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Quantum Conformal Fluctuations Near the Classical Space–Time Singularity

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This paper investigates the behavior of conformal fluctuations of space–time geometry that are admissible under the quantized version of Einstein's general relativity. The approach to quantum gravity is via path integrals. It is shown that considerable simplification results when only the conformal degrees of freedom are considered under this scheme, so much so that it is possible to write down a formal kernel in the most general case where the space–time contains arbitrary distributions of particles with no other interaction except gravity. The behavior of this kernel near the classical space–time singularity then shows that quantum fluctuations inevitably diverge near the singularity. It is shown further that the root cause of this divergence lies in the fact that the Green's function for the conformally invariant scalar wave equation diverges at the singularity. The limitations on the validity of classical general relativity near the space–time singularity are discussed and it is argued that the notion of singularity itself needs to be radically modified once the quantum effects are taken into account.

1. INTRODUCTION

During the 1960s several important theorems were proved within the framework of Einstein's general theory of relativity to show that within the realm of conventional physics a singularity is an inevitable feature of the space–time manifold.[1] The big bang singularity of relativistic cosmology and the singularity terminating the gravitational collapse of a massive object are special cases of these general results.

The response of physicists to these results has been of varying types. The diehard relativist with implicit faith in the correctness of the theory has

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learned to live with the singularity and has even attempted to make a virtue of it. The less committed physicist recognizes the singularity as an indication of the incompleteness of the theory and has looked for a more comprehensive framework which does not admit space–time singularities. Some have argued that conventional physics may break down when matter is very compact, and somewhat unconventional equations of state may avert the space–time singularity.[2]

A point which is frequently mentioned in this context, but not fully explored, relates to the validity of general relativity as a classical theory. Taking the view that classical physics is an approximation to quantum physics, it is natural to ask: what is the correct quantum theory of gravity to which general relativity is an approximation in the macroscopic world? And to follow this question with another: does the space–time singularity have the same inevitable status in quantum gravity as it does in classical general relativity?

The difficulties, both conceptual and operational, of answering the first question have so far prevented a decisive answer to the second question. A few preliminary investigations do show, however, that quantum fluctuations cannot be ignored as one approaches the classical singularity.[3–5]

These investigations have used the path integral approach to quantum gravity. Although this approach has many conceptual and operational difficulties in the most general case, it does simplify for certain simple space–times. In particular, when we limit ourselves to the so-called conformal fluctuations, the problem becomes relatively easy to handle.

In this paper we concern ourselves with conformal fluctuations, although we no longer limit the discussion to simple space–times. In fact we derive a general result of significance to any space–time subject to Einstein's field equations. We show that quantum conformal fluctuations around this classical space–time geometry diverge at the space–time singularity. The classical singularity therefore ceases to be a meaningful concept.

The broad outline of this paper is as follows. We first outline some general properties of conformal fluctuations. We then rederive a result referred to earlier[3] by a new technique. This technique, though illustrated for simple space–times, can be extended to general space–times. When this is achieved the main result follows.

2. CONFORMAL TRANSFORMATIONS

Throughout this paper we assume that the space–time geometry is Riemannian with the signature $(-, -, -, +)$ for the space–time metric

$$ds^2 = g_{ik}\, dx^i\, dx^k \tag{1}$$

The coordinates x^i are, unless otherwise specified, three of space ($i = 1, 2, 3$) and one of time ($i = 4$). A conformal transformation takes the line element (1) to another, given by

$$d\tilde{s} = \Omega \, ds \tag{2}$$

where Ω is a C^2-function of x^i, in the range $0 < \Omega < \infty$. The corresponding transformation of the metric tensor is given by

$$\tilde{g}_{ik} = \Omega^2 g_{ik} \tag{3}$$

We may look upon (2) and (3) as transformations between two space–time geometries definable on the same space–time. Under such a transformation the scalar curvature transforms as

$$\tilde{R} = (R + 6 \, \Box\Omega/\Omega) \, \Omega^{-2} \tag{4}$$

where $\Box$ is the wave operator defined with respect to the metric tensor g_{ik} :

$$\Box\Omega \equiv g^{ik}\Omega_{;ik} \tag{5}$$

Similar transformation formulas can be written down for other quantities, such as R_{ik} , R_{iklm} , etc. In quantum gravity via the path integrals, however, (4) has a special role to play (see Section 3).

In geometrical units $G = 1$, $c = 1$ the Einstein action is given by

$$J = J_E + J_S + J_m \tag{6}$$

where

$$J_E = \frac{1}{16\pi} \int_{\mathscr{V}} R \sqrt{-g} \, d^4x \tag{7}$$

is an integral over the 4-volume of the space–time region $\mathscr{V}$ under consideration. The surface term J_S is an integral over the 3-boundary $\partial\mathscr{V}$ of $\mathscr{V}$, and has been discussed in detail by Hawking.[6] It is designed to eliminate any terms of the form $\delta g_{ik,l}$ in the variation

$$J \rightarrow J + \delta J, \qquad g_{ik} \rightarrow g_{ik} + \delta g_{ik} \tag{8}$$

terms which occur on $\partial\mathscr{V}$. In the following calculation we ignore such surface terms. The last part of the action, J_m , comprises matter terms which are absent in the so-called empty space–time. We return to J_m shortly.

Restricting our attention to J_E, we see that under the conformal transformation (3), it is changed to

$$\tilde{J}_E = \frac{1}{16\pi}\int_{\mathscr{V}} (R\Omega^2 - 6\Omega_i\Omega^i)\sqrt{-g}\,d^4x \tag{9}$$

where surface terms on $\partial\mathscr{V}$ are ignored. Here $\Omega_i = \partial\Omega/\partial x^i$ and indices are raised and lowered with the help of g^{ik} and g_{ik}, respectively.

