

QUANTUM COSMOLOGY VIA PATH INTEGRALS

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Abstract:

The main purpose of this article is to report the progress of the path integral approach to quantum cosmology. Since quantum cosmology is an interdisciplinary field involving inputs from quantum theory, general relativity and cosmology, we begin with a brief survey of classical geometrodynamics and classical cosmology as well as an outline of the problems faced by any quantum theory of gravity. It is against this background that the authors' approach described in sections 3–5 is to be viewed and assessed. The Feynman path integral formalism to the extent necessary for following this approach is described first in section 2.

In section 3 it is shown that the limited goal of quantizing only the conformal part of the space-time metric can be reached with the help of path integral techniques. A case is made as to why this limited approach is still of relevance to quantum cosmology. Explicit examples are worked out to show how meaningful conclusions can be drawn about quantum uncertainty at the classical singularity, the likelihood of singularity-free and horizon-free models in quantum cosmology and the limits on the validity of classical relativity close to the big bang.

In section 4 the existence of stationary states of the universe is discussed. It is shown how the quantization of the conformal degree of freedom leads to stationary states for the quantum analogues of the classical models. The results are generalized and discussed in the framework of the superspace metric.

The difficult problem of the back reaction of quantum conformal fluctuations on the space-time metric is tackled in a semiclassical fashion in section 5. In this approach the conformal part of the metric is treated classically while the conformal fluctuations are replaced by their expectation values. The resulting field equations are solved in a few simple cases and physically interpreted. This preliminary work holds promise for a more complete theory of the future. In the end a solution to the flatness problem of classical cosmology is suggested within the framework of conformal fluctuations.

1. From classical gravity to quantum cosmology

1.1. Introduction

Cosmology deals with the study of the large scale structure of the universe. Quantum theory, on the other hand concerns itself with microscopic physical systems. The subject 'Quantum Cosmology' may therefore appear as the mismatch of two non-overlapping disciplines. The purpose of this article is to correct this erroneous first impression and to demonstrate that quantum cosmology opens up a new branch of cosmology to which quantum theory makes highly relevant contributions.

Since cosmology is studied within the framework of gravity theories, quantum cosmology necessarily forms a branch of quantum gravity. At this stage the practical physicists may ask the pertinent question, whether gravity *needs* to be quantized. Unlike electrodynamics where there are a host of phenomena which cannot be explained without recourse to quantum ideas, there is to date no experimental result or astronomical observation that forces us to quantize gravity.

Nevertheless there are conceptual reasons why gravity must be quantized. Of the four so called basic interactions in physics all except gravity are expressed in the quantum framework. The present unification programme holds out hope for the grand unification of these three interactions. If eventually gravity is also to form part of the "supergrand unified" theory it is essential to know how to describe that interaction in the quantum framework.

Cosmology itself provides another reason. The geometric part of the action in general relativity is given by (cf. section 1.2 for details)

$$S_g \sim \frac{c^3}{16\pi G} \int_{\mathcal{V}} R \sqrt{-g} d^4x \tag{1.1}$$

where \mathcal{V} is the space-time region under consideration. The scalar curvature R of the space-time has dimensions (length)⁻². Taking $R \sim L^{-2}$ for a characteristic length L , and the 4-volume of \mathcal{V} as L^4 , we estimate S_g as

$$S_g \sim \frac{c^3 L^2}{16\pi G}. \quad (1.2)$$

For quantum effects to be important $S_g \leq \hbar$. While this condition is not satisfied by a large margin in the present state of the universe, it must have been relevant in the remote past when the universe was close to its big bang origin. Ignoring the numerical factor 16π , the above quantum condition tells us that at those epochs the characteristic linear size of the universe would have been smaller than

$$L_P = (G\hbar/c^3)^{1/2} \sim 1.6 \times 10^{-33} \text{ cm}. \quad (1.3)$$

L_P is the so called 'Planck length' for gravity.

Experience with electrodynamics warns us not to trust the classical theory once the quantum condition $S \leq \hbar$ is reached for the action S . Likewise we are warned here to look at cosmology within a suitable quantum framework if we want to study the behaviour of the universe when its characteristic size was so small ($\leq L_P$).

In this article we outline such a framework and try to answer some relevant questions of cosmological significance. Before proceeding further we summarize some standard notions of classical gravity and cosmology which we have to draw upon in order to describe quantum cosmology.

1.2. Classical gravity

We shall assume that the equations of classical gravity are given by general relativity [1] and written as

$$R_{ik} - \frac{1}{2}g_{ik}R = -\kappa T_{ik}, \quad \kappa = 8\pi G/c^4. \quad (1.4)$$

Here R_{ik} is the Ricci tensor, R the scalar curvature, g_{ik} the metric tensor and T_{ik} the energy momentum tensor. In our notation the latin indices i, k etc. will take values 0, 1, 2, 3. The index 0 usually implies a timelike component while the space indices 1, 2, 3 are denoted by Greek indices μ, ν etc. The signature of the metric is taken as (+, -, -, -).

Since in this article we shall be concerned mainly with formal aspects we will put $c = 1$, $G = 1$, unless otherwise stated.

The field equations (1.4) are derived from the variation of an action S in a specified 4-volume \mathcal{V} for the variations $g_{ik} \rightarrow g_{ik} + \delta g_{ik}$. The action as originally specified by Hilbert [2] is

$$S = \frac{1}{16\pi} \int_{\mathcal{V}} R \sqrt{-g} d^4x + S_m, \quad g = \det\|g_{ik}\| \quad (1.5)$$

where the energy momentum tensor T_{ik} is related to the variation of S_m by

$$\delta S_m = -\frac{1}{2} \int_{\mathcal{V}} T^{ik} \delta g_{ik} \sqrt{-g} d^4x. \quad (1.6)$$

The geometric part S_g represented by the first term in S suffers from one difficulty. It contains up to second derivatives of g_{ik} . This requires the variations of metric first derivatives $g_{ik,l} \equiv \partial g_{ik} / \partial x^l$ along with δg_{ik} to vanish on $\partial \mathcal{V}$. To obviate this constraint Gibbons and Hawking [3] suggested the addition of a surface term to S given by

$$S_{\text{GH}} = \frac{1}{8\pi} \int_{\partial \mathcal{V}} (\mathcal{H} + \mathcal{L}) (-h)^{1/2} d^3x \quad (1.7)$$

where \mathcal{H} is the second fundamental form for $\partial \mathcal{V}$ and $\{h\}$ the induced metric on it. The form of \mathcal{L} is properly understood only in asymptotically flat space-times.

The Einstein field equations are nonlinear and of second order in g_{ik} . Their general solution in analytic form is not possible. Even the formulation of an initial value problem requires considerable discussion since the equations are constrained by Bianchi identities

$$(R^{ik} - \frac{1}{2}g^{ik}R)_{;k} \equiv 0. \quad (1.8)$$

We refer the reader to standard discussions by Wheeler and his colleagues [4, 5] who refer to the subject as geometrodynamics.

A formulation which we will use in later sections is that of the thin sandwich in which \mathcal{V} is the region between two spacelike hypersurfaces Σ_i and Σ_f . We can choose the coordinate system in such a way that

$$ds^2 = dt^2 + g_{\alpha\beta} dx^\alpha dx^\beta \quad (1.9)$$

and Σ_i, Σ_f are given respectively by $t = t_i, t = t_f$ (t_i, t_f constants). The field equations may be looked upon as determining the sequence of 3-geometries (${}^3\mathcal{G}$) on the spacelike hypersurfaces of constant t for $t_i < t < t_f$.

Isenberg and Wheeler [6] have pointed out that no constraint equations are violated if the initial data are specified not in the form of ${}^3\mathcal{G}$ but in a slightly different form. Their prescription is to specify only the conformal part of ${}^3\mathcal{G}$ and the trace of the extrinsic curvature. The latter makes up for the unspecified scale factor of the former. Such a specification guarantees the existence and uniqueness of solutions.

The action describing the above geometrodynamics can be expressed in the form

$$S_g = \int_{t_1}^{t_2} \int (-{}^3g)^{1/2} \{ {}^3R - (\text{Tr } K)^2 + \text{Tr } K^2 \} d^3x dt \quad (1.10)$$

where 3g is the determinant of the spatial metric and 3R the scalar curvature of ${}^3\mathcal{G}$. $K_{\mu\nu}$ is the extrinsic curvature tensor and Tr stands for trace.

It is interesting (and useful for our later work) to view the progressive change of ${}^3\mathcal{G}$ with time as a dynamical problem in ‘superspace’. Following De Witt’s work [7] we denote by dL^2 the line element in an abstract superspace whose points are the 3-geometries ${}^3\mathcal{G}$. Explicitly we write the superspace metric in the form

$$dL^2 = \int \int G_{\mu\nu\alpha\beta}(x, x') dg^{\mu\nu}(x) dg^{\alpha\beta}(x') d^3x d^3x'. \quad (1.11)$$

Some manipulation with indices then tells us that the action S_0 can be expressed in the form

$$S_0 = \frac{1}{4} \int_{t_1}^{t_2} \left(\frac{dL}{dt} \right)^2 dt + \int_{t_1}^{t_2} \int (-^3g)^{1/2} ({}^3R) d^3x dt \quad (1.12)$$

provided we define

$$G_{\mu\nu\alpha\beta}(x, x') = \delta(x, x') \bar{G}_{\mu\nu\alpha\beta}(x), \quad (1.13)$$

$$\bar{G}_{\mu\nu\alpha\beta} = \frac{1}{2} (-^3g)^{1/2} \{ g_{\mu\alpha} g_{\nu\beta} + g_{\mu\beta} g_{\nu\alpha} - 2g_{\mu\nu} g_{\alpha\beta} \}. \quad (1.14)$$

The time coordinate t acts as a parameter in superspace. The evolution of ${}^3\mathcal{G}$ from Σ_i to Σ_t traces a path in superspace. The paths followed by the solutions of (space-space) parts of Einstein's equations have a special status in this space: it can be shown that they are geodesics in superspace. This result can be expressed in the following way.

Write a single index A for the pair $(\mu\nu)$ so that $g_{\mu\nu}$ are written g_A . Choose a new time coordinate λ defined by

$$d\lambda/dt = (-^3g)^{1/2} ({}^3R). \quad (1.15)$$

Then in terms of λ the variational equations $\delta S_g = 0$ reduce to the geodesic equations

$$\frac{d^2 g^A}{d\lambda^2} + \Gamma_{BC}^A \frac{dg^B}{d\lambda} \frac{dg^C}{d\lambda} = 0, \quad (1.16)$$

where the Christoffel symbols Γ_{BC}^A are defined in the usual way.

These equations can also be expressed as Hamiltonian equations with a suitably defined Hamiltonian H for the action S_g . The constraints on the field equations come from the (time-time) and (time-space) parts of the original Einstein equations. The former appears here as the relation $H = 0$ while the latter are difficult to incorporate in general. However, they can be eliminated in some symmetric space-times which are usually of interest in cosmology. Space-time symmetries also help to reduce the dimensionality of the superspace. We will encounter such 'minisuperspaces' in our later work.

1.3. Classical cosmology

The symmetries of cosmological model referred to in section 1.2 were assumed by theoreticians partly on observational grounds (viz. the present large scale structure of the universe shows a remarkable degree of homogeneity and isotropy) and partly as a means to simplifying Einstein's equations to the extent that they can be solved exactly. The maximally symmetric space-times [8] describing the expanding universe are given by the Robertson-Walker line element [9, 10]:

$$ds^2 = dt^2 - Q^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right]. \quad (1.17)$$

Here $Q(t)$ is the expansion factor and k the curvature parameter taking one of three possible values -1 , 0 or $+1$. It was Friedmann [11] who first obtained solutions of Einstein's equations for the above line element.

The ten Einstein equations reduce in this case to just two:

$$3 \frac{\dot{Q}^2 + k}{Q^2} = 8\pi T_0^0, \quad (1.18)$$

$$2 \frac{\ddot{Q}}{Q} + \frac{\dot{Q}^2 + k}{Q^2} = 8\pi T_1^1. \quad (1.19)$$

In 'standard models' discussed by Friedmann, the energy tensor T_k^i was taken as for dust. Later work by Gamow and his colleagues [12, 13, 14] showed that the "early" universe was radiation dominated so that the form of T_k^i in early epochs should be as for radiation. For details of such models which we shall refer to as Friedmann–Robertson–Walker models (FRW models in brief) and for solutions of (1.18) and (1.19) see standard texts on cosmology [15, 16]. In addition we will also consider solutions of these equations for T_k^i given by a massless scalar field.

The maximally symmetric FRW models assume the universe to be both homogeneous and isotropic. If we drop the latter requirement but retain the former we get a wider class of models in which the groups of motion are three parameter Lie groups acting transitively on spacelike hypersurfaces [17, 18]. Using the language of differential forms we may write the line element of such space-times in the manifestly homogeneous form

$$ds^2 = dt^2 + g_{\alpha\beta} \omega^\alpha \omega^\beta \quad (1.20)$$

where $g_{\alpha\beta}$ depend on t only and ω^α are a set of three one-forms. (For details of (1.20) see ref. [19].) The various possible groups have been classified and lead to different types of homogeneous models first classified by (and hence named after) Bianchi [17].

For our later work we will use the following representation of $g_{\alpha\beta}$:

$$g_{\alpha\beta} = -e^{2\lambda(t)} [e^{-2\beta(t)}]_{\alpha\beta} \quad (1.21)$$

where

$$\beta = e^{-\psi k_3} e^{-\theta k_1} e^{-\theta k_3} \beta_d e^{\theta k_3} e^{\theta k_1} e^{\psi k_3} \quad (1.22)$$

$$\beta_d = \text{diag.} (\beta_1, -\frac{1}{2}\beta_1 + \frac{1}{2}\sqrt{3}\beta_2, -\frac{1}{2}\beta_1 - \frac{1}{2}\sqrt{3}\beta_2),$$

$$k_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad k_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}. \quad (1.23)$$

θ , ψ and β_1 , β_2 , β_3 are functions of t , characterizing the model.

For detailed properties of these models see ref. [19]. The dynamics of such models is completely determined by the geometric action

$$S_g = \frac{1}{16\pi} \int L dt \quad (1.24)$$

where

$$L = -e^{3\lambda} [6\dot{\lambda}^2 - \frac{3}{2}(\dot{\beta}_1^2 + \dot{\beta}_2^2)] + e^{3\lambda} R^* . \quad (1.24)$$

The form of R^* depends on the structure properties of the Lie group. For simpler types (called type A)

$$R^* = \frac{1}{2}e^{-2\lambda} f(\beta_1, \beta_2) \quad (1.25)$$

while for more general types (type B)

$$R^* = c_1 e^{-2\lambda - c_2\beta_1 - c_3\beta_2} . \quad (1.26)$$

The function $f(\beta_1, \beta_2)$ or the constants c_1, c_2, c_3 depend on the Bianchi type chosen for the model and their explicit form or values do not concern us here.

It is interesting to note the forms of the minisuperspace for these symmetric models. For the FRW models the minisuperspace is one-dimensional being given by just one function $Q(t)$. For the diagonal Bianchi model given by $\psi = 0, \theta = 0, \beta_1 \neq 0, \beta_2 \neq 0, \lambda \neq 0$ the minisuperspace has 3 dimensions with the line element given by

$$dL^2 = -24 d\lambda^2 + 6(d\beta_1^2 + d\beta_2^2) . \quad (1.27)$$

This form was first given by Misner [20]. (Since Misner's convention for the superspace metric differs from that of DeWitt used here we have expressed (1.27) according to the latter's convention.)

1.4. Problems of classical cosmology

We now briefly outline three highly pertinent features of classical cosmology, namely space-time singularity, particle horizons and flatness.

The FRW models all have an instant in the past when the function $Q(t)$ was zero. Except for the trivial case $k = -1, Q \propto t$, all such models have space-time singularity at that epoch. Labelling the epoch of singularity by $t = 0$, we find that all curvature invariants diverge at $t = 0$ and world lines of particles or light rays cannot be continued to the past of $t = 0$. It has now become clear that this appearance of singularity is not a consequence of the assumption of maximal symmetry but is a general feature of all cosmological solutions of Einstein's gravitational equations (cf. ref. [21] and the references therein). In the early 1970s Belinskii, Khalatnikov and Lifshitz [22, 23] studied the behaviour of the general cosmological solution of Einstein's equations near the singular epoch $t = 0$. We will refer to this model (BKL model in brief) in section 3.6.

