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On the semiclassical limit of the Wheeler–DeWitt equation

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Abstract. We continue our investigation of approximation schemes for obtaining semiclassical Einstein equations with a backreaction, starting from the Wheeler–DeWitt equation. The analysis is carried out using a toy model with two degrees of freedom, which represents a matter field interacting with gravity. We argue that the backreaction is to be found using the phase of the matter part of the wavefunction. Using a semiclassical Wigner function we find the general condition for the validity of a semiclassical theory: the dispersion in the metric derivative of the phase of the matter wavefunction should be negligible. We then consider a special case of the toy Lagrangian, that of a time-dependent harmonic oscillator, and show that the backreaction is equal to the expectation value of the matter Hamiltonian only if the background 'metric' varies slowly with time. The Wigner function, when applied to a semiclassical cosmological model, shows that the semiclassical approximation is valid only when the quantum contribution to the energy-momentum tensor is small compared to the classical contribution.

1. Introduction

The quantised version of general relativity may be formally expressed through the operator constraints, one of them being the Wheeler–DeWitt equation for the wavefunctional $\Psi[g, f]$ of the metric g and the matter field f. At energies well below the Planck scale, the combined system of matter and gravity is expected to obey the semiclassical equations:

$$i\hbar \frac{\partial \psi(f/g)}{\partial t} = \hat{h}\psi(f/g) \tag{1}$$

$$G_{ik}(g) = 8\pi G \langle \psi(f) | T_{ik} | \psi(f) \rangle. \tag{2}$$

Here gravity is being treated as a classical field and $\psi(f/g)$ is the matter wavefunction in the given background metric g. These represent, of course, quantum field theory in curved space, and semiclassical Einstein equations with a possible backreaction.

If (1) and (2) are the correct description of the gravity–matter system in some range of energy, it should be possible to devise an approximation scheme which yields these equations, starting from the operator equations of quantum general relativity. If f is treated as a *test field* (no backreaction), it is possible to derive equation (1) (field theory in a fixed background), as a limit of these equations (see, e.g., Lapchinsky and Rubakov 1979, Banks 1985, also see Gerlach 1969). However, the problem of obtaining (1) and (2) as a coupled system which incorporates backreaction is more tricky. It was first discussed by Hartle (1987) and pursued by Halliwell (1987) and Padmanabhan (1989).

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We now give a brief summary of previous work on this issue, and the associated difficulties. It is very convenient to work with a quantum mechanical toy model which mimics the relevant features of the gravity+matter system. It consists of a 'heavy' particle Q (gravity) and a 'light' particle q (matter), and is described by the Lagrangian

$$L=\tfrac{1}{2}M\dot{Q}^2-MV(Q)+\tfrac{1}{2}m\dot{q}^2-u(q,Q). \tag{3}$$

Starting from the Schrödinger equation for the wavefunction $\psi(q,Q)$ describing this system one would like to arrive at the semiclassical equations

$$M\ddot{Q}=-M\frac{\mathrm{d}V}{\mathrm{d}Q}-\frac{\partial}{\partial Q}\langle\chi|h(q,Q)|\chi\rangle \tag{4}$$

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\chi(q;Q)=h\chi=\left(-\frac{1}{2m}\frac{\partial^2}{\partial q^2}+u(q,Q)\right)\chi(q;Q) \tag{5}$$

in some systematic approximation. Here Q is a classical quantity, and the expectation value of the Hamiltonian h is evaluated in a quantum state $\chi(q;Q)$ for q in the background $Q(t)$. To obtain the semiclassical limit, the wavefunction $\psi(q,Q)$ is written as

$$\psi(q,Q)=\exp(\mathrm{i}S(q,Q)) \tag{6}$$

and substituted in the Schrödinger equation corresponding to the Lagrangian (3). The exponent is expanded as a power series in M, with the leading term proportional to M. The semiclassical limit is the limit $M\to\infty$, and it is analogous to the limit $G\to 0$ in the Wheeler-DeWitt equation. The following results are obtained: to the leading order in M we get the Hamilton-Jacobi equation for $Q(t)$, but without a backreaction (this is equivalent to (4), with $\langle h\rangle$ set as zero). To the next order in M, we get the Schrödinger equation (5) for $\chi(q)$ (for details, see Padmanabhan 1989, hereafter referred to as I).