Next, suppose that g_{ik} are obtained as solutions of the Einstein equations

$$\delta J = 0 \tag{10}$$

Writing

$$\Omega = 1 + \phi \tag{11}$$

it is easy to see that

$$\tilde{J} = \frac{1}{16\pi}\int_{\mathscr{V}} [(1 + 2\phi + \phi^2)R - 6\phi_i\phi^i]\sqrt{-g}\,d^4x + \tilde{J}_m \tag{12}$$

Turning now to J_m, we need to know the form of matter tensor. In general J_m contains information about fields and particles in interaction with each other and with gravity. Since our interest here is with conformal transformations, we would need to know how these particles and fields behave under conformal transformations. In the present work we limit ourselves to a system of particles moving arbitrarily and with no other mutual interaction except gravity. Later it may well be possible to investigate how crucial this limitation is to the conclusions of this paper: but we will not consider this issue further.

Now, taking J_m in the form of a system of particles $a, b, \ldots$,

$$J_m = \sum_a \int m_a\,ds_a \tag{13}$$

with masses m_a, m_b, ..., and an energy–momentum tensor T_m^{ik}, it is easy to see that

$$R = 8\pi T_m\,,\ \tilde{T}_m = \phi T_m\,, \qquad \tilde{J}_m = -\frac{1}{8\pi}\int_{\mathscr{V}} \phi T_m\sqrt{-g}\,d^4x \tag{14}$$

We therefore get

$$\begin{aligned}\tilde{J} &= \frac{1}{16\pi}\int_{\mathscr{V}} R\sqrt{-g}\,d^4x + \frac{1}{16\pi}\int_{\mathscr{V}} (R\phi^2 - 6\phi_i\phi^i)\sqrt{-g}\,d^4x \\ &= J_E + \frac{1}{16\pi}\int_{\mathscr{V}} (R\phi^2 - 6\phi_i\phi^i)\sqrt{-g}\,d^4x \end{aligned}\tag{15}$$

In the following section we apply this result to study quantum fluctuations.

3. QUANTUM GRAVITY VIA PATH INTEGRALS

3.1. The General Scheme

In general relativity the properties of gravity find their expression through the non-Euclidean character of the space–time geometry. The quantized version of this theory requires the dynamical variables, i.e., the quantities describing the space–time geometry, to be treated quantum mechanically. A direct transition from the classical to the quantum theory is obtainable through the Feynman path integral method.(7) In the case of gravity, the notion of 3-geometries is particularly useful,(8) and is briefly described below.

We assume that the space–time manifold is time-orientable and can be foliated by spacelike hypersurfaces $\{\Sigma\}$ which can be labeled by time coordinates t. Thus along a given Σ, we assume t to be constant. In such a manifold consider a region $\mathscr{V}$ sandwiched between two hypersurfaces

$$\Sigma_A : \quad t = t_A , \quad A = 1, 2 \tag{16}$$

We assume the time evolution to proceed from t_1 to t_2.

In classical general relativity we can look upon the geometry of the 4-dimensional region of the sandwich as a succession of evolving 3-geometries on the hypersurfaces $\{\Sigma\}$ for $t_1 \leqslant t \leqslant t_2$. Symbolically we may denote the 3-geometry on a typical Σ by ${}^{(3)}\mathscr{G}$. Thus the solution of the Einstein problem may be expressed as the sequence of $\{{}^{(3)}\mathscr{G}\}$ satisfying the variational condition

$$\delta J = 0 \tag{17}$$

together with the boundary values

$${}^{(3)}\mathscr{G} = {}^{(3)}\mathscr{G}_A \quad \text{on} \quad \Sigma = \Sigma_A , \qquad A = 1, 2 \tag{18}$$

Assuming that a unique solution to this problem exists, this solution describes a particular "path" in the space of these 3-geometries, a path which we may term the "classical" path.

In the quantum version of the theory we have to admit other, non-classical paths also, along which (17) does not hold. The "propagator" describes the probability amplitude that the system starts with ${}^{(3)}\mathscr{G}_1$ on Σ_1 and ends with ${}^{(3)}\mathscr{G}_2$ on Σ_2. Formally we may express this propagator as

$$K[{}^{(3)}\mathscr{G}_2 , t_2 ; {}^{(3)}\mathscr{G}_1 , t_1] = \sum_{\Gamma} \exp(iJ/\hbar) \tag{19}$$

Here the right-hand side involves a sum over all paths Γ of the amplitude to go from the given 3-geometry on Σ_1 to the given 3-geometry on Σ_2. In the classi-classical limit of $\hbar \to 0$, only the classical path and the paths in its neighborhood contribute to K and we recover the solution to the Einstein problem.

This prescription is simple to state in the above formal way but difficult to translate into a workable theory, involving as it does such problems as defining the paths in a meaningful way and ascribing a measure to the sum in (19). However, if we confine ourselves to only those nonclassical paths that correspond to the conformal transforms of the classical Einstein geometry, then the problem does become simplified and tractable. We consider this special situation next.

3.2. Conformal Fluctuations

Suppose in the above sandwich problem that the metric tensor g_{ik} represents a solution of Einstein's field equations for $t_1 < t < t_2$. A conformal transform

$$\tilde{g}_{ik} = (1 + \phi)^2 g_{ik} \tag{20}$$

will in general *not* be a solution of the classical field equations for an arbitrary-C^2 $\phi(x^i)$, since the equations are not conformally invariant. If we confine ourselves to such metrics, we can look upon $\phi(x^i) \neq \text{const}$ as specifying a typical nonclassical path. Accordingly, we reformulate the problem in Section 3.1 in the following form.