The other feature of FRW models is the existence of a particle horizon. Close to the epoch $t = 0$ the particle horizon is of very small extent. All physical interactions between the distant parts of the

universe at any epoch are limited by the extent of the particle horizon at that epoch. If the presently observed large scale homogeneity of the universe is due to the properties of the early universe, it is hard to see how such an adjustment to uniformity come about between parts of the universe lying outside one another's particle horizons. This problem was highlighted by the discovery of the microwave background and the observation of its large scale isotropy (cf. refs. [15, 16] for a detailed discussion). Since the standard interpretation of microwave background is that it is a relic of the early universe the above observation is difficult to explain away. Misner [24] had attempted a resolution of this difficulty in classical cosmology by proposing the mixmaster universe which has a randomly changing horizonfree direction. However, as shown by Chitre [25] this model is not able to 'churn' the universe rapidly enough to generate large-scale homogeneity. Thus the horizon problem remains unsolved in classical cosmology.

A third problem with the standard FRW models has been pointed out recently, first by Dicke and Peebles [26] and later by Guth [27]. Depending on the curvature (negative or positive) of the spatial sections $t = \text{constant}$, the FRW models are characterised as open ($k = -1$) or closed ($k = +1$) models. Dynamically the mean density ρ of matter in the universe determines whether the model is open or closed. The borderline case $k = 0$ (which has flat spatial sections) has the mean density given by

$$\rho_c = \frac{3H^2}{8\pi G}. \quad (1.28)$$

This is called the closure density. For open models we have $\rho/\rho_c < 1$ while for closed models $\rho/\rho_c > 1$. For any open or closed model the ratio ρ/ρ_c changes with epoch. If the ratio was determined very early in the universe (at say, $t \sim 10^{-33}$ s according to some grand unified theories) then it must have been extremely finely tuned near unity in order that it lies in the (currently estimated) range of 0.05 to 5 at the present epoch. Various 'inflationary' scenarios have been proposed [28] to explain this 'coincidence' within the classical cosmology.

Can quantum cosmology address itself to these problems?

1.5. Approaches to quantum cosmology

There is so far no unique answer to the question "How should gravity be quantized?" Although there are several different approaches they all recognize that the basic problems to be tackled by a quantum theory of gravity are the following.

(i) *Duality*. The main feature which sets Einstein's theory of gravity apart from other field theories is the dual role ascribed to geometry. The metric tensor g_{ik} or the Ricci tensor R_{ik} not only describes the properties of space-time in which the various physical interactions are going on, they also describe the phenomenon of gravity. At the classical level this dual role makes the understanding of gravitational radiation vastly more difficult than that of, say, the electromagnetic radiation: there is no obvious and simple way to disentangle the wave propagation phenomenon from purely geometrical properties of the background space-time. Likewise there is no clear-cut way of isolating a quantum gravity effect from changes in space-time geometry in which that effect takes place. By contrast, in ordinary quantum field theory the background space-time is left undisturbed.

(ii) *Nonlinearity*. Einstein's equations are nonlinear and have the unique property that the field equations contain the equations of motion of the sources. Further there is 'back-reaction' in which any change in the energy momentum tensor of the physical systems acts back to change the space-time

geometry. Thus a linearized theory of gravity such as the one discussed by Feynman [29] cannot give the full picture of the real situation.

What constitutes quantum gravity? Quantization of the right-hand side of Einstein's equations in a given space-time has yielded interesting effects such as the Hawking radiation [30]. However even here the role of back reaction has not been fully understood. In any case it is incorrect to regard such investigations as those of quantum gravity. In our opinion quantum gravity must tackle the two problems mentioned above.

Amongst the many approaches to quantum gravity proper we may mention the manifestly covariant approach of B.S. DeWitt [7, 31, 32], the use of path integral in Euclidean space advocated by Gibbons and Hawking [33, 34], the canonical quantization method of Arnowitt, Deser and Misner [35, 36, 37] and the twistor formalism of Penrose [38, 39]. Although these methods have emphasized the formal problems of quantization and led to many interesting abstract concepts they cannot claim to have delivered a complete and workable theory of quantum gravity. In any case they have not shed any light on the three problems of classical cosmology outlined in section 1.4.

It is against this background that our path integral approach is to be viewed. It is less ambitious and less sophisticated (and probably less rigorous!) than the approaches referred to above. Yet it has the merit of giving direct answers to simply posed questions.

2. The path integral formalism

Before we apply the path integral method to the problem of quantum gravity we review it briefly, stating the results which will be useful to us in sections 3–5. Our presentation here will be application oriented rather than formal, being modelled after Feynman's pioneering paper in 1948 [40]. To this extent we will not pause to give a rigorous justification of all the mathematical steps involved in arriving at the final conclusion. The purist will have to be content with the intuitive elegance of the approach and with the success achieved by it.

In this section we will apply the path integral method to the familiar problems of quantum mechanics and quantum field theory. In sections 3–5 we will find that many of these problems reappear in a different garb in quantum cosmology and those can be handled straight away on the basis of the results derived here.

2.1. The propagator as the sum over paths

A simple example will illustrate the difference between classical mechanics and quantum mechanics. Consider a particle of mass m moving freely. Let the position vector $\mathbf{r} = (x, y, z)$ denote the location of the particle at time t . In Newtonian mechanics, the equation of motion of the particle is given by the second law of motion:

$$m\ddot{\mathbf{r}} = 0. \quad (2.1)$$

Given that the particle was at \mathbf{r}_i at an initial instant t_i and at \mathbf{r}_f at the final instant t_f , the above equation gives us its trajectory at all times t in $t_i < t < t_f$ by the formula

$$\mathbf{r}(t) = \mathbf{r}_i + \left(\frac{t - t_i}{t_f - t_i} \right) (\mathbf{r}_f - \mathbf{r}_i). \quad (2.2)$$

Quantum mechanics tells us that such a deterministic solution to the problem is not possible. At best one can talk of a *probability amplitude* for the particle to be found at \mathbf{r}_f at $t = t_f$ given that it was at \mathbf{r}_i at $t = t_i$. This probability amplitude is given by

$$K(\mathbf{r}_f, t_f; \mathbf{r}_i, t_i) = \left[\frac{m}{2\pi i \hbar (t_f - t_i)} \right]^{3/2} \exp \frac{im(\mathbf{r}_f - \mathbf{r}_i)^2}{2\hbar(t_f - t_i)}. \tag{2.3}$$

This two-point function, known as the particle propagator, is obtained as the Green's function of the inhomogeneous Schrödinger equation

$$-\frac{\hbar^2}{2m} \nabla_f^2 K - i\hbar \frac{\partial K}{\partial t_f} = i\hbar \delta_3(\mathbf{r}_f - \mathbf{r}_i) \delta(t_f - t_i) \tag{2.4}$$

where $\delta(x)$ is the Dirac delta function.

Now at first sight the connection between (2.3) and (2.4) and their classical counterparts (2.2) and (2.1) is not obvious and intuitively it is difficult to see the latter as the limiting forms of the former when $\hbar \rightarrow 0$. This connection appears naturally if we resort to the action functional and the formalism of path integrals.

We first note that Newton's law of motion (2.1) follows naturally from the principle of stationary action

$$\delta S = 0 \tag{2.5}$$

for a suitably defined action S . For the free particle we have

$$S = \int_{t_i}^{t_f} \frac{1}{2} m \dot{\mathbf{r}}^2 dt. \tag{2.6}$$

The prescription (2.5) applied to (2.6) singles out the 'classical path' denoted by (2.2). It was Feynman [40] who first gave the analogous quantum prescription that leads to (2.3). In this prescription (see fig. 1)

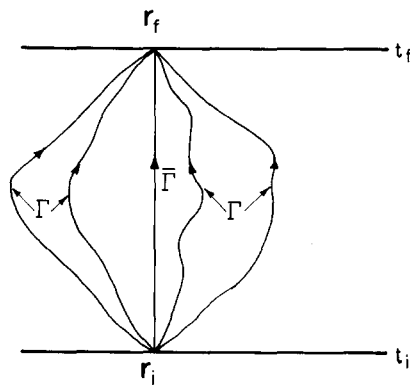


Fig. 1. The classical path for the free particle is $\bar{\Gamma}$. In Feynman's approach all paths Γ like those shown above contribute to the probability amplitude for the particle to go from (\mathbf{r}_i, t_i) to (\mathbf{r}_f, t_f) .

the particle does not necessarily move from (\mathbf{r}_i, t_i) to (\mathbf{r}_f, t_f) along $\bar{\Gamma}$, but may choose any arbitrary path Γ for which t increases monotonically from t_i to t_f . Feynman assigned a probability amplitude

$$P(\Gamma) = \exp[i S(\Gamma)/\hbar] \quad (2.7)$$

for this path Γ to be chosen by the particle. The particle propagator is then simply the sum (over all paths Γ) of the path functionals $P(\Gamma)$. In the continuum distribution of paths this sum becomes a path integral, so that in our example,

$$K[\mathbf{r}_f, t_f; \mathbf{r}_i, t_i] = \int \exp[i S(\Gamma)/\hbar] \mathcal{D}\Gamma. \quad (2.8)$$

This path integral can be evaluated from first principles (see refs. [40, 41]) and the answer shown to be equal to (2.3). In section 2.3 we will show how the answer can be obtained without directly having to evaluate a path integral.

To see the connection with classical mechanics, let $\hbar \rightarrow 0$ in (2.8). The exponent tends to $(\pm i\infty)$. However, the integrand is a point on the unit circle $|z| = 1$ in the complex plane and except in a special situation the integral, being the sum of randomly distributed points on the unit circle, vanishes. The special situation arises for a group of neighbouring paths for which S does not change. All such paths contribute coherently to the net probability amplitude with the result that in the limit the particle moves only along such paths. These are of course the paths near $\bar{\Gamma}$.

The above argument is easily extended to more general systems in mechanics which are describable by the Lagrangian method. With the usual transition from denumerable to nondenumerable degrees of freedom we can formulate a path integral theory for fields [41]. We also see how the fact that $(\hbar \neq 0)$ results in a non-uniqueness of paths and hence in quantum uncertainty.

2.2. The relationship to the Schrödinger equation

The propagator K defined in (2.8) has the transitive property expressed by the relation

$$K[\mathbf{r}_f, t_f; \mathbf{r}_i, t_i] = \int K[\mathbf{r}_f, t_f; \mathbf{r}, t] K[\mathbf{r}, t; \mathbf{r}_i, t_i] d^3\mathbf{r}, \quad (2.9)$$

for any t satisfying $t_i \leq t \leq t_f$. The proof follows from the definition in a straightforward manner, and its generalization to more general Lagrangian systems is equally straightforward [41].

If the state of the particle at time t_i is describable by a wave-function $\psi_i(\mathbf{r}_i, t_i)$, then its final state is given by the wavefunction

$$\psi_f(\mathbf{r}_f, t_f) = \int K[\mathbf{r}_f, t_f; \mathbf{r}_i, t_i] \psi_i(\mathbf{r}_i, t_i) d^3\mathbf{r}_i. \quad (2.10)$$

This relation, which follows from (2.8) and (2.9) tells us how the wave-functions propagate through time. In this context K is often referred to as the kernel.

A comparison of (2.10) and (2.4) suggests that the two are equivalent. In fact by applying (2.10) over infinitesimal time interval $t_f - t_i$, it is possible to derive the Schrödinger equation for a free particle in the

familiar form

$$-\frac{\hbar^2}{2m} \nabla^2 \psi = i\hbar \frac{\partial \psi}{\partial t} \tag{2.11}$$

Again, for a Lagrangian system with a Hamiltonian H , (2.11) can be generalized to

$$H\psi = i\hbar \partial\psi/\partial t, \tag{2.12}$$

where H is expressed in operator form.

Two points should be made here in view of our later applications of the path integral techniques to quantum gravity.

The first point concerns the fact that for a real action S , the path integrand in (2.8) is oscillatory and so the integral may not converge and hence will be unreliable as a physical concept. However, in spite of this formal shortcoming the path integral approach has yielded meaningful results in quantum mechanics. We will therefore attach physical meanings to *similar* path integrals which we shall encounter in quantum cosmology, without worrying about the above formal objection.

The second point is of a more specific nature. In the operator formulation of quantum mechanics the eigenvalue of the operator ($i\hbar \partial/\partial t$) is identified with energy of the system. For mechanical systems and for usual fields, the energy is expected to be positive. In quantum cosmology we will encounter situations where the operator $i\hbar \partial/\partial t$ has negative eigenvalues. This circumstance need not cause us any concern since there the path integrands relate to geometrical parameters of space-time manifold rather than to mechanical systems. We will return to this point in section 4.

2.3. Quadratic integrands in the action

We briefly quote a result which will be useful in our later work. For proof of the result see ref. [41]. Suppose the action S is expressible in the following form

$$S = \int_{t_i}^{t_f} F[q_r, \dot{q}_r, t] dt \tag{2.13}$$

where F is given by

$$F = \sum_r \sum_s \{a_{rs} \dot{q}_r \dot{q}_s + b_{rs} (q_r \dot{q}_s + q_s \dot{q}_r) + c_{rs} q_r q_s\} + \sum_r (g_r q_r + f_s \dot{q}_s) \tag{2.14}$$

where q_1, q_2, \dots are the n Lagrangian coordinates and a, b, c, g, f , are known functions of time.

The classical principle of least action $\delta S = 0$, determines in the above case a classical path $\bar{\Gamma}$ along which

$$q_r(t) = \bar{q}_r(t), \quad \bar{q}_r(t_i) = q_{ri}, \quad \bar{q}_r(t_f) = q_{rf} \tag{2.15}$$

where q_{ri} and q_{rf} are the specified initial and final values of q_r . Let \bar{S} denote the value of S computed for \bar{q}_r , i.e.,

$$\bar{S} = \int_{t_i}^{t_f} F[\bar{q}_r, \dot{\bar{q}}_r, t] dt. \quad (2.16)$$

Then it can be shown that the quantum mechanical propagator for the system is given by

$$K[q_{rf}, t_f; q_{ri}, t_i] = \phi(t_f, t_i) \exp(i\bar{S}/\hbar). \quad (2.17)$$

The function ϕ depends only on t_i and t_f and can usually be determined by using the transitivity property of K . A formal way of determining ϕ is through the van-Vleck determinant Δ , defined in this case by

$$\Delta = \det \left\| \frac{\partial^2 \bar{S}}{\partial q_{ri} \partial q_{sf}} \right\|. \quad (2.18)$$

We then have [42]

$$\phi = (2\pi)^{-n/2} \Delta^{1/2}. \quad (2.19)$$

The importance of this result is that it enables us to compute path integrals explicitly. In our later work we will encounter a number of situations where the path integrals can be transformed so that they can be evaluated by the above method.

Let us illustrate this concept by a simple example. Consider a Lagrangian in the form,

$$\mathcal{L}_1 = \frac{1}{2} f^2(q) \dot{q}^2. \quad (2.20)$$

This does not fall under the class of Lagrangians we have come across. However notice that the coordinate q that appears in the Lagrangian is a generalized coordinate. Instead of q , we can choose the coordinate Q , given by,

$$Q = \int dQ = \int f(q) dq \quad (2.21)$$

in terms of which the Lagrangian reads as,

$$\mathcal{L}_2 = \frac{1}{2} \dot{Q}^2. \quad (2.22)$$

The path integral can now be evaluated trivially. Both \mathcal{L}_1 and \mathcal{L}_2 represent the same physical system (a free particle) in Cartesian and curvilinear coordinates respectively. If the quantization was attempted with \mathcal{L}_1 , we would have encountered factor ordering problems with the Hamiltonian. For example, the Hamiltonian

$$H_1 = \frac{p_q^2}{2f^2(q)} \quad (2.23)$$

can become either,

$$\hat{H}_1 = -\frac{\hbar^2}{2f^2(q)} \frac{\partial^2}{\partial q^2} \quad (2.24)$$

or

$$\hat{H}'_1 = -\frac{\hbar^2}{2} \frac{\partial^2}{\partial q^2} \frac{1}{f^2(q)} \quad (2.25)$$

depending on the ordering of factors in (2.23). No such ambiguities arise in (2.22). Because of this we shall always assume that – wherever possible – the Lagrangian must be brought to quadratic form before quantization is performed.

2.4. Diffusion of a wavepacket

The notion of a wavepacket has played a useful role in understanding how the laws of quantum mechanics operate in practice. Classically, we can always specify the position and momentum of a particle exactly. Quantum mechanics on the other hand limits the accuracy with which these two quantities can be specified, by imposing the uncertainty principle. In the wavepacket we have something of a compromise between the two notions. The wavepacket is given by a wavefunction most of whose support is concentrated in a small region of space, which could be identified as the region of location of the particle. Thus the average values of the position vector $\langle \mathbf{r} \rangle$ and the momentum vector $\langle \mathbf{p} \rangle$ for a wavepacket correspond to the position and momentum of a classical particle. There are, however, dispersions about $\langle \mathbf{r} \rangle$ and $\langle \mathbf{p} \rangle$, denoted by $\Delta \mathbf{r}$ and $\Delta \mathbf{p}$ which remind us that we are dealing with a quantum system, because $\Delta x \Delta p_x \geq \hbar$, $\Delta y \Delta p_y \geq \hbar$ and $\Delta z \Delta p_z \geq \hbar$.