From the analysis carried out in the earlier work, a clear picture regarding the calculation of backreaction does not emerge. In the present paper we argue that the backreaction is to be found using the phase of the matter wavefunction. We then define a semiclassical Wigner function to find the conditions for validity of the semiclassical theory. The results obtained from the Wigner function are then applied to a model from quantum mechanics and to a minisuperspace model from quantum cosmology.

2. Backreaction in the WKB semiclassical approximation

By substituting the form (6) for the wavefunction in the time independent Schrödinger equation

$$-\frac{\hbar^2}{2M}\frac{\partial^2\psi}{\partial Q^2}+MV(Q)\psi-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial q^2}+u(q,Q)\psi=E\psi(q,Q) \tag{7}$$

corresponding to the Lagrangian (3) we get

$$\frac{S'^2}{2M}+\frac{S_q^2}{2m}+MV(Q)+u(q,Q)-\frac{\mathrm{i}\hbar^2}{2M}S''-\frac{\mathrm{i}\hbar^2}{2m}S_{qq}=E \tag{8}$$

($S'=\partial S/\partial Q$, $S_q=\partial S/\partial q$).

To obtain the semiclassical limit we expand $S(q, Q)$ as a power series in M:

$$S(q, Q) = MS_0(q, Q) + S_1(q, Q) + M^{-1}S_2(q, Q) + \dots \tag{9}$$

and separate the terms at different orders in M. Since one half of the Lagrangian scales with M, we assume that the energy E also scales with M and write it as $E = M\varepsilon$.

Substituting (9) in (8) gives, at order M^2

$$(\partial S_0/\partial q)^2 = 0 \tag{10}$$

from which we conclude that S_0 *does not depend on* q. By comparing the terms at order M we conclude that

$$M\varepsilon = \tfrac{1}{2}MS_0'^2(Q) + MV(Q) \tag{11}$$

and that $S_0(Q)$ is real. At this order the wavefunction is

$$\psi(Q) = \exp(\mathrm{i}S_0/\hbar). \tag{12}$$

Equation (11) is the Hamilton-Jacobi (HJ) equation for the Q mode and the defining equation for the classical momentum $MS_0'(Q)$ of the Q mode. From this equation we can write down an equation of motion for Q. The validity of the HJ equation requires that

$$\left|\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\hbar}{MS_0'}\right)\right| \ll 1 \tag{13}$$

which is a statement about the de Broglie wavelength of the Q mode. If this condition is satisfied we can conclude that the Q mode is behaving classically. Taking the $M \to \infty$ limit is equivalent to using the WKB approximation for Q, while leaving q quantum mechanical. This can also be verified with the help of the Wigner function, as discussed below.

To get information about the motion of q we look at the higher order terms in the Schrödinger equation. It can be shown that at order M^0, (7) gives the equation

$$\mathrm{i}\hbar S_0' \frac{\partial f(q, Q)}{\partial Q} = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} + u(q, Q)\right) f(q, Q) \tag{14}$$

where the function $f(q, Q)$ is defined as

$$f(q, Q) = \sqrt{S_0'}\exp(\mathrm{i}S_1/\hbar). \tag{15}$$

Thus at order M^0 the wavefunction of the system is

$$\psi(q, Q) = \frac{1}{\sqrt{S_0'}}\exp(\mathrm{i}MS_0(Q)/\hbar) f(q, Q). \tag{16}$$

