THE ANGULAR APPEARANCE OF WHITE HOLES

J. V. NARLIKAR
Center for Theoretical Physics, University of Texas at Austin, Texas, U.S.A. and Tata Institute of Fundamental Research, Bombay, India

and

R. C. KAPOOR
Indian Institute of Astrophysics, Bangalore, India

(Received 27 July, 1977)

Abstract. It is shown that non-radial light rays emitted from the surface of a white hole can emerge from inside the Schwarzschild barrier. The upper limit on their impact parameter is calculated under the requirement that such rays are blueshifted. The apparent angular size of the white hole determined by blueshifted rays is shown to grow so rapidly in the early stages of its expansion that it produces the appearance of superluminal expansion.

1. Introduction

This is a sequel to the paper by Narlikar and Apparao (1975), hereafter referred to as (I). In that paper the astrophysical consequences of radially emitted photons from spherically symmetric white holes in empty space–time were discussed. It was shown that photons emitted in the early stages of the expansion not only get out of the Schwarzschild barrier but do so with high blueshifts. Such a blueshifted radiation could manifest itself as a gamma-ray burst or as a transient X-ray emission.

In the present paper we consider non-radial emission of photons by white holes. The purpose of this investigation is to examine how the angular appearance of a white hole changes as it expands, especially in the early stages when it is inside the Schwarzschild barrier. The so-called impact parameter (which was zero in the radial case) plays an important role in determining the angular size of the white hole. To facilitate analytical calculations we will use the canonical white hole of (I). The specifications of the canonical model, which will be required for the present calculations, are given briefly below.

The space–time inside the white hole, taken as a spherical homogeneous ball of dust, is given by the Robertson–Walker line element

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - S^2(t)\left[\frac{\mathrm{d}r^2}{1-\alpha r^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right], \tag{1}$$

where r, θ, ϕ are the co-moving coordinates of a typical dust particle and t is its proper time. The parameter α is given by

$$\alpha = \frac{8\pi G\varrho_0}{3c^2}, \tag{2}$$

Astrophysics and Space Science **53** (1978) 155–163.

where G is the constant of gravitation; c the speed of light; and ϱ_0 is the density of matter when the scale factor $S(t) = 1$. The scale factor in fact satisfies the relation

$$\left(\frac{\mathrm{d}S}{\mathrm{d}t}\right)^2 = c^2\left(\frac{1 - S}{S}\right), \tag{3}$$

so that $\dot{S} = \mathrm{d}S/\mathrm{d}t = 0$ when $S = 1$. The white hole explodes from a singularity ($S = 0$) and then comes to rest ($S = 1$). The extent of the white hole is described by $r \leqslant r_b$.

The external solution is given by the familiar Schwarzschild metric

$$\mathrm{d}s^2 = c^2\,\mathrm{d}T^2\left(1 - \frac{2GM}{Rc^2}\right) - \frac{\mathrm{d}R^2}{1 - (2GM/Rc^2)} - R^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \tag{4}$$

The matching of the two metrics (1) and (4) across the boundary of the white hole requires that on $r = r_b$

$$R = r_b S(t), \qquad \left(1 - \frac{2GM}{Rc^2}\right)\cdot\frac{\partial T}{\partial t} = \sqrt{1 - \alpha r_b^2}, \tag{5}$$

and that M is given by

$$M = \frac{\alpha r_b^3 c^2}{2G}. \tag{6}$$

It should be emphasized that the canonical white hole described here is an idealization in many respects. The real explosions are not spherically symmetric. Anisotropics in the form of shear and rotation could be present. The initial state may not be singular, but highly dense. The explosion may not be in empty space–time. Eardley (1974) has argued that accretion may quickly smother an exploding white hole and convert it into a black hole. Lake and Roeder (1976) have emphasized, however, that a white hole of the above type arising out of a big bang universe as a lagging core need not encounter this fate.

For the purpose of the present calculation we will ignore these subtleties. Since exploding objects in the universe of the type discussed in (I) are usually found in a medium of tenuous density, the above model may be considered a crude approximation to reality at least in the not very anisotropic explosions.

2. Non-radial Null Trajectories

Consider a null geodesic emanating from the surface ($r = r_b$) of the white hole in a non-radial direction. The dynamics of test particles in Schwarzschild's geometry gives the following first integrals. First, without loss of generality, choose $\theta = \pi/2$ for the 'plane' of the trajectory. We then have, for an affine parameter λ,

$$R^2\frac{\mathrm{d}\phi}{\mathrm{d}\lambda} = h, \tag{7}$$

$$\left(1 - \frac{2GM}{Rc^2}\right)\frac{\mathrm{d}T}{\mathrm{d}\lambda} = \frac{h}{qc}, \tag{8}$$

where h and q are constants: h is the angular momentum parameter and q is the impact parameter.