The Gaussian wavepacket has the property that the uncertainty product is minimum. For example, for the one-dimensional Gaussian wavepacket

$$\psi(x) = [2\pi(\Delta x)^2]^{-1/4} \exp\left[-\frac{(x - \langle x \rangle)^2}{4(\Delta x)^2} + \frac{i\langle p \rangle x}{\hbar}\right] \quad (2.26)$$

the uncertainty product $\Delta x \Delta p$ is the least and equal to $(\hbar/2)$ (see ref. [43]).

Suppose a free particle moving in one dimension initially has the above wavefunction. How will this wavefunction change with time? This question can be answered by using (2.10) in one dimension. The wavefunction still retains its wavepacket form at $t = t_i$, but its positional uncertainty is found to have increased to

$$(\Delta x)_t = \left[(\Delta x)^2 + \frac{\hbar^2(t_f - t_i)^2}{4m^2(\Delta x)^2} \right]^{1/2}. \quad (2.27)$$

This spread in the wavepacket is often referred to as ‘diffusion’ and it shows that as time proceeds the uncertainty in locating the particle in the neighbourhood of its average position increases. The average position is still that given by classical mechanics:

$$\langle x \rangle_t = \langle x \rangle_i + \frac{\langle p \rangle}{m} (t_t - t_i). \quad (2.28)$$

However, the mean value of a distribution has a significance only as long as the spread is finite. In this example, the spread in (2.27) is finite at all finite times, and hence no difficulty arises. In our later work, we shall come across Lagrangians of the type,

$$\mathcal{L} = \frac{1}{2} m f(t) \dot{q}^2. \quad (2.29)$$

Since this is quadratic, we can evaluate the path integral exactly and obtain a closed form for the kernel. It can be easily shown that this kernel leads to the spread in the form,

$$(\Delta x)_t = \left[(\Delta x)^2 + \frac{\hbar^2}{4m^2 (\Delta x)^2} \left(\int_{t_i}^{t_t} f(t) dt \right)^2 \right]^{1/2}. \quad (2.30)$$

It is possible that the integral in (2.24) diverges at some finite time t_t , making $(\Delta x)_t$ infinite. Under such a circumstance the mean value has no operational significance. In quantum cosmology, classical relativistic models will serve as example of the ‘average’ behaviour while the quantum effect will be seen through the behaviour of diffusion of the wavepacket. Whenever this spread diverges, the ‘average classical value’ loses its significance.

2.5. Quantum stationary states

We discussed briefly (in section 2.2) the connection between the Schrödinger equation,

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi \quad (2.31)$$

and the path integral kernel,

$$K(q_t t_t; q_i t_i) = \int \mathcal{D}q(t) \exp \frac{i}{\hbar} S[q(t)]. \quad (2.32)$$

While the approaches are formally equivalent, the nature of the problem often dictates the choice for use. We will now discuss this interrelationship in somewhat more detail.

Consider the eq. (2.31) for an arbitrary potential $V(q, t)$. The solution $\psi(q, t)$ contains all the information about the system. But in order to solve equation (2.31) uniquely we require the initial condition $\psi(q_i, t_i)$. Thus the wavefunction $\psi(q, t)$ contains all the information about the dynamics (dictated by the potential $V(q, t)$) as well as the initial condition. On the other hand, the kernel in (2.32) can be evaluated, once the potential $V(q, t)$ is given. Thus the kernel contains the information about the dynamics alone.

More often, we will be interested in studying the structure and dynamics of the theory without committing ourselves to any specific initial condition. In situations in quantum cosmology, the initial

condition involves the quantum state of the universe which is largely unknown. The kernel provides the right tool in those situations.

However, the number of cases for which the kernel can be exactly evaluated is very small. It is comparatively easy to attack the Schrödinger equation. This motivates one to see whether the dynamics of the situation can be separated from the initial conditions, in the Schrödinger equation approach. Such a possibility exists – via the concept of quantum stationary states – whenever the potential $V(q, t)$ is independent of time.

When $V(q, t) = V(q)$, the Schrödinger equation admits solutions of the form,

$$\psi(q, t) = e^{-iEt/\hbar} \phi_E(q) \quad (2.33)$$

where $\phi_E(q)$ satisfies the equation,

$$-\frac{\hbar^2}{2m} \nabla^2 \phi_E + V(q) \phi_E = E \phi_E. \quad (2.34)$$

The ‘stationary states’ $\phi_E(q)$ contain all the dynamical information about the system and thus one should be able to construct the kernel in terms of $\phi_E(q)$. This can indeed be done, through the formula,

$$K(q_t t_t; q_i t_i) = \sum_E \phi_E(q_t) \phi_E^*(q_i) \exp\{-iE(t_t - t_i)/\hbar\} \quad (2.35)$$

which can be proved in a straightforward manner, the sum being over the complete set of states labelled by the parameter E . Thus, whenever the path integral cannot be evaluated the stationary states provide an equivalent alternative method.

The stationary states are physically interesting from another point of view as well. When the potential is independent of time, we know that the dynamics – and the physical observables – must be independent of time. Any time dependence in an arbitrary state, $|\psi(q, t)|^2$, must be a complication that arises from the initial conditions. This way, the stationary states $\phi_E(q)$ are the “natural” states for the system, and portray the true dynamics.

Nothing that was said above depends on the physical meaning attributed to the variable q . In our discussions in quantum cosmology q will describe a degree of freedom of space-time geometry. The concept of quantum stationary states exists for the quantum geometry as well. However, there is one difference (which was commented upon earlier): the variable E has the physical meaning of energy in the case of ordinary quantum mechanics. Because of this one expects E to be positive for a free system etc. When q is interpreted as an arbitrary dynamical degree of freedom E does not have any simple meaning. Thus one cannot attach any special significance to its sign.

3. Quantum conformal fluctuations

3.1. Quantum geometrodynamics

We now outline the applications of the path integral formalism developed in section 2, to the problems of quantum cosmology. We begin with recapitulation of geometrodynamics described in

section 1.2. That classical description, as originally given by Wheeler talked of time development of the 3-geometry ${}^{(3)}\mathcal{G}$ on a sequence of spacelike hypersurfaces $\{\Sigma\}$. Suppose we have two hypersurfaces Σ_i and Σ_f which we call the initial and final hypersurfaces respectively, on which the 3-geometries are specified to be ${}^{(3)}\mathcal{G}_i$ and ${}^{(3)}\mathcal{G}_f$. Can we determine the 4-geometry of the spacetime region \mathcal{V} sandwiched between Σ_i and Σ_f ?

In a naive way we may suppose the answer to this question to be in the affirmative, given the classical variational principle for gravity

$$\delta S = 0. \quad (3.1)$$

The Einstein equations which follow from (3.1) determine the dynamical evolution of 3-geometries between Σ_i and Σ_f in much the same way that Hamilton's principle together with the end-point conditions determines the trajectory of a Lagrangian mechanical system.

If this simple analogy holds we can extend it to the quantum domain by introducing the propagator of 3-geometries

$$K[{}^{(3)}\mathcal{G}_f, \Sigma_f; {}^{(3)}\mathcal{G}_i, \Sigma_i] = \sum_{\Gamma} \exp(iS/\hbar) \quad (3.2)$$

where the right-hand side denotes a formal sum over all 'paths' which represent a succession of 3-geometries $\{{}^{(3)}\mathcal{G}\}$ starting with ${}^{(3)}\mathcal{G}_i$ on Σ_i and ending with ${}^{(3)}\mathcal{G}_f$ on Σ_f . Of these paths the (supposedly) unique solution of Einstein's equations with the above boundary conditions represents the classical path $\bar{\Gamma}$. The quantum geometrodynamics is therefore contained in the propagator K .

Apart from the practical difficulties of evaluating the path integral implied in the sum (3.2) there are certain conceptual problems which are serious enough to be discussed right at the outset.

The first problem arises from the oscillating nature of the typical exponential term $\exp(iS/\hbar)$. What meaning can we give to a sum over such terms or to an integral which manifestly does not converge? To circumvent this difficulty Gibbons and Hawking [33] have chosen the so-called Euclidean space-time manifold. (This was referred to earlier in section 1.5.) Formally this procedure has the advantage of changing $\exp(iS/\hbar)$ to a damped exponential $\exp(-S/\hbar)$, and the resulting path integral may be well defined. In the Gibbons-Hawking programme the actual computations are done in this Euclidean (and therefore unrealistic) space-time, with the hope that the results can be continued from the imaginary τ back to the real time coordinate t . While this approach has led to interesting concepts like the space-time foam and gravitational instantons, it has yet to yield a concrete down-to-earth solution of any problem in quantum gravity.

In our approach we will keep to the real space-time with all its problems of summing over nonconvergent terms. This difficulty is also present in Feynman's original approach. It was mentioned in section 2 and tolerated on the grounds that in spite of it, meaningful results can be obtained in quantum mechanics. We give the same justification here, for our path integrals in quantum cosmology will turn out to be remarkably similar to those of quantum mechanics and quantum field theory.

The second difficulty is posed by the lack of proper specification of boundary conditions on Σ_i and Σ_f and the lack of uniqueness of the classical geometrodynamics expressed with the 3-geometries. The classical difficulty was mentioned in section 1.2 where it was pointed out that the initial and final data are better specified through the prescription of Isenberg and Wheeler [6, 44].

In the Isenberg-Wheeler prescription the data to be specified on Σ_i or Σ_f are the conformal part of the 3-geometries and the trace of the extrinsic curvature K^i_j . Such a prescription is unambiguous and

leads to a unique solution of the problem. This notion of conformally related 3-geometries can be seen in the wider context of conformally related 4-geometries which we will highlight in this article.

Finally, it was noted in section 1.2 that the classical Hilbert action for gravity contains second derivatives of the metric tensor. This fact poses a problem for the path integral approach. Normally, in a Lagrangian containing dynamical variables with only the first time derivatives, the paths can be identified by their end-points. In the variational principle, for example, the end-points of a path are kept fixed. If the Lagrangian also has second time derivatives then not only the end-points but the end-slopes of the paths also must be kept fixed. In computing the path integral therefore, simply specifying the end-points would not give the propagator K ; we would also need to specify the end-slopes. This requirement makes the problem of computing path integrals much more complex.

These difficulties are eliminated by the addition of the surface term (1.7) of Hawking and Gibbons [33]. Since the variations $\delta g_{ik,l}$ from the surface term cancel the variations arising from the Hilbert term, the problem effectively involves fixing the surface values of g_{ik} only. In what follows we will tacitly assume that such a surface cancellation does take place.

3.2. Conformal fluctuations

Even though the above conceptual problems are sorted out, the computation of the propagator K in (3.2) for the most general situation, remains an impossibly difficult task. By setting our sights at a less ambitious goal we can, however, make some progress. The remainder of this section will be devoted to a report of this technique and the results produced by it.

Suppose that for a given action (1.5) the space-time metric

$$d\bar{s}^2 = \bar{g}_{ik} dx^i dx^k \tag{3.3}$$

represents the classical solution for a sandwich type region described in section 3.1. The time development of \bar{g}_{ik} between Σ_i and Σ_f denotes the classical path $\bar{\Gamma}$ (see fig. 2).

Consider now a conformal transform of (3.3):

$$ds^2 = \Omega^2 d\bar{s}^2 = \Omega^2 \bar{g}_{ik} dx^i dx^k \tag{3.4}$$

where Ω is an arbitrary function of space and time. Since Einstein's equations are not conformally invariant; except for the trivial case of $\Omega = \text{constant}$, (3.4) describes a nonclassical path Γ between Σ_i and Σ_f . Of course nonclassical paths can be generated in other ways also, but the above method is the only one that preserves the light cone structure of the original classical solution. A change of this structure would lead to a different causal relationship and thereby introduce new physical aspects into the problem. Thus by restricting ourselves to (3.4) we are ensuring that the causal structure of space-time has not changed.

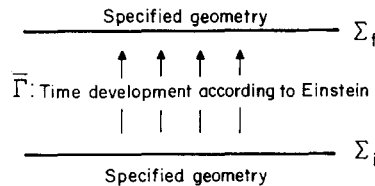


Fig. 2. In the sandwich problem $\bar{\Gamma}$ represents the classical evolution of geometry from a specified value on Σ_i to a specified value on Σ_f .

We may look upon the above restriction as one in which of all possible degrees of freedom available, only the conformal degree of freedom is quantized. This degree of freedom emerges naturally from the work of Isenberg and Wheeler discussed earlier. We may recall that the data required to be specified on Σ_i or Σ_f includes the conformal part of ${}^{(3)}\mathcal{G}$, i.e., the specification of 3-geometry within an arbitrary conformal factor.

Since $\Omega = \text{constant}$ can always be reduced to $\Omega = 1$ by a suitable choice of units we will assume henceforth that $\Omega = 1$ represents the classical path $\bar{\Gamma}$. Any nonconstant Ω between Σ_i and Σ_f denotes a nonclassical path and we will call

$$\phi \equiv \Omega - 1 \quad (3.5)$$

the conformal fluctuation around the classical path. When ϕ is treated as a quantum variable, we will refer to it as the *quantum conformal fluctuation* (QCF).

Under the transformation

$$g_{ik} = (1 + \phi)^2 \bar{g}_{ik} \quad (3.6)$$

the scalar curvature transforms as

$$R = \frac{\bar{R}}{(1 + \phi)^2} + \frac{6\bar{\square}\phi}{(1 + \phi)^3} \quad (3.7)$$

where \bar{R} and the wave operator $\bar{\square}$ are computed for the metric tensor \bar{g}_{ik} . We will now express the action S given by (1.5) in terms of \bar{g}_{ik} and ϕ . Since we will be dealing with formal theoretical concepts only we put $c = 1$, $\hbar = 1$, $G = 1$. A simple calculation gives

$$S = \frac{1}{16\pi} \int_{\mathcal{V}} [(1 + \phi)^2 \bar{R} - 6\phi_i \phi^i] \sqrt{-\bar{g}} d^4x + S_m. \quad (3.8)$$

In arriving at (3.8), we have used Green's theorem to transform the $\bar{\square}\phi$ term to the $\phi_i \phi^i$ term. The surface integral over $\partial\mathcal{V}$ which arises from this operation is cancelled by the change in the surface term (1.7) (introduced by Gibbons and Hawking) under the conformal transformation.

What can we say about the matter term S_m ? If T_m^{ik} denotes the energy tensor for matter, the variational principle gives as in (1.6)

$$\delta S_m = -\frac{1}{2} \int T_m^{ik} \delta g_{ik} \sqrt{-g} \phi d^4x \quad (3.9)$$

for any small variation δg_{ik} in g_{ik} . For small ϕ , we get $\delta g_{ik} = 2\phi \bar{g}_{ik}$ and

$$\delta S_m \cong - \int \bar{T}_m^{ik} \bar{g}_{ik} \sqrt{-\bar{g}} \phi d^4x. \quad (3.10)$$

However, since the barred quantities satisfy Einstein's equations, we have

$$\bar{R} = 8\pi \bar{T}_m^{ik} \bar{g}_{ik}, \quad (3.11)$$

so that the linear term in ϕ disappears from (3.8).

More can be said about finite transformations if the specific forms of S_m are known. For the three forms discussed in section 1.3 we have the following results.

(a) *System of massive particles*: Suppose we have a collection of particles a, b, . . . with restmasses m_a, m_b, \dots , so that S_m is given by

$$S_m = \sum_a \int m_a ds_a. \quad (3.12)$$

In such a case

$$ds_a = (1 + \phi) d\bar{s}_a \quad (3.13)$$

and hence (3.10) holds exactly.

(b) *Radiation*: Since photons are of zero restmass we do not have (3.13) holding for them. In fact electromagnetic theory being conformally invariant, we have in this case

$$S_m = \bar{S}_m. \quad (3.14)$$

(c) *Scalar field*: The Lagrangian for the massless scalar field ψ has the form

$$\mathcal{L} = \frac{1}{2} \partial_i \psi \partial_k \psi g^{ik} \sqrt{-g}. \quad (3.15)$$

Under the conformal transformation g^{ik} goes to $\Omega^{-2} \bar{g}^{ik}$ and $\sqrt{-g}$ to $\Omega^4 \sqrt{-\bar{g}}$ giving, for the matter action

$$S_m = \frac{1}{2} \int (1 + \phi)^2 \psi_i \psi^i \sqrt{-\bar{g}} d^4x. \quad (3.16)$$

In all three cases above the total action S is quadratic in ϕ_i and ϕ . It is this circumstance that makes the explicit evaluation of the path integral possible. Before proceeding with the general case we illustrate the technique with a simple example [45, 46].

