolution has certain universal properties to the extent that we could test it. To begin with, the time dependence has a scaling by a function $g(a)$ which satisfies a particular differential equation. This is one key result of the paper, but has already been discussed adequately in the previous section. In addition, the k-dependence of the power spectrum also seems to be universal within the accuracy of our simulations. This is reminiscent of the universality of the Kolmogorov spectrum in fluid turbulence in the sense that one can grasp the essence of a complex phenomenon by a simple scaling.

We have seen in the preceding section that the effective index of the power spectrum varies between -3 and -0.4 in the range of scales for which the fitting function has been provided. To the lowest order of accuracy, the power spectrum in this range of scales can be approximated by the mean index $n \approx -1$ with $P(k) \propto k^{-1}$. This (approximate) index has a simple interpretation (Bagla & Padmanabhan 1997; Klypin & Mellot 1992). The ensemble-averaged gravitational potential energy of fluctuations per unit volume is given by

$$\mathcal{E} \propto \frac{1}{V}\int_V \mathrm{d}^3\boldsymbol{x}\,\mathrm{d}^3\boldsymbol{y}\,\frac{\langle\delta(\boldsymbol{x})\delta(\boldsymbol{y})\rangle}{|\boldsymbol{x}-\boldsymbol{y}|} \propto \int_0^\infty \mathrm{d}^3 r\,\frac{\xi(r)}{r}$$
$$\propto \int_0^\infty \frac{\mathrm{d}k}{k}\,k P(k). \qquad (10)$$

This energy reaches equipartition (i.e. contributes the same amount per logarithmic band of scales) when $P(k) \propto k^{-1}$. The same result holds for the kinetic energy if the motion is dominated by scale-invariant radial flows. Our result suggests that gravitational power transfer evolves towards this equipartition. In principle, this $\boldsymbol{k}$-dependence of the power spectrum is determined by equation (4); but analytical understanding of this equation is more difficult since the solution depends on the phases of $\phi_{\boldsymbol{k}}$, which do not contribute to the power spectrum. This issue is under investigation.

The current work is based on a limited range of non-linearities probed by the simulation and is for the two initial power spectrum models in an Einstein–de Sitter background universe. It is important to test the theoretical model in simulations with higher dynamic range and resolution and for more varied power spectra. The background cosmology, in our opinion, is not a serious limiting factor but – of course – it will be important when we attempt to use these results in practical situations and compare with observations.

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