Suppose we are given $\phi = \phi_A(x^\mu)$, $\mu = 1, 2, 3$, on Σ_A for $A = 1, 2$. What is the probability amplitude that under quantum conformal fluctuations of the type (20), the space–time geometry is given by $\phi = \phi_1$ at $t = t_1$ and by $\phi = \phi_2$ at $t = t_2$? This is given by the propagator

$$K[\phi_2, t_2; \phi_1, t_1] = \sum_{\Gamma} \exp(iJ/\hbar) \tag{21}$$

Here J is evaluated for a metric of the form (20) for a typical "path" Γ which corresponds to a function $\phi(x^i)$ with

$$\phi(x^i)\,|_{t_A} = \phi_A(x^\mu), \qquad A = 1, 2 \tag{22}$$

Using (15), we see that

$$K[\phi_2, t_2; \phi_1, t_1] = \exp\left(\frac{iJ_E}{\hbar}\right) \sum_{\phi} \exp \frac{i}{16\pi\hbar} \int_{\mathscr{V}} (R\phi^2 - 6\phi_{;i}\phi^{;i}) \sqrt{-g}\, d^4x \tag{23}$$

The circumstance that the integral contains ϕ and ϕ_i only in the quadratic form leads to enormous simplification. To see this let $\bar{\phi}(x^i)$ be a solution of

$$\Box\bar{\phi} + \tfrac{1}{6}R\bar{\phi} = 0 \tag{24}$$

with the boundary conditions

$$\bar{\phi}(x^\mu, t_A) = \phi_A(x^\mu), \quad A = 1, 2 \tag{25}$$

Let

$$\phi = \bar{\phi} + \eta \tag{26}$$

where $\eta(x^i)$ vanishes on Σ_1 and Σ_2. Then

$$\begin{aligned} &\int_{\mathscr{V}} (R\phi^2 - 6\phi_i\phi^i)\sqrt{-g}\, d^4x \\ &\quad = -6\int_{\Sigma_2} \bar{\phi}\bar{\phi}^i\, d\Sigma_i + 6\int_{\Sigma_1} \bar{\phi}\bar{\phi}^i\, d\Sigma_i \\ &\qquad + \int_{\mathscr{V}} (R\eta^2 - 6\eta_i\eta^i)\sqrt{-g}\, d^4x \end{aligned} \tag{27}$$

Note that the first two terms on the right-hand side of (27) are path independent. The third term is path dependent. However, since η vanishes at both ends, the path integral over η can at most contain t_1 and t_2 explicitly. (This result is a generalization of the well-known result for path integrals over single dynamical variables.)

Since the $\exp(iJ_E/\hbar)$ term in (23) can similarly depend on only t_1 and t_2, we can write the solution of (23) in a simple and compact form which involves only ordinary integrals and a function of t_1 and t_2:

$$\begin{aligned} &K[\phi_2, t_2; \phi_1, t_1] \\ &\quad = F(t_1, t_2)\exp\frac{3i}{8\pi\hbar}\left[\int_{\Sigma_1} \bar{\phi}\bar{\phi}^i\, d\Sigma_i - \int_{\Sigma_2} \bar{\phi}\bar{\phi}^i\, d\Sigma_i\right] \end{aligned} \tag{28}$$

K is completely determined once the boundary value problem stated in (24) and (25) is solved.

A result along these lines for conformal fluctuations of flat space–time was obtained by Narlikar and Padmanabhan,[9] who showed how a formal solution in terms of Fourier transforms can be obtained for (28). The problem there in fact is formally the same as that of quantizing a scalar field in Minkowski space–time. The method of Fourier transforms, however, cannot be applied in the present problem, as the space–time is not flat. Since our aim is to study the behavior of K near a space–time singularity, we require a method of writing (28) which will work in highly curved space–times.

Before we proceed to describe this method in the above general context, we will apply it to a simpler problem solved earlier by another method. This is the problem of the collapse of a homogeneous ball of dust first considered by Narlikar.[3] The merit of resurrecting this problem and solving it in a different way lies in the fact that the new method pinpoints the feature that is responsible for the divergence of the quantum conformal fluctuations. The same feature will show up when we consider the more general case of (28).

In all subsequent work we put $\hbar = 1$.

4. THE COLLAPSING DUST BALL

Consider the spherically symmetric collapse of a dust ball under its own gravity. This is the problem originally discussed by Oppenheimer and Snyder.[10] The line element inside the ball in comoving coordinates is given by

$$ds^2 = dt^2 - \left(\frac{t}{t_0}\right)^{4/3} [dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)], \qquad r \leqslant r_0 \tag{29}$$

The time coordinate t tends to zero in the final stages of collapse which culminates in a space–time singularity. The density at any epoch $t < 0$ is given by

$$\rho = 1/(6\pi t^2) \tag{30}$$

The scalar curvature is given by

$$R = 8\pi\rho = 4/(3t^2) \tag{31}$$

We consider conformal fluctuations of the form

$$d\tilde{s} = [1 + \phi(t)]\, ds \tag{32}$$

Thus ϕ is only a function of t. To calculate K of the previous section we therefore have to solve (24), which now reduces to

$$\ddot{\bar{\phi}} + \frac{2}{t}\dot{\bar{\phi}} + \frac{2}{9t^2}\bar{\phi} = 0 \tag{33}$$

with $\bar{\phi} = \phi_1$ at $t = t_1$ and $\bar{\phi} = \phi_2$ at $t = t_2$. Since $t^{-1/3}$ and $t^{-2/3}$ are two linearly independent solutions of (33), it is not difficult to solve this problem by finding the function $\bar{\phi}$ as the suitable combination of these two solutions, as was done in Ref. 3.