3.3. Friedmann cosmology

Consider the homogeneous isotropic cosmologies discussed in section 1.3 at the classical level. Let the classical line element be written as

$$d\bar{s}^2 = dt^2 - \bar{Q}^2(t) \left[\frac{dr^2}{1-kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right]. \quad (3.17)$$

Conformal fluctuations around this classical solution will be given by (3.6). Note, however, that if we restrict ourselves to the class of fluctuations which preserve the homogeneity and isotropy of (3.17) then

ϕ can only depend on t . Defining a new time coordinate τ by

$$d\tau = (1 + \phi) dt, \quad (3.18)$$

we can express the new line element in a manifestly Robertson–Walker form.

We will take for S_m the form (3.12) so that S as given by (3.8) becomes

$$S = \bar{S} + \frac{1}{16\pi} \int_V (\bar{R}\phi^2 - 6\phi; \phi') \sqrt{-\bar{g}} d^4x. \quad (3.19)$$

For \mathcal{V} we choose a coordinate sphere $r \leq r_b$ and a time interval (t_i, t_f) . Since we are interested in the approach towards the classical singularity we will reverse the time direction so that as shown in fig. 3, $0 < t_f < t_i$. Let V be the coordinate volume of the spatial region $r \leq r_b$. As \bar{R} and ϕ depend on t only, we get

$$S = \bar{S} - \frac{3V}{8\pi} \int_{t_i}^{t_f} (\dot{\phi}^2 - \frac{1}{6}\bar{R}\phi^2) \bar{Q}^3 dt. \quad (3.20)$$

We can now formulate our quantum cosmological problem in the following way. We specify the state of the universe by a wavefunction $\psi = \psi(\phi, t)$. If we were considering classical cosmology only, we would be certain that $\phi = 0$ at all time, so that

$$|\psi(\phi, t)|^2 = \delta(\phi). \quad (3.21)$$

Here $\delta(x)$ is the Dirac delta function. In quantum cosmology we cannot be certain that ϕ will always be 0. The ‘best’ we can do in specifying the state of the universe at the initial epoch t_i is to express ψ as a wavepacket. We accordingly write

$$\psi_i(\phi, t_i) = (2\pi\Delta_i^2)^{-1/4} \exp(-\phi^2/4\Delta_i^2). \quad (3.22)$$

The uncertainty in the average value $\phi = 0$ is therefore given by Δ_i . In the limit $\Delta_i \rightarrow 0$, $|\psi|^2$ tends to $\delta(\phi)$.

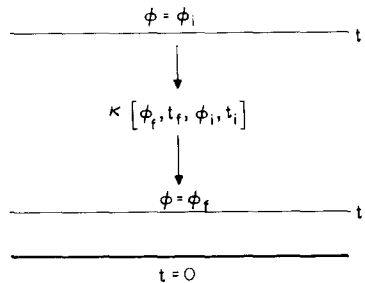


Fig. 3. The time evolution is reversed for the cosmological problem since we are interested in how K behaves as $t_f \rightarrow 0$.

The time development of ψ from ψ_i to ψ_f is given by the propagator $K[\phi_f, t_f; \phi_i, t_i]$:

$$\psi_f(\phi_f, t_f) = \int K[\phi_f, t_f; \phi_i, t_i] \psi_i(\phi_i, t_i) d\phi_i \tag{3.23}$$

where

$$\begin{aligned} K[\phi_f, t_f; \phi_i, t_i] &= \int \exp(iS[\phi]) \mathcal{D}\phi \\ &= \exp i\bar{S} \cdot \int \exp\left\{-\frac{3V}{8\pi} \int_{t_i}^{t_f} (\dot{\phi}^2 - \frac{1}{6}\bar{R}\phi^2) \bar{Q}^3 dt\right\} \mathcal{D}\phi. \end{aligned} \tag{3.24}$$

Our problem is to compute K and then study the behaviour of ψ_f as $t_f \rightarrow 0$.

To evaluate K , note that the path integral in (3.24) is of the form discussed in section 2. Using the result (2.17) we write the formal solution as

$$K[\phi_f, t_f; \phi_i, t_i] = F(t_f, t_i) \exp\left\{-\frac{3V}{8\pi} \int_{t_i}^{t_f} (\dot{\bar{\phi}}^2 - \frac{1}{6}\bar{R}\bar{\phi}^2) \bar{Q}^3 dt\right\} \tag{3.25}$$

where $\bar{\phi}(t)$ is the solution of the equation

$$\frac{d}{dt}(\bar{Q}^3 \dot{\bar{\phi}}) + \frac{1}{6}\bar{R}\bar{\phi} = 0 \tag{3.26}$$

with the boundary conditions

$$\bar{\phi}(t_i) = \phi_i, \quad \bar{\phi}(t_f) = \phi_f. \tag{3.27}$$

For small values of \bar{Q} the curvature term may be neglected. Since for $k = 0$, \bar{R} is given by

$$\bar{R} = 6(\ddot{\bar{Q}}/\bar{Q} + \dot{\bar{Q}}^2/\bar{Q}^2) \tag{3.28}$$

it is easy to write the explicit solution of our problem as

$$\bar{\phi}(t) = \left\{ \phi_i \cdot \frac{\bar{Q}(t_i)}{\bar{Q}(t)} \int_t^{t_i} \frac{du}{\bar{Q}(u)} + \phi_f \cdot \frac{\bar{Q}(t_f)}{\bar{Q}(t)} \int_{t_i}^t \frac{du}{\bar{Q}(u)} \right\} / \int_{t_i}^{t_f} \frac{du}{\bar{Q}(u)}. \tag{3.29}$$

Define the following quantities:

$$T \equiv \int_t^{t_i} \frac{du}{\bar{Q}(u)}, \quad \bar{Q}_i \equiv \bar{Q}(t_i), \quad \bar{Q}_f \equiv \bar{Q}(t_f), \quad H_i \equiv \frac{\dot{\bar{Q}}(t_i)}{\bar{Q}(t_i)}, \quad H_f \equiv \frac{\dot{\bar{Q}}(t_f)}{\bar{Q}(t_f)}. \tag{3.30}$$

We will assume (cf. following section) that T is finite as $t_f \rightarrow 0$. We can now write

$$K[\phi_f, t_f; \phi_i, t_i] = F(t_f, t_i) \cdot \exp \left[\frac{3V_i}{8\pi T} \{ (1 + T\bar{Q}_f H_f) \bar{Q}_f^2 \phi_f^2 + (1 - T\bar{Q}_i H_i) \bar{Q}_i^2 \phi_i^2 - 2\bar{Q}_i \bar{Q}_f \phi_i \phi_f \} \right] \quad (3.31)$$

with

$$F(t_f, t_i) = \left[\frac{3V_i}{8\pi^2 T} (\bar{Q}_i \bar{Q}_f) \right]^{1/2}. \quad (3.32)$$

This completes the first part of our solution. To study the behaviour of ψ_f as $t_f \rightarrow 0$ we use (3.31) and (3.22) in (3.23). The resulting integral is straightforward to evaluate but messy in detail. Apart from an imaginary phase factor ψ_f has a wavepacket form similar to (3.22) with Δ_i replaced by

$$\Delta_f = \frac{2\pi T}{3V\bar{Q}_i\bar{Q}_f} \left[1 + \frac{3V}{2\pi T} \Delta_i^2 \bar{Q}_i^2 (1 + T\bar{Q}_i H_i)^2 \right]^{1/2}. \quad (3.33)$$

Consider what happens as $t_f \rightarrow 0$. We have already seen in section 1.3 that $Q_f \rightarrow 0$ as $t_f \rightarrow 0$. Hence Δ_f diverges as $t_f \rightarrow 0$ as

$$\Delta_f \propto (\bar{Q}_f)^{-1}. \quad (3.34)$$

In the limit of the classical singularity, the quantum conformal fluctuations diverge. Therefore the classical solution which is the 'average' of our wavepacket is no longer reliable (see fig. 4).

This result was first obtained by Narlikar [45] for dust cosmologies ($p = 0$), where it was seen that $\Delta_f \propto t^{-2/3}$. From (3.34) it is clear that the result is valid even for $p \neq 0$. In the limit of radiation ($p = 1/3\varepsilon$) where (3.14) holds it can be shown separately [47] that Δ_f diverges as $(\bar{Q}_f)^{-1}$.

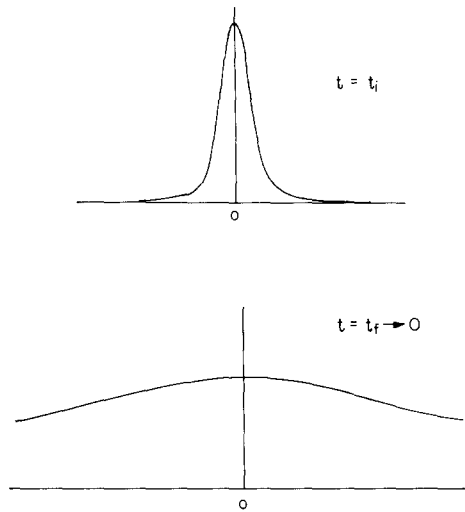


Fig. 4. The spread of the wavepacket (for ϕ) as $t_f \rightarrow 0$.

3.4. Singularity and the particle horizon

Before proceeding to more general solutions of relativistic cosmology it is instructive to deal with two specific questions which can be answered in the context of homogeneous and isotropic universes. The two questions are respectively as follows:

1. Do QCF prevent the existence of a space-time singularity?
2. Do QCF eliminate the appearance of a particle horizon?

We have seen in section 1.4 that the classical Friedmann models have a space-time singularity as well as a particular horizon. In section 3.3 we saw that the QCF diverge at the classical singular epoch which renders the classical solution unreliable. Nevertheless, if we took the full range of quantum ‘paths’ available for $\Omega = 1 + \phi$, can we associate any probability for a ‘yes’ or ‘no’ answer to either of the above two questions? Recent work [48] shows that we can.

The wavepacket solution discussed in section 3.3 suggests the answer. Let us consider the first question first. For a space-time singularity of the line element (3.17) we need $\bar{Q} \rightarrow 0$ as $t \rightarrow 0$.^{*} For a conformal transform of (3.17) we need

$$Q = (1 + \phi) \bar{Q} \rightarrow 0 \tag{3.35}$$

as $t \rightarrow 0$.

At $t = t_f$, $\phi_f = (Q_f - \bar{Q}_f)/\bar{Q}_f$ and hence from (3.34)

$$\phi_f/\Delta_f \approx (Q_f - \bar{Q}_f) \rightarrow 0 \quad \text{as } t_f \rightarrow 0, \tag{3.36}$$

since we know from the classical solution that $\bar{Q}_f \rightarrow 0$. Thus as we approach the singular epoch ($t_f \rightarrow 0$) the wavepacket spreads but the singular solutions are bunched closer and closer to the classical value $\phi_f = 0$ with the result that the probability of a singular origin becomes vanishingly small (see fig. 5).

The second question turns out to be related to the first. First we note that light cones being invariant in a conformal transformation, the conformal transform of (3.17) will continue to have a particle horizon unless, the singular hypersurface $t = 0$ of the classical solution is transformed under (3.18) to a nonsingular hypersurface at a finite value of τ , say $\tau = 0$. For, in that case the null cones can be continued into the past of the $\tau = 0$ hypersurface, and the region of communication extended (see fig. 6)

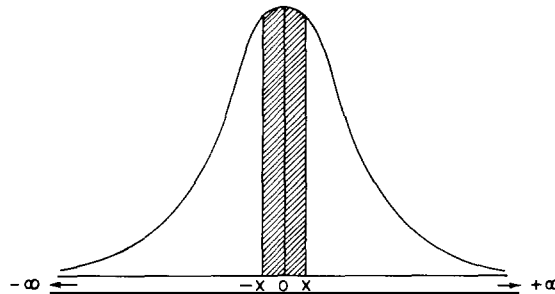


Fig. 5. The shaded region with $X = \phi_f/\Delta_f$ represents the probability of singular models. As $t_f \rightarrow 0$, $X \rightarrow 0$ and hence the probability of a singular origin tends to zero.

^{*} There is one exception, the flat space-time given by $Q \propto t$, $k = -1$.

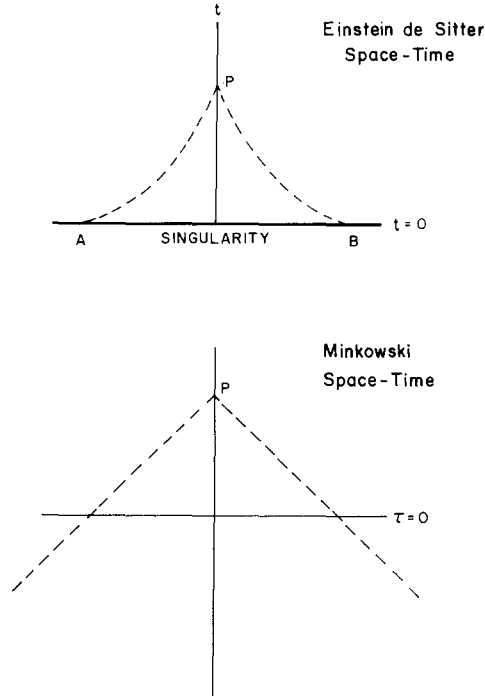


Fig. 6. By a conformal transformation, the Einstein de Sitter model with finite particle horizon is changed to the singularity-free and horizon-free Minkowski space-time as shown above.

to arbitrarily large distance. A simple example of this procedure is found in the conformal transformation of (3.17) for $k = 0$ with $\Omega = (\bar{Q})^{-1}$ which leads us to the horizon-free Minkowski space-time.

For this to happen we therefore need

$$\tau = \int_{t_0}^t (1 + \phi) dt, \quad t_0 = \text{constant}$$

to be finite as $t \rightarrow 0$, for all ϕ such that $(1 + \phi)\bar{Q}$ is finite at $t = 0$ (condition for nonsingularity at $t = 0$, cf. (3.35) above!). If $(1 + \phi)\bar{Q} \rightarrow a > 0$, where a is a constant, we require

$$\tau \sim \int_{t_0}^t \frac{dt}{\bar{Q}} \rightarrow b > 0 \quad \text{as} \quad t \rightarrow 0, \quad (3.37)$$

where b is a constant. It was shown in ref. [48] that (3.37) is satisfied for all solutions of (1.18) and (1.19) for which the pressure is non-negative ($T_1^1 \leq 0$). This is also the condition assumed in the last section, that T defined by (3.30) is finite.

We therefore arrive at the conclusion that in the regime of quantum conformal fluctuations of homogeneous and isotropic universes the probability that the universe originated with a singularity and that sufficiently early on it had a particle horizon, is vanishingly small.

3.5. Universes with arbitrary distributions of massive particles

The divergence of QCF at the classical singular epoch in the homogeneous and isotropic universe is a result which can be generalised in many directions. It was shown by Narlikar [49] that the QCF diverge at the singular epoch of the classical Bianchi Type I model. This work also describes divergence of the quantum fluctuations of the nonconformal degrees of freedom embodied in the shear of the Bianchi type I model. However, if we restrict ourselves to QCF, and assume that the gravitational dynamics of the universe is determined by arbitrary distributions of massive particles, then we can prove a much more general result. This result was obtained by Narlikar [50] and is summarized below.

For the system of massive particles (3.12) describes the classical action and under a conformal transformation (3.10) holds exactly, so that we arrive at (3.19). In section 3.3 we specialized to the case $\phi = \phi(t)$. In the present work we drop this requirement, since we are no longer restricting ourselves to homogeneous and isotropic models.

Although we do not have homogeneity and isotropy, we can still foliate the space-time with spacelike hypersurfaces which are labelled by $t = \text{constant}$. Thus Σ_i and Σ_t are labelled by $t = t_i$ and $t = t_t$ respectively. We then have

$$K[\phi_i, t_i; \phi_t, t_t] = \exp \frac{i\bar{S}}{\hbar} \cdot \sum_{\phi} \exp \frac{i}{16\pi} \int_{\gamma} (\bar{R}\phi^2 - 6\phi_i\phi^i) \sqrt{-\bar{g}} d^4x. \quad (3.38)$$

Now consider a generalization of the result stated in (2.17) which is extended from curvilinear paths $x(t)$ to functional histories $\phi(x^i)$ in the following way.