The equation satisfied by the function f can be interpreted as the Schrödinger equation for the quantum mechanical mode q, in an external classical background Q, and in a 'time coordinate' τ defined by the relation

$$\frac{\partial}{\partial \tau} = S_0' \frac{\partial}{\partial Q}. \tag{17}$$

The momentum for the classical mode Q was defined as the Q derivative of the phase MS_0 in the leading wavefunction (12). To find the backreaction of the q mode on Q it is reasonable to assume that at order M^0 the momentum for Q should be defined as the Q derivative of the phase of the wavefunction (16). Thus we write f as a real amplitude $R(q, Q)$ times a phase $\exp(\mathrm{i}\beta(q, Q)/\hbar)$, so that the wavefunction (16) reads

$$\psi(q, Q) = \frac{1}{\sqrt{S_0'}} \exp[(\mathrm{i}MS_0(Q) + \mathrm{i}\beta(q, Q))/\hbar] R(q, Q). \tag{18}$$

Note that $\beta(q, Q)$ does not involve M, and we may think of this phase as a perturbation on the leading phase $S_0(Q)$.

If we define the classical momentum (P) for Q as the derivative of the total phase we get

$$P = M\frac{\partial S_0}{\partial Q} + \frac{\partial \beta(q, Q)}{\partial Q}. \tag{19}$$

However, this cannot be correct as such, because the left-hand side should be classical, while the right-hand side depends on a quantum variable q. The simplest way to get rid of the dependence on q is to replace $(\partial\beta/\partial Q)$ by its expectation value

$$\langle f|(\partial\beta/\partial Q)|f\rangle \tag{20}$$

in the state $f(q, Q)$ which satisfies the Schrödinger equation (14). It can be shown from a general Schrödinger equation that if the wavefunction ψ is a solution of the equation, then the phase $S(t)$ of ψ and the Hamiltonian of the system are related as $\langle\psi|\dot{S}|\psi\rangle = -\langle\psi|h|\psi\rangle$. In the present context this implies the relation

$$\left\langle f\left|\left(\frac{\partial\beta}{\partial Q}\right)\right|f\right\rangle = -\frac{1}{S_0'}\langle f|h(q, Q)|f\rangle \qquad h = \tfrac{1}{2}m\dot{q}^2 + u(q, Q) \tag{21}$$

and hence that

$$P^2/2M + MV(Q) + \langle f|h|f\rangle = E. \tag{22}$$

Here we have dropped a term of order M^{-1} and used the HJ equation (11) for S_0'.

Equation (22) is an HJ equation for Q; it incorporates the backreaction through $\langle h\rangle$, and is a possible description of the semiclassical theory. In gravity this corresponds to coupling the metric to the expectation value of T_{ik}. The trouble with this semiclassical theory, of course, is that it is *ad hoc*. It does not say why the derivative of the phase is replaced by its expectation value. The criterion for such a replacement is found with the help of a semiclassical Wigner function which helps us decide when a semiclassical theory can be defined, and what the backreaction in such a theory is.

Given a quantum state, the Wigner function defines a probability distribution in phase space, and if this distribution shows a strong correlation between position and momentum, the system can be said to behave quasiclassically in that state. (For a simple discussion of the Wigner function relevant in the present context, see Halliwell (1987).) The standard Wigner function for our system (3) will be defined as

$$F(Q, P, q, p) = \int_{-\infty}^{\infty} \mathrm{d}u\, \mathrm{d}v\, \psi^*(Q - \tfrac{1}{2}\hbar u, q - \tfrac{1}{2}\hbar v) \exp(-\mathrm{i}Pu - \mathrm{i}pv)\psi(Q + \tfrac{1}{2}\hbar u, q + \tfrac{1}{2}\hbar v). \tag{23}$$