Substitution of (7) and (8) into (4) gives, after some manipulation,

$$\frac{\mathrm{d}R}{\mathrm{d}T} = \left(1 - \frac{2GM}{Rc^2}\right) cF(R, q), \tag{9}$$

$$\frac{\mathrm{d}R}{\mathrm{d}\phi} = \frac{R^2}{q} F(R, q), \tag{10}$$

where

$$F(R, q) = \left\{1 - \frac{q^2}{R^2}\left(1 - \frac{2GM}{c^2 R}\right)\right\}^{1/2}. \tag{11}$$

Initially we will set $\phi = \phi_1$, $R = R_1 = r_b S(t_1)$, $T = T(t_1) = T_1$ so that t_1 is the proper time of the surface dust particle at the time of emission. Finally, the light ray arrives at a remote receiving point with $R = R_2 \gg 2GM/c^2$, $\theta = \pi/2$, $\phi = 0$, $T = T_2$ (say). Then from (9) and (10) we get

$$T_2 - T_1 = \frac{1}{c}\int_{R_1}^{R_2} \left[F(R, q)\left(1 - \frac{2GM}{c^2 R}\right)\right]^{-1} \mathrm{d}R, \tag{12}$$

$$\phi_1 = q \int_{R_1}^{R_2} \frac{\mathrm{d}R}{R^2 F(R, q)}. \tag{13}$$

Figure 1 shows the schematic track of a light ray as it winds round the white hole mass before finally coming out at $R = R_2$. At this point the ray makes an angle ε with the radial direction $\phi = 0$. This angle is given by examining (10) in the neighborhood of $R = R_2$. Transforming to a locally cartesian set of coordinates near $R = R_2$ we see that

$$\epsilon \simeq \frac{q}{R_2}. \tag{14}$$

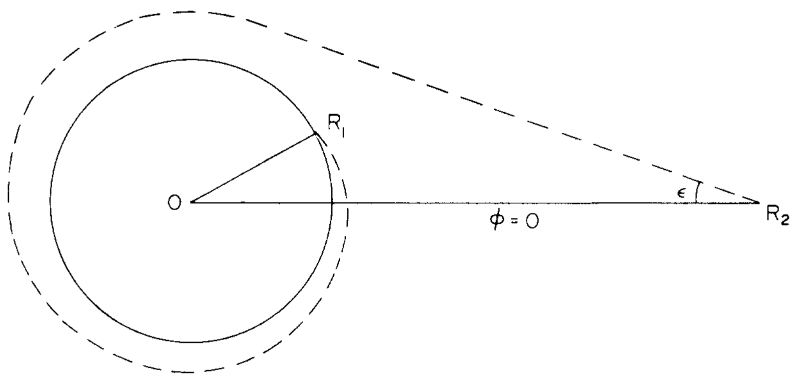

Fig. 1. The track of a non-radial null ray from the surface of a white hole to a remote observer.

Thus the impact parameter essentially determines the angle along which the ray approaches the receiver at $R = R_2$. Note that (14) is derived on the assumption that $\varepsilon \ll 1$, so that $q \ll R_2$. It is not valid for a receiver located very near the white hole. For an Earth-bound observer looking at a galactic or an extragalactic white hole, (14) is a good approximation.

3. The Spectral Shift

We will now compute the spectral shift of the non-radial null trajectory described above. The calculation can be performed either from first principles, by considering two null trajectories leaving $r = r_b$ at $t = t_1$ and $t_1 + \Delta t_1$ and arriving at $R = R_2$ at $T = T_2$ and $T_2 + \Delta T_2$ respectively, or by using the following general formula given by Schrödinger (1950)

$$\frac{\nu_2}{\nu_1} = \frac{(\mathbf{u}_i \mathbf{v}^i)_2}{(\mathbf{u}_i \mathbf{v}^i)_1}, \tag{15}$$

where ν_1 is the frequency of emission, ν_2 the frequency of reception, $\mathbf{v}_1^i$ the source velocity, $\mathbf{v}_2^i$ the receiver velocity, $\mathbf{u}_1^i$ the photon direction at source and $\mathbf{u}_2^i$ the photon direction at the receiver. It is easy to see that the following values obtain for these quantities in the R, T coordinates:

$$\mathbf{v}_1^i = \left[r_b \dot{S}(t_1), 0, 0, \sqrt{1 - \alpha r_a^2}\left(1 - \frac{2GM}{R_1 c^2}\right)\right],$$

$$\mathbf{v}_2^i \simeq [0, 0, 0, 1], \qquad \text{since } 2GM/R_2 c^2 \ll 1,$$

$$\mathbf{u}_A^i = \left[\frac{dR}{d\lambda}, 0, \frac{d\phi}{d\lambda}, \frac{dT}{d\lambda}\right], \qquad (R = R_A, A = 1, 2).$$