Write

$$\phi(x^i) = \bar{\phi}(x^i) + \eta(x^i) \quad (3.39)$$

where

$$\bar{\phi}(x^\mu, t_i) = \phi_i, \quad \eta(x^\mu, t_i) = 0, \quad (3.40)$$

$$\bar{\phi}(x^\mu, t_t) = \phi_t, \quad \eta(x^\mu, t_t) = 0,$$

and $\bar{\phi}(x^i)$ satisfies the wave equation

$$\square \bar{\phi} + \frac{1}{6} \bar{R} \bar{\phi} = 0. \quad (3.41)$$

Then

$$\int_{\gamma} (\bar{R}\phi^2 - 6\phi_i\phi^i) \sqrt{-\bar{g}} d^4x = 6 \int_{\Sigma_i} \bar{\phi} \bar{\phi}^k d\Sigma_k - 6 \int_{\Sigma_t} \bar{\phi} \bar{\phi}^k d\Sigma_k + \int (\bar{R}\eta^2 - 6\eta_i\eta^i) \sqrt{-\bar{g}} d^4x. \quad (3.42)$$

The first two terms on the right-hand side above are path-independent while the last term is independent of end-point values. Therefore the sum over paths in (3.38) is expressible in the form

$$K[\phi_f, t_f; \phi_i, t_i] = F(t_f, t_i) \exp \frac{3i}{8\pi} \left\{ \left(\int_{\Sigma_i} - \int_{\Sigma_f} \right) \bar{\phi} \bar{\phi}^k d\Sigma_k \right\}, \quad (3.43)$$

where F depends only on end-points.

Thus the computation of K reduces to solving the wave equation (3.41) with given boundary conditions on Σ_i and Σ_f . The solution of the wave equation can be expressed with the help of the retarded and advanced Green's functions. Thus the retarded Green's function $G^R(X, B)$ satisfies the wave equation

$$\square G^R(X, B) + \frac{1}{g} R(X) G^R(X, B) = [-\bar{g}(X)]^{-1/2} \delta_4(X, B) \quad (3.44)$$

where the wave operator is with respect to the coordinates of X . The two-point function $G^R(X, B)$ has support inside and on the future light cone of B . The advanced Green's function $G^A(X, B)$ satisfies the same wave equation (3.44) but has support inside and on the past light cone of B . We also have the symmetry condition

$$G^R(X, B) = G^A(B, X). \quad (3.45)$$

Using these Green's functions we can write down the solution for $\bar{\phi}(x^i)$ for the sandwich region enclosed by Σ_i and Σ_f . In particular we get

$$\phi_f(x_f^\mu) = \int_{\Sigma_i} \left[G^R(x_f^\mu, t_f; x_i^\mu, t_i) \dot{\bar{\phi}}(x_i^\mu, t_i) - \bar{\phi}(x_i^\mu, t_i) \frac{\partial}{\partial t_i} G^R(x_f^\mu, t_f; x_i^\mu, t_i) \right] \sqrt{-\bar{g}} d^3 x_i^\mu. \quad (3.46)$$

A similar relation gives $\phi_i(x_i^\mu)$ in terms of a surface integral over Σ_f , involving the advanced Green's function.

We may look upon these relations as linear functional integral equations relating $\phi_f(x_f^\mu)$, $\phi_i(x_i^\mu)$ to the time derivatives $\dot{\bar{\phi}}$ at Σ_i and Σ_f . These equations can be formally inverted so that $\bar{\phi}_i$ and $\bar{\phi}_f$ are expressed linearly in terms of ϕ_i and ϕ_f . For details of this operation see ref. [50]. When we substitute for $\bar{\phi}_i$ and $\bar{\phi}_f$ into (3.43) we get

$$K[\phi_f, t_f; \phi_i, t_i] = F(t_f, t_i) \exp \sum_{(P=i,f)} \sum_{(Q=i,f)} \frac{3i}{8\pi} \int A_{PO}(x_P^\mu, t_P, x_O'^\nu, t_O') \phi_P(x_P^\mu) \phi_O(x_O'^\nu) d^3 x_P^\mu d^3 x_O'^\nu. \quad (3.47)$$

The coefficients A_{ii} and A_{ff} are related to the advanced and retarded Green's functions in a somewhat implicit form. The cross coefficient, however, has a simple interpretation:

$$A_{if}^{-1} = A_{fi}^{-1} = G^R(f, i). \quad (3.48)$$

It will be seen that (3.47) is a formal generalization of (3.31), for the purely time dependent function $\bar{\phi}$.

It is, however, the cross coefficient that carries the 'memory' of the initial state to the final state. For, suppose we denote the initial state by a wavefunctional $\Psi(\phi_i)$ and the final state by the wavefunctional $\Psi(\phi_f)$. Then the functional integral

$$\Psi_f(\phi_f) = \int K[\phi_f, t_f; \phi_i, t_i] \Psi_i(\phi_i) \mathcal{D}\phi_i \tag{3.49}$$

tells us how to compute the final state from the given initial state. However, the integral over ϕ_i becomes trivial if $A_{if} = 0$, and we get

$$\Psi_f(\phi_f) \propto F(t_f, t_i) \exp \frac{3i}{8\pi} \int \int A_{ff} \phi_f(x_f^\mu) \phi_f(x_f^{\nu'}) d^3x_f^\mu d^3x_f^{\nu'}. \tag{3.50}$$

Thus the final state is characterized by an imaginary phase factor irrespective of what the initial state was. It is this situation that occurred in section 3.3 where it was identified with the divergence of quantum fluctuations at the classical singularity.

We now show that the same situation prevails in the present general situation, because at the classical singularity the Green's function diverges. We end this section by a demonstration of this result.

Given a singular space-time manifold $\bar{\mathcal{M}}$ of general relativity with metric \bar{g}_{ik} we can construct another manifold \mathcal{M} which is nonsingular and whose metric \tilde{g}_{ik} is a conformal transform of \bar{g}_{ik} , say,

$$\tilde{g}_{ik} = \Omega^2 \bar{g}_{ik}. \tag{3.51}$$

Explicit demonstration of such transformations has been given by Kembhavi [51]. The conformal function Ω then diverges at the singular epoch t_s .

The Green's function in \mathcal{M} is related to the Green's function in $\bar{\mathcal{M}}$ by the following simple relation:

$$\Omega(A) \Omega(B) \tilde{G}(A, B) = \bar{G}(A, B). \tag{3.52}$$

Now since \mathcal{M} is nonsingular at $t = t_s$, \tilde{G} stays well behaved if either of the points A, B lies on the hypersurface $t = t_s$. However, $\Omega(A)$ or $\Omega(B)$ diverges as A or B approaches the singular hypersurface at $t = t_s$. Hence $\bar{G}(A, B)$ diverges at $t = t_s$ and A_{if} tends to zero as Ω^{-1} at the singular epoch. This is why the QCF diverge at the classical singularity.

3.6. Further generalizations

If instead of material particles of nonzero restmass the principal content of the universe were pure radiation, then because of (3.14) and because $\bar{T} = 0$ for radiation, the expression in (3.38) is simplified to the extent that $\bar{R} = 0$. The subsequent argument carries through as far as the remarks following (3.50). The demonstration that $\bar{G}(A, B)$ diverges at the classical singularity cannot, however, be given in the form described at the end of section 3.5 because the operator $\bar{\square}$ is not conformally invariant (as $\bar{\square} + \bar{R}/6$ was) when applied on a scalar function.

Padmanabhan and Narlikar [52] have, however, pointed out that the above case can be dealt with under a more general framework which considers QCF around the classical cosmological singularity in its 'most general form'. We recall here the classic work of Belinskii, Khalatnikov and Lifshitz [22, 23] (abbreviated henceforth to BKL), mentioned in section 1.3. BKL have argued that near the singularity the explicit form of S_m does not make any significant difference to the mode in which the singularity is approached. (We are assuming as in section 3.5 that the time axis is turned backwards and that the singular hypersurface is at $t = 0$.)

If we take the BKL model then simple algebra shows that close to $t = 0$, the Green's function takes

the form

$$G(x_f^\mu, t_f; x_i^\mu, t_i) \cong \frac{\delta_3(x_f^\mu - x_i^\mu)}{f(x_f^\mu)} \ln \left| \frac{t_f}{t_i} \right| \quad (3.53)$$

where f is some well-behaved function.

Taking the limit $t_f \rightarrow 0$ we therefore see that G diverges. Because the infinite dimensional matrix in (3.53) is diagonal we can argue that $A_{if} \rightarrow 0$ and hence the QCF diverge at the singularity. This general result includes various specific approaches to singularity with different forms for S_m . In particular it includes the result of Padmanabhan [53] that the QCF diverge around the classical Friedmann solution containing massless scalar field.

Finally we mention another generalization of the result on singularities described in section 3.4. Using the techniques of section 3.5, Narlikar [54] has shown that amongst the entire range of conformal transforms of the classical solutions of Einstein's equations for non-interacting massive particles, the models with singularity occur with vanishing likelihood. Thus the conditions of homogeneity and isotropy of the Friedmann models are not necessary for establishing the unlikelihood of space-time singularity.

3.7. Concluding remarks

The explicit results of sections 3.3–3.6 are sufficient to establish our confidence in the path integral technique as applied to quantum cosmology. The fact that QCF diverge at the epoch when the classical relativistic models have singularity, tells us that quantum cosmology has a nontrivial contribution to make to the subject of strong gravitational fields. In particular the classical conclusion that singularity and horizons are inevitable, has to be modified to the quantum cosmological result that it is highly unlikely that the universe originated in a singularity or had particle horizon sufficiently early on. We may compare this contrast of conclusions with the contrast between the conclusions drawn by classical and quantum electrodynamics for the fate of the orbiting electron in the H-atom.

To complete the above comparison we will next study the possible existence of stationary states in quantum cosmology.

4. Quantum stationary geometries

4.1. The need for a new approach

The discussion in the previous section has clearly pointed out the limited validity of classical relativity at the singular epoch. Starting with a wavefunction for the conformal factor, which is strongly peaked at the classical value, we were led back to a completely uncertain state at the singularity. The general proof for this result shows that questions regarding singularity just cannot be discussed within classical relativity.

Thus, to proceed further, we have to introduce a formalism for quantum gravity. We have already discussed, in a previous section, how the various approaches to gravity lead to divergent views regarding the question of singularity. The existence of a singularity implies the end of predictive power for physics. In this sense, it is reasonable to assume that any worthwhile model for quantum gravity must resolve the difficulty of singularities.

What guidelines do we have to proceed towards such a model? To begin with, it is clear that technical sophistication alone is unlikely to bear fruit. It is already apparent that virtually every known method has been used in the field of quantum gravity without any significant success. We take this to be indicative of the need for new physical input.

A study of conventional methods of quantum gravity also suggests a probable line of attack. All the previous methods of attack treated gravity either completely as a field or completely as space-time geometry. The truth might lie in-between; classical gravity plays the dual role of field and space-time geometry. *It sounds plausible that the field aspect of gravity should be quantized while the geometrical aspects remain as C-numbers.* This would be the basic philosophy that we will follow.

To proceed further we should make quantitative what is meant by the ‘geometrical’ and ‘field’ aspects of gravity. Our discussion in the previous sections has shown how well the conformal degree of freedom lends itself to quantum treatment. Also the geometrical and causal properties of the space-time, decided by the light cone structure, remain invariant under quantum fluctuations. In other words, by writing a metric tensor g_{ik} as,

$$g_{ik}(x) = \Omega^2(x) \bar{g}_{ik}(x) \quad (4.1)$$

one has effected a neat separation of geometrical and field aspects of the space-time metric.

The separation in (4.1) is generally covariant. Under coordinate transformations, the form (4.1) is retained as long as Ω transforms as a scalar and g_{ik} transforms as a tensor.

Having decided on these preliminaries, one should produce a consistent system of interpretation for our approach. To begin with, notice that since $\Omega(x)$ is going to be a q -number, g_{ik} in (4.1) (as it stands) is a quantum variable. In order to give the standard interpretation to the line element, we have to use the expectation value for (4.1) in a suitable form and write,

$$\langle g_{ik} \rangle = \langle \Omega^2(x) \rangle \bar{g}_{ik}. \quad (4.2)$$

This section and the next (section 5) will concentrate on studying the expectation value $\langle \Omega^2 \rangle$ and determining \bar{g}_{ik} .

4.2. *Quantum stationary geometries*

The classical description of geometry uses well-defined functions; for example, the FRW universes are described by the function $Q(t)$ which has a singularity at $t = 0$. We want to replace this description by a set of suitably chosen quantum states. The analogy of hydrogen atom mentioned before helps one in arriving at the choice.

Classically, the electron in the hydrogen atom is described by a trajectory $q(t)$ that spirals down to the nucleus ($r = 0$) in a short time. Quantum mechanics prevents this catastrophe by an appeal to uncertainty principle: complete localization at $r = 0$ involves infinitely large fluctuations in momentum. A compromise must be reached between the classical pull of the potential and the quantum fluctuations that arise from the uncertainty principle. These ‘compromise states’ are the standard stationary states of the hydrogen atom. There arises a lower bound to the electron orbit in the form of the Bohr radius.

This suggests that one should look for corresponding *quantum stationary geometries* (QSGs) for the conformal factor. The expectation value in equation (4.2) can be taken to be in these stationary states, leading to the line elements for various QSGs. The choice of stationary states is also ‘natural’ in the sense that we are not making any assumptions about the initial conditions for the quantum universe.

One might wonder, at this stage, about the background metric \bar{g}_{ik} . So far, this choice was dictated by classical solutions. We shall formulate, in the next section, the complete equations of quantum gravity. Till then, the form of \bar{g}_{ik} will be taken to be externally prescribed.

We shall now proceed to discuss specific examples for the stationary geometries.

4.3. QSGs in flat space

The simplest example for QSG occurs in the flat space itself [55]. Consider the situation where the background metric is taken to be Lorentzian, so that

$$g_{ik} = \Omega^2 \bar{g}_{ik} = \Omega^2 \eta_{ik}. \quad (4.3)$$

The action for the conformal factor has the form,

$$S = -\frac{3}{8\pi G} \int \Omega^i \Omega_i d^4x. \quad (4.4)$$

Making a Fourier transform (with $G = 1$),

$$\Omega(\mathbf{x}, t) = \int q_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}} \frac{d^3\mathbf{k}}{(2\pi)^3}, \quad (4.5)$$

we can write this action as

$$S = - \int_{t_i}^{t_f} dt \int \{ |q_{\mathbf{k}}|^2 - \omega_{\mathbf{k}}^2 |q_{\mathbf{k}}|^2 \} \frac{d^3\mathbf{k}}{(2\pi)^3}, \quad (4.6)$$

which corresponds to a set of independent harmonic oscillators, except for the overall minus sign. This minus sign can be incorporated into the ‘energy’ of the stationary states and thus does not affect the probability amplitude. Of special interest is the ground state analogous to the vacuum state in quantum field theory. The ‘vacuum functional’ for gravity can be written as

$$\Psi[\Omega(x)] = N \exp \left\{ -\frac{3}{8\pi^3} \frac{1}{L_p^2} \int \int \frac{(\nabla_x \Omega) \cdot (\nabla_y \Omega)}{|\mathbf{x} - \mathbf{y}|^2} d^3\mathbf{x} d^3\mathbf{y} \right\} \quad (4.7)$$

which gives the probability amplitude for detecting a conformal factor $\Omega(\mathbf{x})$ in the flat space. Quite clearly large deviations from the flat space can occur at Planck length scales, and these lead to the often talked about ‘space-time foam’ structure of the universe.

4.4. QSGs in a maximally symmetric universe

Since our main motivation for quantum gravity arises from cosmology, we shall now turn to discuss the stationary geometries for a Robertson–Walker model. We shall write the metric (with $c \neq 1$) as

$$d\bar{s}^2 = c^2 dt^2 - \bar{Q}^2(t) \left\{ \frac{dr^2}{1-r^2/a^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right\}. \quad (4.8)$$

In section 3, while considering the QCFs, we studied the class of metrics of the form

$$ds^2 = \Omega^2(t) d\bar{s}^2 \quad (4.9)$$

and determined the Kernel for $\Omega(t)$. Following this analogy one would have liked to consider the stationary states for $\Omega(t)$. However the Hamiltonian for $\Omega(t)$ will contain $\bar{Q}(t)$ as an externally prescribed function and hence it will be a time-dependent; thus no stationary states can exist. When the stationary states are not available, we can still evaluate the expectation value

$$\langle g_{ik} \rangle = \langle \Omega^2 \rangle \bar{g}_{ik} \quad (4.10)$$

in any suitably chosen evolving state. For example, if the state is chosen to be a Gaussian wavepacket [see (3.22)] in $(\Omega - 1)$, then (4.10) is equivalent to replacing $\bar{Q}^2(t)$ by,

$$Q^2(t) = \langle (1 + \phi)^2 \rangle \bar{Q}^2(t) = [1 + \Delta^2(t)] \bar{Q}^2(t). \quad (4.11)$$

As the singularity is approached, we know that $\bar{Q}^2(t)$ goes to zero. However, it was shown in (3.34) that $\Delta(t)$ goes as $\bar{Q}^{-1}(t)$ as $t \rightarrow 0$. Thus,

$$Q^2(t) \rightarrow \Delta^2(t) \bar{Q}^2(t) = \text{constant} \quad (4.12)$$

near the singularity. In other words, our model with the metric replaced by its quantum expectation value remains nonsingular [56].