Before we put this function to any use we must ask about the relation of its definition to the purpose of its construction. The parameter $\hbar$ appears in the Wigner function in this particular form because we have in mind the $\hbar \to 0$ limit. Recall, for example, that to interpret the Wigner function for the WKB quasiclassical state, we expand the state in powers of $\hbar$. Since the semiclassical limit is the limit $M \to \infty$ and not the limit $\hbar \to 0$ it is reasonable to construct a modified Wigner function which involves a parameter M^{-1}. For a single degree of freedom this function F_M may be defined as

$$F_M(Q, P) = \frac{1}{M} \int_{-\infty}^{\infty} du\, \psi^*\left(Q - \frac{\hbar}{2M} u\right) \exp(-iPu/M) \psi\left(Q + \frac{\hbar}{2M} u\right). \tag{24}$$

This function is obtained by just a redefinition of the variable of integration u in the ordinary Wigner function, and hence preserves the standard properties of the Wigner function. Thus it is really the same function as before. However, it is useful to write it like this and take the limit $M \to \infty$ in the wavefunction when the need arises. Formally the limit $M \to \infty$ will coincide with the limit $\hbar \to 0$ in this definition. That should be expected because of the equivalence of $\hbar \to 0$ and $M \to \infty$ for a single particle moving in a potential $MV(Q)$. The difference arises when we try to define a modified Wigner function for two particles. Thus for the two-particle system we define the modified Wigner function as

$$\begin{aligned} F_M(Q, P, q, p) &= \frac{1}{Mm} \int_{-\infty}^{\infty} du\, dv\, \psi^*\left(Q - \frac{\hbar}{2M} u, q - \frac{\hbar}{2m} v\right) \\ &\quad \times \exp\left[-i\left(\frac{Pu}{M}\right) - i\left(\frac{pv}{m}\right)\right] \psi\left(Q + \frac{\hbar}{2M} u, q + \frac{\hbar}{2m} v\right) \end{aligned} \tag{25}$$

where M and m are the masses of the particles respectively. We see that now the limits $M \to \infty$ and the limit $\hbar \to 0$ are not equivalent. However, this function is obtained from (23) by a redefinition of u and v, and hence preserves the properties of the Wigner function.

We now calculate F_M for specific states. Of course our interest is in the states we have been using to define the semiclassical limit of the two-particle Lagrangian (3). Consider first the state (12), which is obtained at order M^1 in the expansion in powers of M. This state does not depend on q and for it F_M is equal to

$$\begin{aligned} F_M(Q, P, p) &= \frac{1}{M} \delta(p) \int_{-\infty}^{\infty} du \exp[-i(Pu/M)] \\ &\quad \times \exp\left[iMS_0\left(Q + \frac{\hbar}{2M} u\right) \hbar^{-1} - iMS_0\left(Q - \frac{\hbar}{2M} u\right) \hbar^{-1}\right]. \end{aligned} \tag{26}$$

By expanding the wavefunction in powers of $(1/M)$ and retaining up to order M^0 we get

$$F_M = \delta(p)\delta(P - MS_0'(Q)) \tag{27}$$

which gives the expected correlation between the position and momentum of Q, and hence indicates that Q is behaving classically at this order of approximation.

Now consider the wavefunction at order M^0, namely equation (16). It is convenient to write it as an amplitude times a phase, as in (18). The Wigner function for this state is

$$
\begin{aligned}
&F_{\mathrm{M}}(Q, P, q, p) \\
&\quad = \frac{1}{Mm} \int_{-\infty}^{\infty} \mathrm{d}u \, \mathrm{d}v \frac{1}{[S_0'(Q + hu/2M)]^{1/2}} \frac{1}{[S_0'(Q - hu/2M)]^{1/2}} \\
&\quad \times R\left(Q - \frac{\hbar}{2M} u, q - \frac{\hbar}{2m} v\right) R\left(Q + \frac{\hbar}{2M} u, q + \frac{\hbar}{2m} v\right) \\
&\quad \times \exp\left[-\mathrm{i}\left(\frac{Pu}{M}\right) - \mathrm{i}\left(\frac{pv}{m}\right)\right] \\
&\quad \times \exp\left(\frac{\mathrm{i}}{\hbar}\right)\left[\beta\left(Q + \frac{\hbar}{2M} u, q + \frac{\hbar}{2m} v\right) - \beta\left(Q - \frac{\hbar}{2M} u, q - \frac{\hbar}{2m} v\right)\right] \\
&\quad \times \exp\left(\frac{\mathrm{i}M}{\hbar}\right)\left[S_0\left(Q + \frac{\hbar}{2M} u\right) - S_0\left(Q - \frac{\hbar}{2M} u\right)\right].
\end{aligned} \tag{28}
$$