The derivatives $dR/d\lambda$, $d\phi/d\lambda$, $dT/d\lambda$ are given by (7), (8) and (9). A simple calculation gives

$$\frac{\nu_2}{\nu_1} = \frac{1 - \dfrac{2GM}{c^2 R_1}}{\sqrt{1 - \alpha r_b^2} - \dfrac{r_b \dot{S}(t_1)}{c} F(R_1, q)}. \tag{16}$$

The approach from first principles [which was used in (I)] gives the same answer.

It is interesting to note that ν_2/ν_1 is well behaved at all $R_1 > 0$, even at $R_1 = 2GM/c^2$. When $R_1 = 2GM/c^2$, the denominator as well as the numerator vanish, but the ratio ν_2/ν_1 is finite. We in fact have from (3), (6) and (11)

$$1 - \alpha r_b^2 - \frac{r_b^2 \dot{S}^2 F^2(R_1, q)}{c^2} = 1 - \alpha r_b^2 - \frac{r_b^2 \dot{S}^2}{c^2} + \frac{r_b^2 \dot{S}^2}{c^2}\{1 - F^2(R_1 q)\}$$

$$= 1 - \frac{2GM}{c^2 R_1} + \frac{r_b^2 \dot{S}^2}{c^2} \cdot \frac{q^2}{R_1^2}\left(1 - \frac{2GM}{c^2 R_1}\right).$$

Hence, Equation (16) reduces to the expression

$$\frac{\nu_2}{\nu_1} = \frac{\sqrt{1 - \alpha r_b^2} + \dfrac{r_b \dot{S}}{c} F(R_1, q)}{1 + \dfrac{q^2 r_b^2 \dot{S}^2(t_1)}{c^2 R_1}}. \tag{17}$$

At $R_1 = 2GM/c^2$, this becomes

$$\left[\frac{\nu_2}{\nu_1}\right]_{R_1 = 2GM/c^2} = \frac{2\sqrt{1 - \alpha r_b^2}}{1 + \dfrac{q^2}{R_1^2}(1 - \alpha r_b^2)}. \tag{18}$$

Examination of these formulae shows the expected result that the ratio ν_2/ν_1 for non-radial geodesics is smaller than for radial geodesics ($q = 0$). We now look for the possible upper limit on q so that the corresponding null geodesic shows a blueshift ($\nu_2 > \nu_1$).

Setting $\nu_2 = \nu_1$ and solving for q we get the following upper limit on q for a blueshift

$$q^2 < \frac{2\sqrt{1 - \alpha r_b^2} - 2 + \dfrac{\alpha r_b^2}{S_1}}{\alpha\left(\dfrac{1 - S_1}{S_1^3}\right)}, \tag{19}$$

where $S_1 = S(t_1)$. For small S_1 ($\ll \alpha r_b^2 < 1$), we have

$$q^2 \leqslant r_b^2 S_1^2 = R_1^2. \tag{20}$$

For $S_1 > \alpha r_b^2$ there is another requirement on q; namely, that F should be real, i.e.

$$\frac{1}{q^2} \geqslant \left(1 - \frac{2GM}{R_1 c^2}\right) \frac{1}{R_1^2}. \tag{21}$$

The right-hand side has a maximum for $R_1 = 3GM/c^2$. Thus, for $q \leqslant 3\sqrt{3}\, GM/c^2$, F is always real. [Note that (20) is automatically satisfied for $R_1 < 2GM/c^2$, i.e. for $S_1 < \alpha r_b^2$.] We shall, however, be concerned mainly with (20) which applies in the very early stages of the white hole expansion.

4. The Rate of Growth of Angular Size

How does the white hole appear to a remote observer as it expands from $S = 0$? In (I) we considered the brightness, spectrum, flex density, etc., by using the radially outgoing null trajectories. In the present section we will look at the apparent rate of growth of the angle subtended by the white hole. In §2 we saw that this angle is proportional to q, the impact parameter of a non-radial ray leaving the white hole surface. What is the maximum value of q at any given time T_2 and how does it change with q_2? To answer this question we will assume that only the blueshifted rays are easily detectable by a remote observer. The redshifted rays will be too faint to observe. This is only a crude approximation to reality where the actually measured angular size will depend on a number of parameters related to the measuring instrument, such as its sensitivity, its response to specific wavelengths, the signal to noise ratio, etc. All that is implied in the following calculation is that quanta that are blueshifted are relatively easier to detect than quanta that are redshifted.