Though the above result is interesting, it is model dependent. If one considers conformal fluctuations in cosmologies which are not isotropic – for example in a Kasner model – then $\Delta(t)$ goes as $(\ln \bar{Q})$ near $\bar{Q} \sim 0$. In these cases $Q(t)$, the final metric, still remains singular [57, 58].

Because of the above reason, we will arrive at the concept of stationary states for the FRW universe via a different route. We will write the metric in (4.8) as,

$$d\bar{s}^2 = \bar{Q}^2(t) \left[c^2 dt^2 - \frac{dr^2}{1-r^2/a^2} - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right]. \quad (4.13)$$

(The coordinate t used here is different from that used in (4.8).) When conformal fluctuations are considered we would take into account *all* metrics of the form,

$$ds^2 = \Omega^2(t) \bar{Q}^2(t) \left[c^2 dt^2 - \frac{dr^2}{1-r^2/a^2} - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right]. \quad (4.14)$$

It is clear that $\Omega \bar{Q}$ can be treated as a single quantum variable. Suppose we consider the class of geometries,

$$ds^2 = \Omega^2(t) ds_c^2 \quad (4.15)$$

with the background metric,

$$ds_c^2 = \left[c^2 dt^2 - \frac{dr}{1 - r^2/a^2} - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad (4.16)$$

it is clear that all space-times of (4.14) have been incorporated in (4.16). However, this metric ds_c^2 is static and thus one can introduce stationary states for the conformal factor. The action, governing the conformal fluctuations in this static background is,

$$S_g = \frac{Vc^4}{16\pi G} \int_{t_1}^{t_2} (\bar{R}\Omega^2 - 6\dot{\Omega}^2) dt \quad (4.17)$$

$$= -\frac{1}{2}M \int_{t_1}^{t_2} (\dot{q}^2 - \omega^2 q^2) dt \quad (4.18)$$

where we have used the combinations,

$$q = a\Omega; \quad M = \frac{3}{2}\pi(ac^2/G); \quad \omega = c/a; \quad V = \int \sqrt{-g} d^3x. \quad (4.19)$$

(V is the volume of the closed universe.) The action corresponds to a negative energy harmonic oscillator with mass M and frequency ω .

The total action is obtained by adding the matter term of the action to S_g in (4.18). We shall assume that matter terms are conformally invariant and hence independent of q . This is true of the radiation dominated early universe. Thus the stationary state structure is completely decided by (4.18). The 'Schrödinger equation',

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi = \frac{\hbar^2}{2M} \frac{\partial^2 \psi}{\partial q^2} - \frac{1}{2}M\omega^2 q^2 \psi \quad (4.20)$$

can be separated by the substitution,

$$\psi(q, t) = \exp\left\{\frac{i\varepsilon}{\hbar} t\right\} \cdot \phi_\varepsilon(q), \quad (4.21)$$

leading to,

$$-\frac{\hbar^2}{2M} \frac{\partial^2 \phi}{\partial q^2} + \frac{1}{2}M\omega^2 q^2 \phi = \varepsilon \phi. \quad (4.22)$$

The solutions are expressible in terms of the Hermite polynomials as,

$$\phi_n(q) = (2^n n!) \left(\frac{M}{\pi\hbar}\right)^{1/4} H_n\left(\frac{M\omega q}{\hbar}\right) \exp\left(-\frac{M\omega}{2\hbar} q^2\right). \quad (4.23)$$

The metric in the n th stationary state for quantum geometry will read as,

$$\begin{aligned} ds_n^2 &= \langle \Omega^2 \rangle_n \left[c^2 dt^2 - \frac{dr^2}{1 - r^2/a^2} - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] \\ &= \left(\frac{2}{3\pi} \right) \left(\frac{G\hbar}{c^3} \right) \left(n + \frac{1}{2} \right) [d\eta^2 - d\chi^2 - \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2)] \end{aligned} \quad (4.24)$$

where we have put $r = a \sin \chi$ and $\eta = \omega t$. We see that $\langle \Omega^2 \rangle$ has a lower bound of the order of the Planck length [59, 60]. Also notice that length intervals are separated by distances of the order of the Planck length. Such effects can be seen only at about 10^{19} GeV.

The action in (4.18) leads to the standard harmonic oscillator equation in the classical limit, with the solution,

$$q(t) = q_0 \sin \omega t = q_0 \sin \eta. \quad (4.25)$$

This is the right form of the solution for a radiation universe. Of course, since we are treating the source as externally specified, we will not be able to derive—at this state—any relation between the geometrical variables and the matter variables.

The fully quantized metric in (4.24) merely gives information about the quantum dynamics. In order to extract information about the actual universe, we have to specify the quantum state of the universe at some time t , and evaluate the expectation value of $\langle q^2 \rangle$ in that state.

What could be the best possible choice for the state of the universe, in order to produce the classical limit in (4.25)? One choice is to choose it to be a stationary state with large n , so that the profile of $|\phi_n(q)|^2$ will reproduce the classical evolution of the universe. However, as long as the state is a stationary state, $\langle q^2 \rangle$ will be independent of time and hence our semiclassical metric will not be time-dependent.

This motivates one to choose the state to be a coherent state [43, 60] with amplitude q_0 . These states are the nondispersive Gaussian wave packets for the harmonic oscillator with a probability distribution,

$$|\psi(q, t)|^2 = N \exp \left\{ -\frac{1}{L_p^2} (q - q_0 \sin \omega t)^2 \right\}, \quad (4.26)$$

so that the classical solution arises as the expectation value,

$$\langle q \rangle = q_0 \sin \omega t. \quad (4.27)$$

However, the metric contains $\langle q^2 \rangle$ and thus picks up quantum corrections:

$$\langle q^2 \rangle = q_0^2 \sin^2 \omega t + L_p^2. \quad (4.28)$$

Quite clearly the universe never contracts below the Planck length L_p in its evolution. This semiclassical approximation in (4.28) connects up the quantum universe of equation (4.24) with the classical limit of equation (4.25). In this sense, quantum stationary geometries ‘explain’ the conclusions reached in section 3 based on conformal fluctuations. Classical evolution must give way to a description in terms of quantum stationary geometries near the singularity.

4.5. QSGs in other cosmological models

There are two main reasons for concentrating on the Robertson–Walker cosmologies: (i) our physical universe is very well described by an isotropic, homogeneous universe, (ii) the conformal degree of freedom is the only degree of freedom in the Robertson–Walker model. All the same, one can discuss the structure of conformal fluctuations in other cosmologies as well. In particular, one can consider meaningfully the extension of conformal fluctuations to the homogeneous Bianchi cosmologies.

When we try to develop the QSGs for the homogeneous models, however, we face the same kind of problem as faced in the last section: The metric contains time-dependent functions and thus stationary states cannot exist. In the case of Robertson–Walker models we got around this difficulty because the dynamical degree of freedom was conformal (see eqs. (4.13, 14, 15)). For an arbitrary Bianchi model this cannot be done. The concept of stationary states has to be generalized taking other degrees of freedom into account.

In this section we shall examine such an extension of the theory. Consider the metric for a diagonal Bianchi model (see eqs. (1.21–23)),

$$ds^2 = dt^2 + g_{\mu\nu} dx^\mu dx^\nu \quad (4.29)$$

with,

$$-g_{\mu\nu}(t) = e^{2\lambda(t)} [e^{-2\beta(t)}]_{\mu\nu} \quad (4.30)$$

$$\beta = \text{dia}[\beta_1, -\frac{1}{2}\beta_1 + \frac{1}{2}\sqrt{3}\beta_2, -\frac{1}{2}\beta_1 - \frac{1}{2}\sqrt{3}\beta_2]. \quad (4.31)$$

Thus the geometry is described by 3 functions of time $\lambda(t)$, $\beta_1(t)$, $\beta_2(t)$. The action can be expressed in terms of these functions (cf. eq. (1.24)) as,

$$S = \frac{1}{16\pi G} \int_{t_1}^{t_2} \{-e^{3\lambda} [6\dot{\lambda}^2 - \frac{3}{2}(\dot{\beta}_1^2 + \dot{\beta}_2^2)] + e^{3\lambda} R^*\} dt. \quad (4.32)$$

We write down the Hamiltonian for these models, once the form of R^* is given. Let us begin with Bianchi class B models for which R^* is given by (1.26). After some algebra, the Hamiltonian can be written as [61, 62]

$$\hat{H} = -\frac{\hbar^2}{4q^2} \left[\frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2} \right] + \frac{\hbar^2}{4} \frac{\partial^2}{\partial q^2} - c_1 q^{2/3} e^{-c_2 u_1} \quad (4.33)$$

where

$$q = \left(\frac{8}{3}\right)^{1/2} \exp\left(\frac{3}{2}\lambda\right) \quad (4.34)$$

$$u_1 = \left(\frac{8}{3}\right)^{1/2} \left[\frac{3}{2(1-3h)} \right]^{1/2} (\beta_1 + \sqrt{-3h} \beta_2) \quad (4.35)$$

$$u_2 = \left(\frac{8}{3}\right)^{1/2} \left[\frac{3}{2(1-3h)} \right]^{1/2} (\sqrt{-3h} \beta_1 - \beta_2) \quad (4.36)$$

$c_1, c_2 = \text{constants}$.

The solution to the eigenvalue equation,

$$\hat{H}\psi = E\psi \tag{4.37}$$

can be, in general, quite complicated. However, we are interested in the behaviour near $q = 0$, where the conformal part becomes singular. Near this point, the equation (for any finite E) reads as,

$$-\left(\frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2}\right)\phi(u_1, u_2, q) + q^2 \frac{\partial^2 \phi}{\partial q^2} = 0. \tag{4.38}$$

Putting

$$\phi(u_1, u_2, q) = e^{i(ku_1 + pu_2)} f(q) \tag{4.39}$$

we obtain, for $f(q)$ the following equation,

$$q^2 \frac{d^2 f}{dq^2} = -(p^2 + k^2) f = -\frac{1}{4}\alpha^2 f, \text{ say.} \tag{4.40}$$

When $\alpha \neq 0$, this has the solution,

$$f_\alpha(q) = q^{1/2} [A_\alpha q^{(1/2)\sqrt{1-\alpha^2}} + B_\alpha q^{(-1/2)\sqrt{1-\alpha^2}}] \tag{4.41}$$

where A_α and B_α are constants. It is easy to see that $f_\alpha(q)$ goes to zero as q goes to zero (irrespective of whether $\alpha \geq 1$). When $\alpha = 0$, the solution is just

$$f(q) = Aq + B, \tag{4.42}$$

which may have nonzero value at $q = 0$. However, $\alpha = 0$ implies that the state is completely independent of u_1 and u_2 or rather β_1 and β_2 . From the metric it is clear that (β_1, β_2) represent the deviations from isotropy while q represents the conformal part. The above result therefore shows that states with an anisotropy give zero probability at $q = 0$. In some sense this may be interpreted by saying that conformal fluctuations predominate over other degrees of freedom near the singularity. A similar, lengthy analysis will prove that the results are valid for class A space-times as well.

4.6. The use of superspace metric

The concept of quantum stationary geometries, and in fact, the path integral approach itself can be presented in an elegant manner using the concept of superspace. We shall describe the connection briefly [61].

We showed in section 1.2 that Einstein's equations can be derived from an action functional defined in the superspace, in the form

$$S_0 = \int dt \int \sqrt{-g} R d^3x + \int dt \int G^{AB} \dot{g}_A \dot{g}_B d^3x. \tag{4.43}$$

The classical solution – giving the 3-metric $g_{\mu\nu}$ at every hypersurface labelled by t – is obtained from

$\delta S_0 = 0$. As usual, we can formulate a quantum theory by the Feynman postulate,

$$K[g_{\Lambda}^f, t_f; g_{\Lambda}^i, t_i] = \int \mathcal{D}g_{\Lambda}(t) \exp \frac{i}{\hbar} [S_0(g_{\Lambda})] \quad (4.44)$$

where the action S_0 is the action evaluated in the superspace. We can proceed with $S_0(g_{\Lambda})$ in exactly the same way as for the action in the case of standard quantum mechanics.

In particular, following the ideas described in section 1.3 we find the superspace action for Bianchi cosmologies to be

$$S_0 = \int [R - 6\dot{\lambda}^2 + \frac{3}{2}(\dot{\beta}_1^2 + \dot{\beta}_2^2)] e^{3\lambda} dt, \quad (4.45)$$

which agrees with the form we have used in our discussion. The wavefunction $\psi(q, u_1, u_2)$ in (4.37) may be considered more formally as a wavefunction for the 3 geometry $\psi(^3\mathcal{G})$ defined in superspace. The stationary state represents a configuration whose probability density remains the same at all hypersurfaces.

At the simple level in which we are working formal techniques based on superspace may be redundant. However, more technical questions such as stability, topological structure etc. can probably be better tackled through a superspace approach.

4.7. Conclusion

The concept of QSGs provided the necessary alternative description for space-time near classical singularity. However, the formalism has, at this stage, one major drawback: it does not replace the classical theory completely because the background metric \bar{g}_{ik} is still put in by hand. It is necessary that a complete theory determines both (a) the quantum evolution of $\Omega(t)$ and (b) the classical dynamics of \bar{g}_{ik} . We shall address ourselves to this task in the next section.

5. A model for quantum gravity

We have outlined in section 4 an approach to quantum gravity in which the conformal part is treated quantum mechanically and the background metric classically. The model is not complete as it stands, because of the fact that we have not determined the 'background metric' in a self-consistent manner, taking the back reactions of the conformal fluctuations into account. We shall make this extension [63, 55] now.

Let us again begin with the Einstein action (see (1.5, 1.7), with $c = 1$)

$$S_g = \frac{1}{16\pi G} \int \bar{R} \sqrt{-g} d^4x + \frac{1}{8\pi G} \int \bar{K} \sqrt{h} d^3x \quad (5.1)$$

written out for a metric tensor in the form,

$$\bar{g}_{ik} = \Omega^2(x) g_{ik}(x)$$

as,

$$S_g = \frac{1}{16\pi G} \int (R\Omega^2 - 6\Omega_i\Omega^i) \sqrt{-g} d^4x. \quad (5.2)$$

Adding the matter part of the action, we get, for the total action of the system,

$$S = S_g + S_m = \frac{1}{16\pi G} \int \Omega^2 R \sqrt{-g} d^4x + \frac{3}{4\pi G} \int \mathcal{L}_{NE} \sqrt{-g} d^4x + \int \mathcal{L}_m \sqrt{-g} d^4x. \quad (5.3)$$

Here \mathcal{L}_m stands for the matter part of the Lagrangian and \mathcal{L}_{NE} stands for

$$\mathcal{L}_{NE} = -\frac{1}{2}\Omega_i\Omega^i \quad (5.4)$$

which has the form of the Lagrangian for a negative energy scalar field.

In the previous sections we have used this action as a functional of $\Omega(x)$ and have treated $\Omega(x)$ as a quantum variable. This led to the concept of quantum states – especially, the stationary states – for the variable $\Omega(x)$, described by the wavefunctions $\Psi[\Omega, t]$. In doing these calculations, we have assumed a particular form for g_{ik} (which is usually taken to be the solution of classical Einstein’s equations). Now, however, we want to write down a set of C -number equations for the g_{ik} taking into account the quantum conformal fluctuations.

This can be done in the following natural fashion: Varying g_{ik} in (5.3) will certainly yield a set of equations for g_{ik} ; but these will not be C -number equations because $\Omega(x)$ is a quantum variable. Therefore, we shall replace S in (5.3) by an ‘effective action’ in which Ω^2 , \mathcal{L}_{NE} etc. will be taken to be the expectation values in the quantum state. Thus we have,

$$S_{\text{eff}} = \langle S \rangle = \frac{1}{16\pi G} \int \langle \Omega^2 \rangle R \sqrt{-g} d^4x + \frac{3}{4\pi G} \int \langle \mathcal{L}_{NE} \rangle \sqrt{-g} d^4x + \langle S_m \rangle. \quad (5.5)$$

Here the expectation values are calculated in the quantum state of the conformal factor. In the matter part of the action, we have written S_m in order to take into account any dependence of S_m on Ω . If one believes that matter action must be conformally invariant, then S_m will be independent of Ω and $\langle S_m \rangle$ may be replaced by just S_m .

Varying g_{ik} in eq. (5.5) we are led to the following set of equations,

$$\langle \Omega^2 \rangle (R_{ik} - \frac{1}{2}g_{ik}R) + 6t_{ik} = -8\pi GT_{ik} + [g_{ik}\square - \nabla_i\nabla_k] \langle \Omega^2 \rangle \quad (5.6)$$

where

$$t_{ik} = -\langle \Omega_i\Omega_k \rangle + \frac{1}{2}g_{ik}\langle \Omega_a\Omega^a \rangle \quad (5.7)$$

and T_{ik} is the energy momentum tensor for matter. We have assumed that the matter Lagrangian is conformally invariant (otherwise T_{ik} will be replaced by $\langle \Omega^n \rangle T_{ik}$ with a suitable n).