In this function q is a quantum mechanical degree of freedom at all orders, and we do not expect a correlation between the position and momentum of q. On the other hand, we expect a correlation between Q and P, in some approximation. To check for such a correlation we should obtain a marginal probability distribution $F_{\mathrm{M}}(Q, P)$ by integrating $F_{\mathrm{M}}(Q, P, q, p)$ over q and p. That is, we restrict ourselves to the Q, P section of the phase space. Integrating the F_{M} of (28) over p and q gives

$$
\begin{aligned}
F_{\mathrm{M}}(Q, P) &= \frac{1}{M} \int_{-\infty}^{\infty} \mathrm{d}q \, \mathrm{d}u \frac{1}{[S_0'(Q + hu/2M)]^{1/2}} \frac{1}{[S_0'(Q - hu/2M)]^{1/2}} \\
&\quad \times R\left(Q - \frac{\hbar}{2M} u, q\right) R\left(Q + \frac{\hbar}{2M} u, q\right) \\
&\quad \times \exp\left[-\mathrm{i}\left(\frac{Pu}{M}\right)\right] \exp\left(\frac{\mathrm{i}}{\hbar}\right)\left[\beta\left(Q + \frac{\hbar}{2M} u, q\right) - \beta\left(q - \frac{\hbar}{2M} u, q\right)\right] \\
&\quad \times \exp\left(\frac{\mathrm{i}M}{\hbar}\right)\left[S_0\left(Q + \frac{\hbar u}{2M}\right) - S_0\left(Q - \frac{\hbar u}{2M}\right)\right].
\end{aligned} \tag{29}
$$

Next, we expand R, β and S_0 in powers of M^{-1} and retain only the leading terms. Such an expansion is equivalent to taking the limit $M \to \infty$ in the wavefunction. The result is

$$
F_{\mathrm{M}}(Q, P) = \frac{1}{S_0'(Q)} \int_{-\infty}^{\infty} \mathrm{d}q \, R^2(Q, q) \delta[P - MS_0'(Q) - \beta'(Q, q)]. \tag{30}
$$

This probability distribution can be interpreted as follows. If the particle q is at a well-defined position, it will contribute an amount $\partial\beta/\partial Q$ to the momentum for Q. However, the probability for it to be at q is $R^2(q, Q)$, and hence we average over all positions for q. We can say that a semiclassical theory holds if this averaging yields a correlation between Q and P. It then also becomes clear that the existence of a semiclassical limit depends on the quantum state for q and that the phase determines the backreaction.

To find out when a correlation exists between Q and P we use the integral representation of the Wigner function to rewrite (30) as

$$F_{\mathrm{M}}(Q, P)=\frac{1}{2\pi S_0'}\int_{-\infty}^{\infty} \mathrm{d}\lambda\ \exp(-\mathrm{i}\lambda[P-MS_0'])\int_{-\infty}^{\infty} \mathrm{d}q\, R^2(q, Q)\exp(\mathrm{i}\lambda\,\beta'(q, Q)). \tag{31}$$

It is easy to see that in general, the integral over q will not give the result $\exp[-\mathrm{i}\lambda\langle\beta'\rangle]$. It is also easy to see that the sufficient condition for obtaining this result from the integration is

$$\langle\beta'^n\rangle=\langle\beta'\rangle^n \qquad n=2, 3, \ldots. \tag{32}$$