Nevertheless here we solve the problem by using Green's functions. We define the retarded Green's function by the following properties:

$$
\begin{aligned}
&G^{\mathrm{R}}(t, t_1) = 0 \qquad \text{for} \quad t < t_1 \\
&\ddot{G}^{\mathrm{R}}(t, t_1) + \frac{2}{t}\,\dot{G}^{\mathrm{R}}(t, t_1) + \frac{2}{9t^2}\,G^{\mathrm{R}}(t, t_1) = \frac{1}{t_1^2}\,\delta(t - t_1)
\end{aligned}
\tag{34}
$$

The function $G^R(t, t_1)$ is continuous across $t = t_1$, but its time derivative has a step function behavior. A simple calculation shows that for $t \geqslant t_1$

$$
G^{\mathrm{R}}(t, t_1) = G(t, t_1) \equiv 3(t_1^{-2/3} t^{-1/3} - t_1^{-1/3} t^{-2/3}) \tag{35}
$$

We also define the advanced Green's function by

$$
G^A(t, t_2) = G^R(t_2, t) \tag{36}
$$

By considering the time derivatives of

$$
\dot{\bar{\phi}}(t)\, G^R(t, t_1) - \bar{\phi}(t)\, \dot{G}^R(t, t_1)
$$

and

$$
\dot{\bar{\phi}}(t)\, G^A(t, t_2) - \bar{\phi}(t)\, \dot{G}^A(t, t_2)
$$

it is easy to show that

$$
t_1^2 \dot{\bar{\phi}}(t_1)\phi_1 = G(t_2, t_1)^{-1}\phi_1\phi_2 + G(t_2, t_1)^{-1}\,\frac{\partial G(t_2, t_1)}{\partial t_1}\, t_1^2\phi_1^2 \tag{37}
$$

$$
t_2^2 \dot{\bar{\phi}}(t_2)\phi_2 = -G(t_2, t_1)^{-1}\phi_1\phi_2 + G(t_2, t_1)^{-1}\,\frac{\partial G(t_2, t_1)}{\partial t_2}\, t_2^2\phi_2^2 \tag{38}
$$

Returning now to (28), we see that since $\bar{\phi}$ is a function of t only,

$$
\int_{\Sigma_1} \bar{\phi}\bar{\phi}^i\, d\Sigma_i - \int_{\Sigma_2} \bar{\phi}\bar{\phi}^i\, d\Sigma_i = V[t_1^2\phi_1\dot{\bar{\phi}}(t_1) - t_2^2\phi_2\dot{\bar{\phi}}(t_2)] \tag{39}
$$

where V is the coordinate volume of the dust ball:

$$
V = \tfrac{4}{3}\pi\, r_0^3 \tag{40}
$$

Thus we recover the expression obtained in Ref. 3 for the propagator in the form

$$K[\phi_2, t_2; \phi_1, t_1] = F(t_1, t_2) \exp i(A_{11}\phi_1^2 + 2A_{12}\phi_1\phi_2 + A_{22}\phi_2^2) \quad (41)$$

with

$$A_{11} = \frac{3V}{8\pi} G(t_2, t_1)^{-1} \frac{\partial G(t_2, t_1)}{\partial t_1} t_1^2 \quad (42)$$

$$A_{22} = -\frac{3V}{8\pi} G(t_2, t_1)^{-1} \frac{\partial G(t_2, t_1)}{\partial t_2} t_2^2 \quad (43)$$

$$A_{12} = \frac{3V}{8\pi} G(t_2, t_1)^{-1} \quad (44)$$

If the initial state (at t_1) of geometry is descreibed by a wave function $\psi_1(\phi_1)$, the final state (at t_2) is given by

$$\psi_2(\phi_2) = \int K[\phi_2, t_2; \phi_1, t_1]\psi_1(\phi_1)\, d\phi_1 \quad (45)$$

In Ref. 3 it was shown that if $\psi_1(\phi_1)$ is a compact wave packet centered on the classical average $\phi_1 = 0$, $\psi_2(\phi_2)$ is also a wave packet centered on on $\phi_2 = 0$ but with a spread which diverges as $t_2 \to 0$. This result was taken to imply that the quantum fluctuations around the classical average diverges at the space–time singularity of the classical solution.

In terms of our present notation we may restate this result as follows. Suppose

$$\psi_1(\phi_1) = (2\pi\Delta_1^2)^{-1/4} \exp(-\phi_1^2/4\Delta_1^2) \quad (46)$$

Then (45) gives

$$|\psi_2(\phi_2)|^2 = (2\pi\Delta_2^2)^{-1/2} \exp(-\phi_2^2/^2\Delta_2^2) \quad (47)$$

where

$$\Delta_2^2 = \Delta_1^2[A_{11}^2 + 1/(16\Delta_1^4)]A_{12}^{-2} \quad (48)$$

Thus the divergence of Δ_2 is guaranteed by the divergence of A_{12}^{-2}

$$\Delta_2 > \frac{1}{4\Delta_1} | A_{12}^{-1} | = \frac{2\pi}{3V\Delta_1} | G(t_2, t_1)| \sim | t_2 |^{-2/3} \to \infty \quad (49)$$

In other words, the divergence of quantum fluctuations appears to come from the divergence of the Green's function of (33), as the classical space–time singularity is approached.

Although this divergence is explicitly seen in the above example of the wave packet, we can look at it in general terms through (45). Note that with K given by (45), as we approach the space–time singularity, $A_{12} \to 0$. This makes the cross-term $(\phi_1\phi_2)$ less and less important, with the result that in the limit

$$\begin{aligned}\psi_2(\phi_2) &\sim F(t_1, t_2) \exp(i A_{22}\phi_2^2) \int \psi(\phi_1) \exp(iA_{11}\phi_1^2)\, d\phi_1 \\ &\sim \exp(iA_{22}\phi_2^2) \qquad (50)\end{aligned}$$

Thus in the limiting situation ψ_2 depends on ϕ_2 only through a phase factor, implying infinite dispersion. So the crucial role is played by the coefficient of the cross-term, which in this case happens to be the reciprocal of the Green's function.

We will now show that these features are found in the general case outlined in Section 3.