We see that t_{ik} has the form of the energy-momentum tensor for a negative energy scalar field. This term arises from the variation of g_{ik} in \mathcal{L}_{NE} . The only peculiarity is the last term in the right-hand side

of (5.6). This comes about in the following manner: consider the variation of g_{ik} in the term

$$S_1 = \frac{1}{16\pi G} \int \langle \Omega^2 \rangle R \sqrt{-g} d^4x. \quad (5.8)$$

We have,

$$\begin{aligned} \delta S_1 &= \frac{1}{16\pi G} \int \langle \Omega^2 \rangle \delta(R_{ik} g^{ik} \sqrt{-g}) d^4x \\ &= \frac{1}{16\pi G} \int \langle \Omega^2 \rangle (R_{ik} - \frac{1}{2} g_{ik} R) \sqrt{-g} \delta g^{ik} d^4x + \frac{1}{16\pi G} \int \langle \Omega^2 \rangle g^{ik} \delta R_{ik} \sqrt{-g} d^4x. \end{aligned} \quad (5.9)$$

The term with δR_{ik} can be written as

$$\int \langle \Omega^2 \rangle g^{ik} \sqrt{-g} \delta R_{ik} d^4x = \int \langle \Omega^2 \rangle \partial_k (\sqrt{-g} W^k) d^4x, \quad (5.10)$$

where

$$W^k = g^{ik} \delta \Gamma^l_{il} - g^{il} \delta \Gamma^k_{il}. \quad (5.11)$$

Provided $\langle \Omega^2 \rangle$ is a constant, this expression can be converted into a total divergence and ignored. Now $\langle \Omega^2 \rangle$ can (in general) be a function of x^i and hence this procedure is not possible. It is this contribution, that appears in the second term in the right-hand side of equation (5.6).

Thus the full formulation of quantum gravity requires us to solve a coupled set of equations; equation (5.6) for the metric g_{ik} and the functional integral (or Schrödinger equation) for the quantum state. We shall now attempt the solution for various special cases.

5.1. Static solutions to quantum gravity equations

It is clear from the previous section that the equations of quantum gravity are much more intricate in structure than the classical Einstein's equations. Since general solutions are totally out of question, we shall try to concentrate on situations pertinent to cosmology [63, 55].

We shall begin by demonstrating the selfconsistency of our discussion in section 4. In section 4.4 we assumed a metric of the form (we shall use standard units $c \neq 1$ in this section)

$$ds^2 = \langle \Omega^2 \rangle \left[c^2 dt^2 - \frac{dr^2}{1 - r^2/a^2} - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] \quad (5.12)$$

and treated Ω to be a quantum variable. In order to be selfconsistent it is essential that we can show the background metric in (5.12) to be a solution to our coupled equations. Let us begin by considering the action for $\Omega(t)$,

$$S = -\frac{1}{2} M \int_{t_1}^{t_2} (\dot{q}^2 - \omega^2 q^2) dt \quad (5.13)$$

with

$$q = a\Omega ; \quad M = \frac{3}{2}\pi(ac^2/G) ; \quad \omega = c/a . \quad (5.14)$$

We have assumed, as in section 4 that the source consists of radiation and thus S_m is independent of q . The Schrödinger equation for the action in (5.13) is just the harmonic oscillator equation,

$$-i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2M} \frac{\partial^2 \psi}{\partial q^2} + \frac{1}{2}M\omega^2 q^2 \psi . \quad (5.15)$$

We now look for a selfconsistent solution with the universe in one of the stationary states, with $\langle \Omega^2 \rangle = \text{constant}$. If the harmonic oscillator is in the n th state,

$$\langle \Omega^2 \rangle = \frac{\hbar}{M\omega a^2} (n + \frac{1}{2}) . \quad (5.16)$$

Also,

$$\langle \dot{\Omega}^2 \rangle = \frac{\langle P_{\dot{\Omega}}^2 \rangle}{M^2} = \frac{\hbar}{c^2 M a^2} (n + \frac{1}{2}) . \quad (5.17)$$

One should now satisfy the Einstein's equations (with quantum corrections, (5.6)) with these stationary state values. Since $\langle \Omega^2 \rangle$ is constant, the equation reads,

$$\langle \Omega^2 \rangle (R^i_k - \frac{1}{2}\delta^i_k R) = -8\pi G T^i_k - 6t^i_k . \quad (5.18)$$

Now for our model,

$$T^i_k = \varepsilon (1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}) \quad (5.19)$$

$$t^i_k = \frac{1}{2}\langle \dot{\Omega}^2 \rangle (-1, 1, 1, 1) \quad (5.20)$$

$$= \frac{\hbar\omega}{2Ma^2c^2} (n + \frac{1}{2}) (-1, 1, 1, 1) . \quad (5.21)$$

For the maximally symmetric space-time, there are only two independent equations in (5.18) which may be conveniently taken to be the 'trace equation' and ' $\binom{0}{0}$ equation'. The trace equation reads,

$$-\langle \Omega^2 \rangle R = -6t^i_i = -6\langle \dot{\Omega}^2 \rangle . \quad (5.22)$$

Using the fact that, $R = 6/a^2$, we get,

$$\langle \dot{\Omega}^2 \rangle = \omega^2 \langle \Omega^2 \rangle \quad (5.23)$$

which is identically satisfied in the stationary states. Thus we only have

to worry about the $\langle \hat{\Omega} \rangle$ component equation, which becomes

$$\langle \Omega^2 \rangle \left(-\frac{3}{2} \frac{1}{a^2} \right) - 3 \langle \dot{\Omega}^2 \rangle + 8\pi G \varepsilon = 0, \quad (5.24)$$

i.e.,

$$\varepsilon = \frac{9c^4}{16\pi G} \frac{\langle \Omega^2 \rangle}{a^2} = \frac{9c^4}{16\pi G} \left(\frac{\hbar}{M\omega} \right) \frac{1}{a^4} \left(n + \frac{1}{2} \right) \quad (5.25)$$

$$= \frac{3}{8\pi^2} \left(\frac{\hbar c}{a^4} \right) \left(n + \frac{1}{2} \right). \quad (5.26)$$

In other words, our solution is selfconsistent provided the energy density of matter satisfies the quantum condition in (5.26). This is the bonus that arises from the consideration of the back reaction.

Notice that this is a purely quantum condition on the matter source, arising from quantum gravity. One is reminded of the property of Einstein's equations that they also determine the classical dynamics of the source. Quantum gravity, it seems, can put nontrivial requirements on the quantum nature of the source.

The above solution thus vindicates the choice of the background metric made in section 4. However, this solution is static while the observed universe is not. We shall now turn to the Robertson-Walker solutions with nontrivial time dependence.

5.2. Evolutionary models for the universe

In order to introduce nontrivial time dependence, we can either make the background metric time-dependent, or choose a time-dependent quantum state for the conformal factor [64]. Because of the natural significance of the stationary states, we will continue to consider $\langle \Omega^2 \rangle$ to be constant, and put the time dependence into g_{ik} .

The most general maximally symmetric space-time will have the metric ($k = 0, \pm 1$)

$$ds^2 = \langle \Omega^2 \rangle \left[c^2 dt^2 - Q^2(t) \left\{ \frac{dr^2}{1 - kr^2/a^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right\} \right]. \quad (5.27)$$

We shall begin by considering the quantum dynamics of Ω governed by the action,

$$S = -\frac{1}{2} M \int_{t_1}^{t_2} Q^3(t) [\dot{q}^2 - \omega^2(t) q^2] dt \quad (5.28)$$

where,

$$q = a\Omega; \quad M = \frac{2}{3}\pi\beta(ac^2/G); \quad \nu = c/a, \quad (5.29)$$

$$\omega^2(t) = \frac{\ddot{Q}}{Q} + \frac{1}{Q^2} (\dot{Q}^2 + k\nu^2). \quad (5.30)$$

We have introduced a parameter β , which is the ratio between the volume of the region of the universe under consideration to the volume of the universe for $r \leq a$. Our results are not sensitively dependent on the actual value of β . The Schrödinger equation for this action reads as,

$$-i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2M Q^3(t)} \frac{\partial^2 \psi}{\partial q^2} + \frac{1}{2} M \omega^2(t) Q^3(t) q^2 \psi. \quad (5.31)$$

It looks as though stationary state solutions cannot exist for this equation because of the time dependence. In general, this is true. However there is one particular choice of $Q(t)$ for which we do have solutions of the form,

$$\psi(q, t) \sim [\exp if(t)] \cdot \phi(q). \quad (5.32)$$

We shall exploit this fact. Let us suppose, for a moment, that $Q(t)$ satisfies the equation,

$$Q^6(t) \omega^2(t) = \alpha^2, \quad (5.33)$$

i.e.

$$Q^6(t) \left\{ \frac{\ddot{Q}}{Q} + \frac{\dot{Q}^2 + k\nu^2}{Q^2} \right\} = \alpha^2 \quad (5.34)$$

with some (positive) constant α^2 . Then eq. (5.31) becomes,

$$-i\hbar Q^3(t) \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2M} \left(\frac{\partial^2 \psi}{\partial q^2} \right) + \frac{1}{2} M \alpha^2 q^2 \psi \quad (5.35)$$

which has the solution,

$$\psi(q, t) = \exp \left\{ \frac{iE_n}{\hbar} \int \frac{dt}{Q^3(t)} \right\} \cdot \phi_n(q), \quad (5.36)$$

where $\phi_n(q)$ is the n th eigenfunction of a harmonic oscillator with frequency α and mass M . The expectation value for q^2 and Ω^2 are

$$\langle q^2 \rangle_n = \frac{\hbar}{M\alpha} \left(n + \frac{1}{2} \right), \quad \langle \Omega^2 \rangle = \frac{\hbar}{M\alpha a^2} \left(n + \frac{1}{2} \right). \quad (5.37)$$

The expectation value $\langle \dot{q}^2 \rangle$ can be computed by noticing that the generalized momentum for q is

$$\hat{p} = \partial L / \partial \dot{q} = M Q^3(t) \dot{q} \quad (5.38)$$

so that,

$$\langle \dot{q}^2 \rangle = \frac{1}{M^2 Q^6(t)} \langle \hat{p}^2 \rangle = \frac{1}{M^2 Q^6(t)} \int_{-\infty}^{+\infty} \left\{ \psi^* \left(-\hbar^2 \frac{\partial^2}{\partial q^2} \right) \psi \right\} dq = \left(\frac{\hbar \alpha}{M} \right) \frac{1}{Q^6(t)} \left(n + \frac{1}{2} \right). \quad (5.39)$$

We now have to show that Einstein's equations are satisfied with these conditions. Since $\langle \Omega^2 \rangle$ is, again, constant the equations read

$$\langle \Omega^2 \rangle (R^i_k - \frac{1}{2} \delta^i_k R) + 6t^i_k = -8\pi G T^i_k. \quad (5.40)$$

We shall assume the source to be conformally invariant, leading to an energy momentum tensor,

$$T^i_k = \varepsilon(t) (1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}). \quad (5.41)$$

The trace equation in (5.40) again reads,

$$\langle \dot{\Omega}^2 \rangle = \omega^2(t) \langle \Omega^2 \rangle. \quad (5.42)$$

Since from eqs. (5.39) and (5.37),

$$\langle \dot{\Omega}^2 \rangle / \langle \Omega^2 \rangle = \alpha^2 / Q^6(t), \quad (5.43)$$

we see that eq. (5.42) is identically satisfied because of the fact that $\omega^2(t) Q^6(t) = \alpha^2$.

Thus we only have to satisfy the $\binom{0}{0}$ component equation,

$$3 \frac{\dot{Q}^2 + k\nu^2}{Q^2} = \frac{8\pi G a^2}{\langle q^2 \rangle c^2} \varepsilon(t) - 3 \frac{\langle \dot{q}^2 \rangle}{\langle q^2 \rangle}. \quad (5.44)$$

Let us see what this equation implies for $\varepsilon(t)$. First, notice that (5.34) can be integrated once, to give,

$$\dot{Q}^2 / Q^2 = -k\nu^2 / Q^2 - \alpha^2 / Q^6 + \rho^2 / Q^4, \quad (5.45)$$

where ρ^2 is an integration constant. This can be substituted into (5.44) to give,

$$\frac{8\pi G a^2}{\langle q^2 \rangle c^2} \varepsilon(t) = 3 \frac{\dot{Q}^2 + k\nu^2}{Q^2} + 3 \frac{\langle \dot{q}^2 \rangle}{\langle q^2 \rangle} = 3 \left\{ -\frac{\alpha^2}{Q^6} + \frac{\rho^2}{Q^4} \right\} + \frac{\alpha^2}{Q^6} = \frac{3\rho^2}{Q^4(t)}. \quad (5.46)$$

Thus the matter energy density evolves as,

$$\varepsilon(t) = \frac{3\rho^2 c^2}{8\pi G a^2} \left(\frac{\hbar}{M\alpha} \right) \frac{1}{Q^4(t)} \left(n + \frac{1}{2} \right) = \frac{9}{16\pi^2 \beta} \left(\frac{\hbar \rho^2}{a^3 \alpha} \right) \frac{1}{Q^4(t)} \left(n + \frac{1}{2} \right). \quad (5.47)$$

This completes our solution formally. We have (5.33) for $Q(t)$, (5.36) specifying the quantum state $\psi(q, t)$ and (5.47) giving the $\varepsilon(t)$. The solutions are parametrized by the variables α^2 and ρ^2 . Physically ρ^2 – appearing in (5.47) – sets the scale of $\varepsilon(t)$ while α denotes the deviation from the classical limit. This can be easily seen by noting that (5.45) goes over to

$$(\dot{Q}^2 + k\nu^2) / Q^2 = \rho^2 / Q^4 \quad (5.48)$$

when α goes to zero. (5.48) is the classical equation in the Robertson–Walker space-time. We shall now discuss the solutions for $Q(t)$ under various conditions.

5.3. The quantum universes

The space-time metric, when the universe is in the n th quantum state, is given by

$$ds_n^2 = \langle \Omega^2 \rangle_n \left[c^2 dt^2 - Q^2(t) \left\{ \frac{dr^2}{1 - kr^2/a^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right\} \right]. \quad (5.49)$$

It is more convenient to write this in the form,

$$ds_n^2 = \langle \Omega^2 \rangle_n Q^2(\tau) \left[c^2 d\tau^2 - \frac{dr^2}{1 - kr^2/a^2} - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] \quad (5.50)$$

where $Q(\tau)$ satisfies the equation (see (5.45))

$$\left(\frac{dQ}{d\tau} \right)^2 = \rho^2 - \alpha^2/Q^2 - k\nu^2 Q^2. \quad (5.51)$$

We shall begin with the case where there is no matter ($\rho^2 = 0$). The expansion of the universe is completely sustained by the vacuum. From (5.51) it follows that, when $\rho^2 = 0$, k must be (-1) , giving,

$$(dQ/d\tau)^2 = \nu^2 Q^2 - \alpha^2/Q^2 \quad (5.52)$$

which has the solution,

$$Q^2(\tau) = (\alpha/\nu)[1 + 2 \sinh^2 \nu\tau]. \quad (5.53)$$

In the geometric coordinates, the metric reads as,

$$ds_n^2 = L_n^2(\nu/\alpha) Q^2(\tau) [d\eta^2 - d\chi^2 - f^2(\chi) (d\theta^2 + \sin^2 \theta d\phi^2)] \quad (5.54)$$

(where $f(\chi) = \sin \chi$, χ or $\sinh \chi$ and $L_n^2 = (G\hbar/c^3)(3/2\pi\beta)(n + \frac{1}{2})$) and in the case of $Q(\tau)$ in (5.53), we have,

$$ds_n^2 = L_n^2(1 + 2 \sinh^2 \eta) [d\eta^2 - d\chi^2 - \sinh^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2)]. \quad (5.55)$$

We see that the model is nonsingular. In fact, as we go along we will find that all the solutions to our equations are nonsingular. At large η , the above metric goes into,

$$ds_n \approx 2L_n^2 \sinh^2 \eta [d\eta^2 - d\chi^2 - \sinh^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2)] \quad (5.56)$$

which is the form for a classical, open, radiation filled model with energy density,

$$\varepsilon = \frac{9}{8\pi^2\beta} \left(\frac{\hbar c}{a^4} \right) \frac{1}{Q^4(t)} \left(n + \frac{1}{2} \right). \quad (5.57)$$

In the terminology popularized by Wheeler we can call this ‘matter without matter’. This is a purely quantum gravitational solution.