If these conditions hold then we get from (31) that

$$F_{\mathrm{M}}(Q, P)=\frac{1}{S_0'}\,\delta(P-MS_0'-\langle f|\beta'|f\rangle). \tag{33}$$

This implies that the momentum P is peaked at the value which will lead to the semiclassical Einstein equations, as may be seen using (21) and (22). We can now conclude that if the dispersion conditions (32) hold, a semiclassical theory can be defined, and the source for backreaction is $\langle h\rangle$. It is very likely that the dispersion conditions are not only sufficient, but also necessary for a semiclassical theory to be definable. This is justified with an example below.

For a generic quantum state, the dispersion requirements can never be met exactly but only approximately. In other words the dispersion in β' should be small. The most relevant of these conditions is the leading one ($n=2$), that is

$$\langle\beta'^2\rangle=\langle\beta'\rangle^2. \tag{34}$$

We thus conclude that for obtaining the backreaction as $\langle h\rangle$ the dispersion in the metric derivative of the phase should be small. (Similar suggestions have been made earlier by Ford (1982) and Halliwell (1987).)

It may be shown from the Schrödinger equation that (34) is equivalent to requiring

$$\langle h\rangle^2-\langle h^2\rangle=\hbar^2\langle\ddot{R}/R\rangle \tag{35}$$

where h is the Hamiltonian for q. This is not quite the requirement that the dispersion in $\langle h\rangle$ should be small, but a relation between the dispersion and the time variation of R. An interesting special case is that of the stationary state—in this case the amplitude R is a constant, and $\langle h^2\rangle$ is identically equal to $\langle h\rangle^2$, so that the requirement (34) is naturally met. The same is true of a quasi-stationary state for which the wavefunction is of the form

$$f(q, t)=\exp\left(-\mathrm{i}\int_0^t E(t')\,\mathrm{d}t'/\hbar\right)\xi(q, t). \tag{36}$$

Here $\xi(q, t)$ is assumed to be the stationary eigenfunction of the instantaneous Hamiltonian:

$$h(q, t)\xi(q, t)=E(t)\xi(q, t). \tag{37}$$

It is possible to show (Schiff 1968) under the adiabatic approximation that if the initial state is a quasi-stationary state as above, the amplitude for transition to a state other than the initial state is oscillatory, and does not increase with time. Thus the leading time dependence of $f(t)$ is in the phase $(-\int E(t')\,\mathrm{d}t')$ and the dispersion condition is again satisfied. The dispersion conditions are natural requirements on a quantum state of the matter field, so that a semiclassical theory may exist. However, contrary to an intuitive guess, they are not directly a constraint on the dispersion in the Hamiltonian (see (35) above) but a constraint on the metric dependence of the phase. As we will show below, these conditions are quite powerful, and may incorporate a wide variety of physical situations which are a prerequisite for a semiclassical theory.

3. The example of the harmonic oscillator

We now consider the example of a harmonic oscillator q whose frequency depends on Q, and the interaction potential is

$$u(q, Q)=\tfrac{1}{2}\omega^2(Q)q^2. \tag{38}$$

We shall consider a particular solution of the Schrödinger equation (14) in this potential and for convenience convert all functions of Q into functions of τ, using (17). We assume that the initial state for q is a Gaussian of the form

$$f(q, \tau)=N(t)\exp[-B(\tau)q^2+\mathrm{i}\theta(\tau)/\hbar] \tag{39}$$

where $N(\tau)$ and $\theta(\tau)$ are real functions of τ, while $B(\tau)$ is complex. The mean position and mean momentum of the q mode have been set as zero in this state. Essentially, this state is the ground state of the oscillator. Since the potential is quadratic, it propagates this state to another Gaussian. Thus at all times the state is of the form (39).