5. CONFORMAL FLUCTUATIONS IN GENERAL

5.1. Green's Functions

We return to the wave equation (24) together with the boundary conditions (25). Guided by our simple example of Section 4, we construct the advanced and retarded Green's functions of the equation

$$\Box_x G(X, B) + \tfrac{1}{6}R(X)G(X, B) = [-g(X)]^{-1/2}\, \delta_4(X, B) \qquad (51)$$

Green's functions of wave equations in curved space–time have been discussed by others in different contexts.[11,12] The retarded Green's function $G^R(X, B)$ has a support only inside and on the future light cone of B, while the advanced Green's function has its support on and inside the past light cone of B. We also have

$$G^R(X, B) = G^A(B, X) \qquad (52)$$

By considering the integral

$$\int_{\mathscr{V}} \bar{\phi}\, \Box G - G\, \Box\bar{\phi})\ \sqrt{-g}\, d^4x$$

over the sandwich region of Section 3 for the advanced and the retarded Green's functions it is easy to show that

$$\phi_2(x_2^\mu) = \int_{\Sigma_1} [G^{\mathrm{R}}(x_2^\mu, t_2 ; x_1^\mu, t_1)\dot{\phi}(x_1^\mu, t_1)$$

$$- \frac{\partial}{\partial t_1} G^{\mathrm{R}}(x_2^\mu, t_2 ; x_1^\mu, t_1)\phi_1(x_1^\mu)] \sqrt{-g}\, d^3x_1^\mu \tag{53}$$

and

$$\phi_1(x_1^\mu) = - \int_{\Sigma_2} [G^{\mathrm{A}}(x_1^\mu, t_1 ; x_2^\mu, t_2)\dot{\phi}(x_2^\mu, t_2)$$

$$- \frac{\partial}{\partial t_2} G^{\mathrm{A}}(x_1^\mu, t_1 ; x_2^\mu, t_2)\phi_2(x_2^\mu)] \sqrt{-g}\, d^3x_2^\mu \tag{54}$$

Here we have assumed that the normals to Σ_1, Σ_2 are only in the time direction. If this condition is not satisfied, the above expressions become more involved, without, however, altering the general conclusion to be established here.

The expressions (53) and (54) are integral equations which determine $\dot{\phi}(x_1^\mu, t_1)$ and $\dot{\phi}(x_2^\mu, t_2)$ in terms of ϕ_1 and ϕ_2. Once these are known the propagator (28) is in principle determined. The situation has obvious analogy to the simpler expressions of Section 4, to (37) and (38). Before we proceed any further it is necessary to understand what a propagator like (28) in fact means.

5.2. The Propagator in the Function Space

A relation of the form (45) is easy to interpret when ϕ_1 and ϕ_2 belong to $\mathscr{R}^1$, the Euclidean line. However, what does a relation like

$$\Psi_2[\phi_2(x_2^\mu)] = \int K[\phi_2(x_2^\mu), t_2 ; \phi_1(x_1^\mu), t_1]\Psi_1[\phi_1(x_1^\mu)]\, \mathscr{D}\phi_1(x_1^\mu) \tag{55}$$

mean ? The ordinary integration over ϕ_1 in (45) is now replaced by a functional integration over $\phi_1(x_1^\mu)$. We can understand this transition as a limit from a discrete to a continuum set of dynamical variables, much as a field is looked upon as a dynamical variable with infinite degrees of freedom.

Suppose, for example, instead of the single variable ϕ in (45) we had a discrete set of variables $\phi^{(1)}, \phi^{(2)},..., \phi^{(n)}$ describing the state of a system. Then the probability amplitude is described by a wave function of the type

$\psi[\phi^{(1)},..., \phi^{(n)}]$. For such a system the propagator relation (45) is easily generalized to

$$\psi_2[\phi_2^{(1)},..., \phi_2^{(n)}]$$
$$= \int \cdots \int K[\phi_2^{(1)},..., \phi_2^{(n)}, t_2 ; \phi_1^{(1)},..., \phi_1^{(n)}, t_1]$$
$$\times \psi_1[\phi_1^{(1)},..., \phi_1^{(n)}]\, d\phi_1^{(1)} \cdots d\phi_1^{(n)} \tag{56}$$

In going from (56) to (55) the number of variables is increased from n to a continuum infinity, so that a discrete label r, $r = 1,..., n$, is replaced by a continuum variable x^μ. The functional integration of (55) is similarly the continuum limit of the n-tuple integral in (56). Such transitions are commonly used in evaluating path integrals (see Ref. 7).

With these limiting ideas in mind we go back to (53) and (54). The integrals therein are then limits of sums over quantities labeled by $x_1{}^\mu$. To state this more explicitly we replace x^μ by the vector $\mathbf{x}$ and suppose $\mathbf{x}$ to take a discrete set of values. Then we can rewrite (53) in the following form, with $g(x_1{}^i) = g_1$ and $g(x_2{}^i) = g_2$:

$$\phi_2(\mathbf{x}_2) = \sum_{\mathbf{x}_1} [G(\mathbf{x}_2 , \mathbf{x}_1)\dot{\phi}(\mathbf{x}_1 , t_1) - G_{t_1}(\mathbf{x}_2 , \mathbf{x}_1)\phi_1(\mathbf{x}_1)] \sqrt{-g_1} \tag{57}$$

$$\phi_1(\mathbf{x}_1) = \sum_{\mathbf{x}_2} [-G(\mathbf{x}_2 , \mathbf{x}_1)\dot{\phi}(\mathbf{x}_2 , t_2) + G_{t_2}(\mathbf{x}_2 , \mathbf{x}_1)\phi_2(x_2)] \sqrt{-g_2} \tag{58}$$

Here we have simplified our notation further by writing

$$G^A(x_1{}^\mu, t_1 ; x_2{}^\mu, t_2) = G^R(x_2{}^\mu, t_2 ; x_1{}^\mu, t_1) = G(\mathbf{x}_2 , \mathbf{x}_1) \tag{59}$$

and denoting by subscripts t_1 and t_2 the time derivatives with respect to t_1 and t_2, respectively.