Now we consider models with matter density. The simplest is the one with $k = 0$, for which (5.51) reads

$$(dQ/d\tau)^2 = \rho^2 - \alpha^2/Q^2. \quad (5.58)$$

This has the solution

$$Q^2(\eta) = \alpha^2/\rho^2 + \rho^2\tau^2. \quad (5.59)$$

The metric is given by

$$ds^2 = L_n^2(\alpha^2/\rho^2 + \rho^2\tau^2) [d\eta^2 - d\chi^2 - \chi^2 (d\theta^2 + \sin^2\theta d\phi^2)]. \quad (5.60)$$

We find that the model is nonsingular and goes to the minimum radius $L_n^2(\alpha/\rho)$ in a phase of contraction and re-expansion. The solutions for $k = \pm 1$ are more complicated and are given by,

$$Q^2(\eta) = \left[\frac{\rho^4}{4\nu^4} + \frac{\alpha^2}{\nu^2} \right]^{1/2} \cosh 2\eta - \frac{\rho^2}{2\nu^2} \quad (k = -1) \quad (5.61)$$

$$Q^2(\eta) = \frac{\rho^2}{2\nu^2} - \left[\frac{\rho^4}{4\nu^4} - \frac{\alpha^2}{\nu^2} \right]^{1/2} \cos 2\eta \quad (k = +1). \quad (5.62)$$

These expansion factors go over to corresponding classical expressions as $\alpha \rightarrow 0$. However there is one curiosity about the $k = +1$ model. This model can exist only if there is 'sufficient matter' in the sense that,

$$\rho^2 > 2\alpha\nu. \quad (5.63)$$

Classically α goes to zero and thus this inequality is trivially satisfied. The physical interpretation of this result is that the universe can be closed only when the positive energy density predominates over the negative energy of conformal fluctuations.

When $\rho^2 = 2\alpha\nu$ (5.62) gives a constant $Q(\eta)$. One recalls that (5.45) was obtained from the fundamental equation (5.34) by multiplying both sides by \dot{Q} . This procedure is invalid when Q is constant. For all $\rho^2 \leq 2\alpha\nu$ we can only have $k = -1$ model.

In all the above cases the matter density begins at a constant value and evolves as $1/Q^4(t)$. The constant α cannot be determined by the theory without extra input. This parameter goes into deciding the quantum state of the universe which one expects to be related to 'the measurement of the quantum space-time'. Since we do not have any idea about the physical meaning of such a terminology we cannot say anything further at this stage.

More complicated solutions to our equations can exist, many of which will have $\langle \Omega^2 \rangle$ *not* a constant. This would make the mathematical structure of the equations intractable, and only a detailed numerical analysis will lead to definite information. However such solutions contain some interesting possibilities. It is possible that there exists solutions for which $\varepsilon(t)$ starts at zero value, increases to some point and then dies down as $1/S^4(t)$. Such a model allows for the 'creation of the universe'. In a less esoteric manner we can start with a ($\rho = 0$) matterless universe and produce particles by the expansion of the universe, making a transition into the universe with matter. These are considerations for the future.

We also mention in passing that the difficulty of particle horizons, discussed at length in section 3 is resolved in the above models. It can be demonstrated by explicit calculation that the above models are horizon-free. For details we refer the reader to ref. [47].

5.4. The flatness problem in quantum cosmology

We mentioned in section 1.4 that classical cosmology faces three difficulties viz. those of singularity, horizon and flatness. The introduction of quantum ideas seems to eliminate the first two difficulties. This section is intended to give a brief idea of how the third difficulty may also be tackled within the present framework.

Quantum conformal fluctuations allow transitions between conformally related space-times. It can be verified directly (by computing the Weyl tensor) that all the Robertson–Walker space-times are conformally flat and hence can be represented by a metric of the form,

$$ds^2 = \Omega^2(\mathbf{x}, t) [dt^2 - dx^2 - dy^2 - dz^2]. \quad (5.64)$$

In the case of a $k = 0$ universe, the conformal factor will depend only on time; i.e. $\Omega(\mathbf{x}, t) = \Omega(t)$. For $k = \pm 1$ we have different forms of $\Omega(\mathbf{x}, t)$. Thus any one of the three types of FRW universes (with $k = 0, \pm 1$) could have originated from the flat Lorentz space-time [with $\Omega = \Omega_0$, constant] via conformal fluctuations. (In fact, it can be shown that the flat space is unstable to conformal fluctuations [65].) It is possible to calculate the transition probability between the flat Lorentz space-time and the general space-time described by eq. (5.64). This probability is given by (see for details, ref. [66])

$$\mathcal{P} = N \exp \left\{ -\frac{3}{8\pi} \int \int \frac{\nabla \Omega(\mathbf{r}_1) \cdot \nabla \Omega(\mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^2} d^3 \mathbf{r}_1 d^3 \mathbf{r}_2 \right\}. \quad (5.65)$$

It is clear that the probability is maximum for $\Omega(\mathbf{x}, t) = \Omega(t)$, i.e. for the spatially flat ($k = 0$) universe. In other words if the universe originated via a quantum conformal fluctuation, it is most likely to be flat spatially, thus providing a simple explanation to the spatial flatness of the observed universe.

6. Concluding remarks

General relativity and quantum theory are both remarkable achievements of the present century. Although both theories have had profound influence on the thinking of modern physicists, they differ considerably in the way they were created and in the way they evolved. General relativity was presented as a complete and finished product by Einstein in 1915. Many new ideas and subtleties which were originally not thought of were subsequently discovered in this theory. The theory did not arise in response to experimental or observational conundrums; it did so out of conceptual problems. The observational tests verifying its conclusions came later. By contrast the quantum theory arose in response to experimental challenges. It did not attain a ‘perfect’ form right away; rather it evolved to a working theory through stages with inputs from several theoreticians. Even today it is far from being in a final form.

What should be the mode of development of quantum gravity, the off-spring of a marriage between general relativity and quantum theory?

DeWitt [67] has emphasized the formal aspect “we shall here adopt an uncompromisingly formal stance, as being the most likely to survive future developments, given the known tendency for pure formalism to acquire and maintain a consistency and logic of its own”. This attitude is more in sympathy with the mode of development of general relativity rather than that of quantum theory. Had quantum theorists waited to produce a formally well-established framework they would even now be waiting on the sidelines with nothing concrete to offer.

We feel that the pragmatic approach of the quantum theorist is needed in the present state of quantum gravity. The difficulties discussed in section 1.5 will not go away in a single stroke. Just as the early approaches of Planck, Bohr, Schrödinger, Heisenberg and Dirac represent increasingly more sophisticated steps in formulating the quantum theory, so do we have to begin with a somewhat simple crude approach and build up towards an increasingly comprehensive theory. As with the initial developments of quantum theory, we have to face criticisms implied by adjectives ‘ad hoc’, ‘highly limited’, ‘simple minded’ etc. with our approach to quantum gravity.

The work described from section 3 onwards is motivated by this pragmatic viewpoint. We do not suggest that the ideas presented here are formally the last word on quantum cosmology. Rather the reverse! These are ideas on which a better theory of quantum cosmology may be built in future. Formal objections on the grounds of oscillating path integrals, lack of a precise measure, limitation of quantizable degrees of freedom etc. will have to await a better theory for a possible resolution.

Even in ordinary quantum mechanics considerable discussion is still going on about the foundations of the theory. The role of the observer, the collapse of the wavefunction, the many-universe concept, the part played by consciousness are still not clarified. These questions assume even greater difficulty in quantum cosmology. For example, what do we mean by quantum uncertainty for a global property? Where is the observer in all this? To what extent do measurements ‘disturb’ the universe? Such questions will continue to form the subjects of discussion on the foundations of quantum cosmology.

On the credit side, the present method yields many interesting and concrete results in quantum cosmology. By limiting ourselves to conformal degrees of freedom we are able to formulate a quantization procedure which highlights the differences between quantum and classical cosmologies. For example, the properties of singularity and horizon of Friedmann cosmology are shown to be highly unlikely in the models obtainable in quantum cosmology. The quantum uncertainty in the form of conformal fluctuations diverges near the classical space-time singularity. This has been demonstrated under very general conditions in section 3.

We started this article with the analogy of the H-atom. In sections 4 and 5 we have demonstrated that the analogy holds in quantitative terms in the sense that in contrast to the singular solutions of classical theory nonsingular stationary states exist in quantum theory. The demonstration that quantum stationary states with characteristic linear size L_p exist for the universe reinforces our view of section 3 that classical cosmology is not reliable for characteristic sizes $\leq L_p$.

In section 5 we have given a simple working model for quantum gravity in which the conformal degree of freedom is treated quantum mechanically and the rest of the metric is treated classically. A system of quasiclassical field equations is obtained with the operator for the conformal variable replaced by its expectation value. A selfconsistent solution of the coupled equations is explicitly given, a solution which is singularity-free and horizon-free. We have ended with a brief description of how our approach may help resolve the flatness problem of classical cosmology.

Reasons can be given (cf. section 3) for singling out the conformal degree of freedom for quantum treatment. That scale transformation plays a special role in physics has been noted from time to time and in different contexts. Ours is one more in that list. To what extent nonconformal degrees of

freedom in space-time geometry are important to the overall problem of quantum gravity remains to be understood at present.

In short, this is a status report on a continuing series of investigations in quantum cosmology, with a few explicit demonstrations of the working of a specific technique in quantum gravity. Modest though its aims are, we hope that this technique will aid in our understanding of this complex subject.

References

- [1] A. Einstein, *Preuss. Akad. Wiss. Berlin, Sitzber.* (1915) 778 and 844.
- [2] D. Hilbert, *Konigl. Gesell. d. Wiss. Gottingen* (1915) p. 395.
- [3] G.W. Gibbons and S.W. Hawking, *Phys. Rev. D*15 (1977) 2738.
- [4] C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation* (Freeman & Co., San Francisco, 1973) Chap. 15.
- [5] J.A. Wheeler, in: *Battelle Rencontres* eds. C.M. DeWitt and J.A. Wheeler (Benjamin, New York, 1968).
- [6] J. Isenberg and J.A. Wheeler, in: *Relativity, Quanta and Cosmology*, eds. M. Pantaleo and F. de Finis (Johnson Reprint Corporation, New York, 1979) p. 267.
- [7] B.S. DeWitt, *Phys. Rev.* 160 (1967) 1113.
- [8] L.P. Eisenhart, *Riemannian Geometry* (Princeton University Press, Princeton, N.J., 1926).
- [9] H.P. Robertson, *Ap. J.* 82 (1935) 248; 83 (1935) 187.
- [10] A.G. Walker, *Proc. Lond. Math. Soc.* 42 (1936) 90.
- [11] A. Friedmann, *Z. Phys.* 10 (1922) 377.
- [12] G. Gamow, *Phys. Rev.* 70 (1946) 572.
- [13] R.A. Alpher and R.C. Herman, *Nature* 162 (1948) 774.
- [14] R.A. Alpher, H.A. Bethe and G. Gamow, *Phys. Rev.* 73 (1948) 80.
- [15] S. Weinberg, *Gravitation and Cosmology* (Wiley, 1972).
- [16] J.V. Narlikar, *Introduction to Cosmology* (Jones and Bartlett Publishers, Inc., 1983).
- [17] L. Bianchi, *Mem-di-Mat. Soc. Ital. Sci.* 11 (1897) 267.
- [18] M.A.H. MacCallum, in: *General Relativity – An Einstein Centenary Survey*, eds. S.W. Hawking and W. Israel (Cambridge, 1979).
- [19] M.P. Ryan L.C. Shepley, *Homogeneous Relativistic Cosmologies* (Princeton University Press, Princeton, New Jersey, 1975).
- [20] C.W. Misner, in: *Magic without magic*, ed. J. Klauder (Freeman, San Francisco, 1972).
- [21] S.W. Hawking and G.F.R. Ellis, *The large scale structure of the universe* (Cambridge, 1973).
- [22] V.A. Belinskii, I.M. Khalatnikov and E.M. Lifshitz, *Adv. Phys.* 19 (1972) 525.
- [23] V.A. Belinskii, I.M. Khalatnikov and E.M. Lifshitz, *Sov. Phys. JETP* 35 (1972) 838.
- [24] C.W. Misner, *Phys. Rev. Lett.* 22 (1969) 1071.
- [25] D.M. Chitre, Ph.D. Dissertation, University of Maryland (1972).
- [26] R.H. Dicke and P.J.E. Peebles, in: *General Relativity: An Einstein Centenary Survey*, eds. S.W. Hawking and W. Israel (Cambridge, 1979).
- [27] A.H. Guth, *Phys. Rev. D*23 (1981) 347.
- [28] J.D. Barrow and M.S. Turner, *Nature* 298 (1982) 801.
- [29] R.P. Feynman, *Acta Phys. Polon.* 24 (1963) 697.
- [30] S.W. Hawking, *Comm. Math. Phys.* 43 (1975) 199.
- [31] B.S. DeWitt, *Phys. Rev.* 162 (1967) 1195.
- [32] B.S. DeWitt, *Phys. Rev.* 162 (1967) 1239.
- [33] G.W. Gibbons and S.W. Hawking, *Phys. Rev. D*15 (1977) 2752.
- [34] S.W. Hawking, in: *General Relativity – An Einstein Centenary Survey*, eds. S.W. Hawking and W. Israel (Cambridge, 1979).
- [35] R. Arnowitt, S. Deser and C.W. Misner, *Phys. Rev.* 121 (1961) 1556.
- [36] R. Arnowitt, S. Deser and C.W. Misner, in: *Gravitation – An introduction to current research*, ed. L. Witten (John Wiley, New York, 1962).
- [37] R. Arnowitt, S. Deser and C.W. Misner, *Phys. Rev.* 122 (1961) 997.
- [38] R. Penrose, in: *Quantum Gravity – An Oxford Symposium*, eds. C.J. Isham, R. Penrose and D.W. Sciama (Oxford University Press, 1975).
- [39] G. Sparling, in: *Quantum Gravity – An Oxford Symposium*, eds. C.J. Isham, R. Penrose and D.W. Sciama (Oxford University Press, 1975).
- [40] R.P. Feynman, *Rev. Mod. Phys.* 20 (1948) 367.
- [41] R.P. Feynman and A.R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).
- [42] C. DeWitt, *Ann. Phys.* 97 (1976) 367.
- [43] L. Schiff, *Quantum Mechanics* (McGraw-Hill, New York, 1968).
- [44] J. Isenberg, Ph.D. Dissertation.
- [45] J.V. Narlikar, *Mon. Not. Roy. A. Soc.* 183 (1979) 159.
- [46] J.V. Narlikar, *Nature* 269 (1977) 129.

- [47] T. Padmanabhan, Ph.D. Dissertation (1982).
- [48] J.V. Narlikar, *Phys. Lett.* 96A (1983) 107.
- [49] J.V. Narlikar, *Gen. Rel. Grav.* 10 (1979) 883.
- [50] J.V. Narlikar, *Found. Phys.* 11 (1981) 473.
- [51] A.K. Kembhavi, *Mon. Not. Roy. A. Soc.* 185 (1978) 807.
- [52] T. Padmanabhan and J.V. Narlikar, *Nature* 295 (1982) 677.
- [53] T. Padmanabhan, *Phys. Rev. D*26 (1982) 2162.
- [54] J.V. Narlikar, *Found. Phys.* (1983) to appear.
- [55] T. Padmanabhan, Quantum conformal fluctuations and the stationary states, *Int. J. Theor. Phys.* (1983) to appear.
- [56] A. Maheshwari, *Phys. Lett.* 73A (1979) 295.
- [57] T. Padmanabhan, *Gen. Rel. Grav.* 13 (1981) 451.
- [58] T. Padmanabhan, T.I.F.R. Preprint (1981).
- [59] T. Padmanabhan and J.V. Narlikar, *Phys. Lett.* 84A (1981) 361.
- [60] T. Padmanabhan, *Phys. Lett.* 87A (1982) 226.
- [61] T. Padmanabhan, *Gen. Rel. Grav.* 14 (1982) 549.
- [62] T. Padmanabhan, Quantum conformal fluctuations and avoidance of singularities, submitted to *Jour. Class. Quan. Grav.* (1984).
- [63] T. Padmanabhan, Quantum cosmology and stationary states – Essay selected for ‘Honorable Mention’ in Gravity Essay Contest (1982).
- [64] T. Padmanabhan, Universe before Planck time, *Phys. Rev. D*28 (1983) 756.
- [65] T. Padmanabhan, *Phys. Lett.* 93A (1983) 116.
- [66] J.V. Narlikar and T. Padmanabhan, The problems of singularity, particle horizon and flatness in quantum cosmology, *Ann. Phys.* (1983) to appear.
- [67] B.S. DeWitt, in: *General relativity – An Einstein Centenary Survey*, eds. S.W. Hawking and W. Israel (Cambridge, 1979).