Consider a typical blueshifted ray which has an impact parameter q and which leaves the surface of the white hole at $t = t_q$ in such a way as to arrive at $R = R_2$ at $T = T_2$. The answer to our question is then obtained by calculating the largest value of q for a given T_2, subject to (20), – i.e.,

$$q \leqslant r_b S(t_q) = R_q. \tag{22}$$

Applying (12) to this null ray we have

$$T_2 = T(t_q) + \frac{1}{c}\int_{R_q}^{R_2} \left[F(R, q)\left(1 - \frac{2GM}{c^2 R}\right)\right]^{-1} \mathrm{d}R.$$

As we change q, t_q changes. The variation of the above relation for a fixed T_2 gives

$$\Delta t_q = -\frac{1}{c}\left(1 - \frac{2GM}{c^2 R_q}\right)\cdot\frac{\int_{R_q}^{R_2} \frac{q}{R^2}\,[F(R, q)]^{-3}\,\mathrm{d}R}{\sqrt{1 - \alpha r_b^2} - \frac{r_b \dot{S}(t_q)}{cF(R_q, q)}}\,\Delta q. \tag{23}$$

It can be verified that in the early stages ($R_q \ll 2GM/c^2$) and for

$$q < \frac{R}{\sqrt{1 - \alpha r_b^2}}, \tag{24}$$

t_q decreases as q increases. That is, geodesics with larger impact parameters have to start earlier than those with smaller q, in order to arrive simultaneously at $R = R_2$. Thus, the radial null geodesic ($q = 0$) starts last. As q increases from $q = 0$, geodesics have to start at earlier epochs with smaller S. Clearly, q can increase only up to the limit set by (22) [which is consistent with (24)]. Therefore, the largest angular size is obtained by setting $q = R_q$ in the specification of the null geodesic. This gives

$$T_2 = T(t_q) + \frac{1}{c}\int_q^{R_2} \left(1 - \frac{2GM}{Rc^2}\right)^{-1}\left[1 - \frac{q^2}{R^2}\left(1 - \frac{2GM}{Rc^2}\right)\right]^{-1/2} \mathrm{d}R. \tag{25}$$

This implicit relation determines q as a function of T_2. In the appendix it is shown that for $q \ll 2GM/c^2$, (25) gives a rate of the type

$$\frac{\mathrm{d}q}{\mathrm{d}T_2} = KT_2^{-1/4}, \tag{26}$$

where $K =$ constant. The corresponding angular radius of the white hole appears to increase at a rate

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}T_2} = KR_2^{-1}T_2^{-1/4}. \tag{27}$$

For small T_2, (26) will predict an apparent superluminal expansion speed.

It is of interest to note that apparent superluminal expansion speeds have been observed in the radio components of many quasistellar sources (Kellermann, 1976). In a typical example, two components appear to separate from each other at a velocity $v > c$. If these speeds are 'real' they contradict either the special theory of relativity or the cosmological hypothesis of the distance of a QSO from us. Although the above example refers to a spherical white hole, it points out a likely explanation of the above phenomena. To obtain quantitative estimates for comparison with observations it is, however, necessary to consider linearly expanding white holes. Using the result derived in the appendix we see that when the value of $q = R_1 = \eta \times$ Schwarzschild radius of the white hole, the observed expansion speed is given by

$$v = \frac{\mathrm{d}q}{\mathrm{d}T_2} \sim \frac{c}{(2\eta)^{1/3}}. \tag{28}$$

Hence, for $v > c$ we need the white hole to be considerably smaller in extent than implied by the Schwarzschild radius.

In (13) we have the integral determining the bending of a light ray emitted by a white hole. The integral for ϕ_1 gives a finite value except when $F^2(R, q) = 0$ has a double root in R. This happens when $q = 3\sqrt{3}\, GM/c^2$. As q tends to this value, ϕ_1 diverges, i.e. the light ray keeps going round and round the object at $R = 3GM/c^2$. For q-values close to this critical value, rings and multiple-imaging occur for light emerging from collapsing or highly collapsed objects [see Ames and Thorne (1968), Das (1975)]. These phenomena are not expected to occur in the case of white holes for two reasons. First, as we saw earlier, the interesting values of q for white holes are considerably smaller than the critical value. For these ϕ_1 is not very large. Secondly, the expanding white hole cannot permit even the rays with $q \approx 3\sqrt{3}\, GM/c^2$ to circulate forever because it soon occupies the region $R = 3GM/c^2$ where this could occur.