Treating (57) and (58) as linear equations, we solve for $\dot{\phi}(\mathbf{x}_1 , t_1)$ and $\dot{\phi}(\mathbf{x}_2 , t_2)$ by using the "inverse" of the matrix $G(\mathbf{x}_2 , \mathbf{x}_1)$. Denoting the inverse by $G(\mathbf{x}_2 , \mathbf{x}_1)^{-1}$, we assume it to satisfy the following relations:

$$\sum_{\mathbf{x}_1} G(\mathbf{x}_2 , \mathbf{x}_1)^{-1} G(\mathbf{x}_2{}', \mathbf{x}_1) = \delta_{\mathbf{x}_2\mathbf{x}_2{}'} \tag{60}$$

$$\sum_{\mathbf{x}_2} G(\mathbf{x}_2 , \mathbf{x}_1)^{-1} G(\mathbf{x}_2 , \mathbf{x}_1{}') = \delta_{\mathbf{x}_1\mathbf{x}_1{}'} \tag{61}$$

where the right-hand sides are Kronecker deltas, which become delta functions in the continuum limit. The details are discussed in the Appendix.

Using these properties of the inverse, we get

$$\dot{\bar{\phi}}(\mathbf{x}_1, t_1)\sqrt{-g_1} = \sum_{\mathbf{x}_2} G(\mathbf{x}_2, \mathbf{x}_1)^{-1}\phi_2(\mathbf{x}_2)$$

$$+ \sum_{\mathbf{x}_2}\sum_{\mathbf{x}_1'} G_2(\mathbf{x}_2, \mathbf{x}_1)^{-1}G_{t_1}(\mathbf{x}_2, \mathbf{x}_1')\phi_1(\mathbf{x}_1')\sqrt{-g_1'} \tag{62}$$

and

$$\dot{\bar{\phi}}(\mathbf{x}_2, t_2)\sqrt{-g_2} = -\sum_{\mathbf{x}_1} G(\mathbf{x}_2, \mathbf{x}_1)^{-1}\phi_1(\mathbf{x}_1)$$

$$+ \sum_{\mathbf{x}_1}\sum_{\mathbf{x}_2'} G(\mathbf{x}_2, \mathbf{x}_1)^{-1}G_{t_2}(\mathbf{x}_2', \mathbf{x}_1)\phi(\mathbf{x}_2')\sqrt{-g_2'} \tag{63}$$

Therefore

$$\frac{3}{8\pi}\left[\sum_{\mathbf{x}_1}\phi_1(\mathbf{x}_1)\,\dot{\bar{\phi}}(\mathbf{x}_1, t_1)\sqrt{-g_1} - \sum_{\mathbf{x}_2}\phi_2(\mathbf{x}_2)\,\dot{\bar{\phi}}(\mathbf{x}_2, t_2)\sqrt{-g_2}\right]$$

$$= \sum_{\mathbf{x}_1}\sum_{\mathbf{x}_1'} A_{11}(\mathbf{x}_1, \mathbf{x}_1')\phi_1(\mathbf{x}_1)\phi_1(\mathbf{x}_1')$$

$$+ \sum_{\mathbf{x}_2}\sum_{\mathbf{x}_2'} A_{22}(\mathbf{x}_2, \mathbf{x}_2')\phi_2(\mathbf{x}_2)\phi_2(\mathbf{x}_2')$$

$$+ 2\sum_{\mathbf{x}_1}\sum_{\mathbf{x}_2} A_{12}(\mathbf{x}_1, \mathbf{x}_2)\phi_1(\mathbf{x}_1)\phi_2(\mathbf{x}_2) \tag{64}$$

where

$$A_{11}(\mathbf{x}_1, \mathbf{x}_1') = \frac{3}{8\pi}\sum_{\mathbf{x}_2} G(\mathbf{x}_2, \mathbf{x}_1)^{-1}G_{t_1}(\mathbf{x}_2, \mathbf{x}_1')\sqrt{-g_1'} \tag{65}$$

$$A_{22}(\mathbf{x}_2, \mathbf{x}_2') = -\frac{3}{8\pi}\sum_{\mathbf{x}_1} G(\mathbf{x}_2, \mathbf{x}_1)^{-1}G_{t_2}(\mathbf{x}_2', \mathbf{x}_1)\sqrt{-g_2'} \tag{66}$$

$$A_{12}(\mathbf{x}_1, \mathbf{x}_2) = \frac{3}{8\pi} G(\mathbf{x}_2, \mathbf{x}_1)^{-1} \tag{67}$$

The similairity between the three relations (42)–(44) of Section 4 and the above three relations is obvious.

In the continuum limit the above summations are replaced by integrations and we may write the formal solution for the propagator in the form:

$$K[\phi_2, t_2; \phi_1, t_1]$$
$$= F(t_1, t_2) \exp i \left(\iint A_{11}(\mathbf{x}_1, \mathbf{x}_1')\phi_1(\mathbf{x}_1)\phi_1(\mathbf{x}_1')\, d^3\mathbf{x}_1\, d^3\mathbf{x}_1' \right.$$
$$+ \iint A_{22}(\mathbf{x}_2, \mathbf{x}_2')\phi_2(\mathbf{x}_2)\phi_2(\mathbf{x}_2')\, d^3\mathbf{x}_2\, d^3\mathbf{x}_2'$$
$$\left. + 2 \iint A_{12}(\mathbf{x}_1, \mathbf{x}_2)\phi_1(\mathbf{x}_1)\phi_2(\mathbf{x}_2)\, d^3\mathbf{x}_1\, d^3\mathbf{x}_2 \right) \tag{68}$$

Notice that the exponential in the propagator is still quadratic in $\phi_1$and ϕ_2, as expected from the quadratic dependence of the action on ϕ. It is this fortunate circumstance that has enabled us to write down the *exact* solution for the propagator in the above manner.

6. THE BEHAVIOR OF FLUCTUATIONS NEAR A SPACE–TIME SINGULARITY

We now consider (55) with K given by (68). The relation tells us how Ψ has changed from $\Psi_1(\phi_1)$ to $\Psi_2(\phi_2)$. In general we cannot conclude anything about how this evolution proceeds. There is, however, a special situation which enables us to draw a definite conclusion. This situation arises when $A_{12}(\mathbf{x}_1, \mathbf{x}_2)$ vanishes.

As seen in the special case of Section 4, when $A_{12}(\mathbf{x}_1, \mathbf{x}_2)$ vanishes there is no "mixing" of ϕ_1 and ϕ_2. The functional integral over ϕ_1 can be carried out and it can only give a function of t_1 and t_2. Therefore we get

$$\Psi_2(\phi_2) \sim f(t_1, t_2) \exp i \iint A_{22}(\mathbf{x}_2, \mathbf{x}_2')$$
$$\times\ \phi_2(\mathbf{x}_2)\phi_2(\mathbf{x}_2')\, d^3\mathbf{x}_2\, d^3\mathbf{x}_2' \tag{69}$$

where f is a function of t_1 and t_2. However, $\Psi_2(\phi_2)$ contains ϕ_2 only through an imaginary phase factor. Hence the final state has complete quantum uncertainty. In other words, the quantum fluctuations diverge provided $A_{12}(\mathbf{x}_1, \mathbf{x}_2)$ vanishes.

This condition is achieved in the curvature singularities of general relativity, as can be seen from the following argument.

Suppose we have a space–time manifold with a curvature singularity at $t = t_0$ (say). We can then find a conformal transformation

$$\tilde{g}_{ik} = \Omega^2 g_{ik} \tag{70}$$

such that in the space with the metric tensor $\tilde{g}_{ik}$ there is no singularity.[13] The effect of the conformal transformation is to "stretch" the space–time so that the divergence in the curvature invariants is removed. To achieve this the conformal function Ω must diverge at $t = t_0$.

Now the wave equation (51) is conformally invariant. Hence if $G(X, B)$ is the Green's function for the space–time with the metric tensor g_{ik}, the Green's function for the conformally transformed space–time is given by

$$\tilde{G}(X, B) = G(X, B)/\Omega(X)\Omega(B)$$

i.e.,

$$G(X, B) = \Omega(X)\Omega(B)\tilde{G}(X, B) \tag{71}$$

Now in general $\tilde{G}(X, B)$ is finite, since the conformally transformed space–time has no singularity at $t = t_0$. Hence, from (71) we see that $G(X, B)$ must diverge as $\Omega(X)$ when X approaches the singular epoch. Correspondingly, its inverse must tend to zero.

Thus from (67) we see that as $t_2 \to t_0$, the coefficient A_{12} must vanish for all $\mathbf{x}_2$, which proves the result.

In the case of the collapsing dust ball, it is well known that the line element (29) is conformal to the Minkowski line element. Hence in (70) we have $\tilde{g}_{ik} = \eta_{ik}$, the Minkowski metric tensor, while $\Omega \sim | t |^{-2/3}$. Thus in (71), $G(X, B) \sim | t |^{-2/3}$ and it diverges as $| t | \to 0$.

7. DISCUSSION

During this work we have tacitly assumed the existence of $G(\mathbf{x}_2, \mathbf{x}_1)^{-1}$ and have not entered into the mathematically pedantic discussion of the conditions (necessary and sufficient) under which such a two-point function meaningfully exists. From a physical standpoint we may argue that just as the Green's function determines the "field" in terms of its given sources, so its inverse determines the source in terms of the given field. Except in pathological situations, we expect there to be a unique relationship between the source and its retarded field, i.e., between the Green's function and its inverse.

Since the emphasis of this paper is on the physics of the problem, the above mathematical aspect has been ignored in this work, just as many mathematical aspects of path integrals are often slurred over so long as the resultant physical result is sound (see Ref. 7 for many examples of this approach). Nevertheless we expect that future work on this mathematical aspect of the Green's function will be undertaken and will justify the steps to the final solution in Section 5. In the Appendix we have given some discussion of the inverse of the Green's function in flat space–time.

Granted this leeway, what does our result of Section 6 imply?

First, it shows that classical general relativity is no longer a reliable theory to tell us what happens when the quantum condition

$$|J| \lesssim \hbar \tag{72}$$

is satisfied. This, of course, is hardly surprising, since this relation marks the difference between the classical and the quantum world. However, for relativity it has more profound implications. The classical result of the inevitability of a singularity is shown here to be drowned in the sea of quantum (nonclassical) solutions as the singularity is approached. In other words, we cannot assert that space–time has a singularity. Rather, we have a picture wherein a finite region may collapse like the dust ball, until the condition (72) is reached, after which the space–time geometry makes a transition to a state which again makes $|J| \gg \hbar$ and restores the validity of the classical theory. Looked at in this way, the Universe as a whole may not have a unique big-bang origin at all. Rather, it is made of a large (probably infinite) number of compact regions which at times go through highly dense phases.

Second, we have limited our discussion to conformal fluctuations primarily because this limitation enables us to derive exact conclusions. Thus, unlike many other results of quantum gravity, these calculations are not limited to weak fields. Rather, they are geared to bringing out the effects of strong fields near the space–time singularity.

Third, while it would be desirable to explore the full range of variations of geometry, it is already clear from this work that the conformal fluctuations alone are sufficient to show up the limited validity of classical general relativity. Had our result turned out to point the other way by showing that the conformal fluctuations are insignificant, then the full investigations of quantum gravity would have become more imperative. The exploration of other, nonconformal, degrees of freedom would no doubt show further richness of quantum gravity. But it is too much to expect that these degrees of freedom would introduce additional uncertainties that neatly cancel the divergent conformal uncertainties.