5. Conclusion

We have shown that, for the canonical white hole, the nonradial rays can emerge from inside the Schwarzschild barrier and reach a remote observer. The ratio of received to emitted frequencies for non-radial rays is smaller than that for radial rays. It is still possible to have such rays blueshifted in the early stages of the expansion, provided their impact parameter is smaller than the Schwarzschild radial coordinate of the emitting surface at the time of emission.

Assuming that only blueshifted rays are seen by the remote observer, the apparent angular size of the white hole grows very fast in the early stages giving the impression of a superluminal expansion.

Acknowledgement

One of us (J.V.N.) wishes to express his grateful thanks to Professor John A. Wheeler for hospitality at the Center for Theoretical Physics Research, University of Texas at Austin, where part of this work was done.

Appendix

Differentiate (25) with respect to q and use the relation (5) to get

$$\frac{\mathrm{d}T_2}{\mathrm{d}q} = \frac{1}{c}\sqrt{1-\alpha r_b^2}\left(1-\frac{2GM}{qc^2}\right)^{-1}\frac{\mathrm{d}t_q}{\mathrm{d}q} - \frac{1}{c}\left(1-\frac{2GM}{qc^2}\right)^{-1}\cdot\sqrt{\frac{2GM}{qc^2}} + $$
$$+\frac{1}{c}\int_q^{R_2}\frac{q}{R^2}\left\{1-\frac{q^2}{R^2}\left(1-\frac{2GM}{Rc^2}\right)\right\}^{-3/2}\mathrm{d}R.$$

Since $q = r_b S(t_q)$, we have $\mathrm{d}q = r_b\dot{S}(t_q)\,\mathrm{d}t_q$. Hence the above relation simplifies to

$$\frac{\mathrm{d}T_2}{\mathrm{d}q} = \frac{1}{c}\left(1-\frac{2GM}{qc^2}\right)^{-1}\left\{\frac{\sqrt{1-\alpha r_b^2}}{r_b\dot{S}(t_q)} - \sqrt{\frac{2GM}{qc^2}}\right\} + $$
$$+\frac{1}{c}\int_q^{R_2}\frac{q}{R^2}\left\{1-\frac{q^2}{R^2}\left(1-\frac{2GM}{Rc^2}\right)\right\}^{-3/2}\mathrm{d}R.$$

For $q \ll 2GM/c^2$, $\dot{S}^2 \approx \alpha c^2/S$. Hence, we get

$$\frac{\mathrm{d}T_2}{\mathrm{d}q} = \frac{1}{c}(1-\sqrt{1-\alpha r_b^2})\cdot\left(\frac{qc^2}{2GM}\right)^{3/2} + $$
$$+\frac{1}{c}\int_q^{R_2}\frac{q}{R^2}\left\{1-\frac{q^2}{R^2}\left(1-\frac{2GM}{Rc^2}\right)\right\}^{-3/2}\mathrm{d}R.$$

For small q use the following approximations, with $x = R^{-1}$, $y = (2GMc^{-2}b^2)^{1/3}x$:

$$\frac{1}{c}\int_q^{R_2}\frac{q}{R^2}\left\{1-\frac{q^2}{R^2}\left(1-\frac{2GM}{Rc^2}\right)\right\}^{-3/2}\mathrm{d}R \approx$$
$$\approx \frac{q}{c}\int_0^{q^{-1}}\left(1-q^2x^2+\frac{2GM}{c^2}q^2x^3\right)^{-3/2}\mathrm{d}x$$
$$\approx \frac{1}{c}\left(\frac{qc^2}{GM}\right)^{1/3}\int_0^{\infty}\left\{1-\left(\frac{qc^2}{GM}\right)^{2/3}y^2+2y^3\right\}^{-3/2}\mathrm{d}y$$
$$\approx \left(\frac{q}{GMc}\right)^{1/3}\int_0^{\infty}(1+2y^3)^{-3/2}\,\mathrm{d}y.$$
$$\approx A\left(\frac{q}{GMc}\right)^{1/3},$$

where A is a constant of the order of unity. For small q the behavior of q with respect to T_2 is therefore determined by the equation

$$\frac{\mathrm{d}T_2}{\mathrm{d}q} \sim A\left(\frac{q}{GMc}\right)^{1/3}.$$

which gives

$$T_2 \sim \frac{3A}{4(GMc)^{1/3}} q^{4/3}.$$

Hence, we get

$$\frac{dq}{dt} \sim KT^{-1/4}$$

with $K = (3/4)^{1/4} A^{5/4} (GMc)^{-5/12}$.

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