The difference between the classical and the quantum behavior of the electron in the hydrogen atom led to the important concept of stationary states. It would be interesting to look for stationary states of space–time geometry in lieu of the space–time singularity of classical general relativity.

Finally, although we have limited ourselves to matter tensors in the form of a system of particles, our conclusions clearly remain valid so long as *any* matter tensor satisfies the rules (14). It is, however, possible to think of matter tensors (e.g., for massive scalar fields) which do not satisfy (14). It would be interesting to investigate whether the presence of such terms makes any serious difference to the conclusions of this paper.

APPENDIX

The relations (60) and (61) when written out in full and in the continuum form are

$$\iiint_{\Sigma_1} H(x_2^\mu, t_2\,; x_1^\mu, t_1)G(x_2'^\mu, t_2\,; x_1^\mu, t_1)\,d^3x_1^\mu = \delta_3(x_2^\mu - x_2'^\mu) \tag{A1}$$

$$\iiint_{\Sigma_2} H(x_2^\mu, t_2\,; x_1^\mu, t_1)G(x_2^\mu, t_2\,; x_1'^\mu, t_1)\,d^3x_2^\mu = \delta_3(x_1^\mu - x_1'^\mu) \tag{A2}$$

where H is the inverse $G(\mathbf{x}_2, \mathbf{x}_1)^{-1}$. These relations allow us to invert integral relations in the following way:

$$\alpha(x_2^\mu) = \iiint_{\Sigma_1} G(x_2^\mu, t_2\,; x_1^\mu, t_1)\beta(x_1^\mu)\,d^3x_1^\mu \tag{A3}$$

$$\beta(x_1^\mu) = \iiint_{\Sigma_2} H(x_2^\mu, t_2\,; x_1^\mu, t_1)\alpha(x_2^\mu)\,d^3x_2^\mu \tag{A4}$$

Alternatively, if we require that (A3) and (A4) hold, then (A1) implies (A2) and vice versa. We will now look at these relations more closely in flat space–time.

Taking x^μ to be Cartesian coordinates, we can then write, for $t_2 > t_1$,

$$G(x_2^\mu, t_2\,; x_1^\mu, t_1) = \delta(t_2 - t_1 - |\, x_2^\mu - x_1^\mu \,|)/4\pi\,|\, x_2^\mu - x_1^\mu \,| \tag{A5}$$

How can we express its inverse in the above sense ? We can give it a meaning provided we bear in mind the fact that G is a generalized function and its operations can be defined only with a suitable test function in the background.

Using Fourier transforms, it is not difficult to establish that, with $Q = |\, Q_\mu \,|$,

$$\frac{\delta(t_2 - t_1 - |\, x_2^\mu - x_1^\mu \,|)}{4\pi\,|\, x_2^\mu - x_1^\mu \,|} = \iiint \theta(t_2 - t_1)\,\frac{\sin Q(t_2 - t_1)}{4\pi Q}\,\exp[iQ_\mu(x_2^\mu - x_1^\mu)]\,\frac{d^3Q_\mu}{(2\pi)^3} \tag{A6}$$

The inverse has a Fourier transform which is simply the reciprocal of the above Fourier transform:

$$H(x_2^\mu, t_2\,; x_1^\mu, t_1) = \iiint \left[\theta(t_2 - t_1)\,\frac{\sin Q(t_2 - t_1)}{4\pi Q}\right]^{-1} \exp[iQ_\mu(x_2^\mu - x_1^\mu)]\,\frac{d^3Q_\mu}{(2\pi)^3} \tag{A7}$$

Of course as an ordinary integral (A7) is divergent and hence undefined, just as (A6) is divergent and undefined. Nevertheless, with suitable test functions $\alpha(x_2{}^\mu)$ and $\beta(x_1{}^\mu)$, (A6) and (A7) can be given a meaning.

Take for instance,

$$\beta(x_1{}^\mu) = \exp\,(ik_\mu x_1{}^\mu) \tag{A8}$$

Then using (A6), we get, for $t_2 > t_1$,

$$\alpha(x_2{}^\mu) = \int \cdots \int \frac{\sin Q(t_2 - t_1)}{4\pi Q} \exp[iQ_\mu(x_2{}^\mu - x_1{}^\mu)] \times \exp(ik_\mu x_1{}^\mu) \frac{d^3Q_\mu}{(2\pi)^3}\, d^3x_1{}^\mu$$

Performing the x_1 integration first, we get $\delta_3(Q_\mu - k_\mu)$. The resulting Q_μ integration then gives

$$\alpha(x_2{}^\mu) = \frac{\sin k(t_2 - t_1)}{4\pi k} \exp(ik_\mu x_2{}^\mu) \tag{A9}$$

with $k = |k_\mu|$. To invert the relation, we have

$$\begin{aligned}
&\iiint_\Sigma H(x_2{}^\mu, t_2\,; x_1{}^\mu, t_1)\alpha(x_2{}^\mu)\, d^3x_2{}^\mu \\
&\quad = \int \cdots \int \frac{4\pi Q}{\sin Q(t_2 - t_1)} \frac{\sin k(t_2 - t_1)}{4\pi k} \\
&\qquad \times \exp[iQ_\mu(x_2{}^\mu - x_1{}^\mu)] \exp(+ik_\mu x_2{}^\mu)\, d^3x_2{}^\mu \frac{d^3Q_\mu}{(2\pi)^3} \\
&\quad = \iiint \frac{Q \sin k(t_2 - t_1)}{k \sin Q(t_2 - t_1)} \delta_3(Q_\mu + k_\mu) \exp(-iQ_\mu x_1{}^\mu)\, d^3Q_\mu \\
&\quad = \exp(ik_\mu x_1{}^\mu) \\
&\quad = \beta(x_1{}^\mu)
\end{aligned}$$

which proves the result.

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