Using the form (39) in the Schrödinger equation for q and comparing different powers of q gives the equations

$$\begin{aligned}\dot{\theta}&=-(\hbar^2/m)B_{\mathrm{R}}\\ \dot{B}_{\mathrm{R}}&=(4\hbar/m)B_{\mathrm{R}}B_{\mathrm{I}}\\ \dot{B}_{\mathrm{I}}&=-(2\hbar/m)(B_{\mathrm{R}}^2-B_{\mathrm{I}}^2)+\omega^2/2\hbar.\end{aligned} \tag{40}$$

A dot indicates a derivative with respect to τ. B_{R} and B_{I} are respectively the real and imaginary parts of B.

The expectation value of the Hamiltonian $h(q)$ in this state is

$$\langle f|h(q)|f\rangle=\frac{\omega^2}{8R}+\frac{\hbar^2}{2mB_{\mathrm{R}}}(B_{\mathrm{R}}^2+B_{\mathrm{I}}^2). \tag{41}$$

It is easy to check that this is equal to the expectation value of the time derivative of the phase $(\hbar B_{\mathrm{I}}q^2+\theta)$, up to a sign

$$\langle f|h(q)|f\rangle=-\langle f|(\hbar\dot{B}_{\mathrm{I}}q^2+\dot{\theta})|f\rangle. \tag{42}$$

To find the backreaction in this state we compute the modified Wigner function F_{M} of (31) for it. By substituting expression (39) in (31) and doing the q integration we get

$$F_{\mathrm{M}}(Q, P)=\frac{1}{2\pi S_0'}\int_{-\infty}^{\infty}\mathrm{d}\lambda\left(1+\frac{\mathrm{i}\lambda B_{\mathrm{I}}'}{2B_{\mathrm{R}}}\right)^{-1/2}\exp[-\mathrm{i}\lambda(P-MS_0'-\theta')]. \tag{43}$$

If B_{I}' is zero, we find that a semiclassical theory can be defined because F_{M} is a delta function and the momentum P is peaked at the value $MS_0'+\theta'$. Hence we can write

$$P=MS_0+\theta'. \tag{44}$$

Upon squaring this equation, and using (42) we conclude that the backreaction is equal to $\langle h\rangle$. From the equation for B_I in (40) we find that setting B_I' equal to zero implies

$$\omega^2=(4\hbar^2/m)(B_R^2-B_I^2)=(4\hbar^2/m)(R_0^2\exp(8B_0\tau)-B_0^2). \tag{45}$$

Here, B_0 is the constant value of B_I. In particular, if B_0 is zero, we get that ω is a constant. This equation is a condition on the time dependence of the frequency if the semiclassical theory is to hold. We also notice from (43) that B_I' as zero is a *necessary* condition for the semiclassical theory to be valid. As we see below, this is equivalent to saying that the dispersion conditions are not only sufficient, but also necessary.

What is more interesting is the case when B_I' is not zero, but small compared with B_R, so that in (43) we may write

$$(1+\mathrm{i}\lambda B_I'/2B_R)^{-1/2}\simeq\exp(-\mathrm{i}\lambda B_I'/4B_R). \tag{46}$$

In this case also, (43) implies that the momentum is peaked, but now at the value

$$P=MS_0'+\theta'-B_I'/4B_R. \tag{47}$$

One can again show, using the equations of motion in (40), that the backreaction is equal to $\langle h\rangle$.

The backreaction equals $\langle h\rangle$ when $\dot{B}_I$ is exactly zero, and it is approximately equal to $\langle h\rangle$ if $\dot{B}_I$ is nearly zero. From the equation for B_I in (40) we see that if $\dot{B}_I$ is always to be small compared with B_I then we require B_I to be nearly constant at a value which is much smaller than B_R, and that

$$\omega\simeq(2\hbar/\sqrt{m})B_R. \tag{48}$$

This equation implies that

$$\dot{\omega}/\omega\ll(4\hbar/m)\omega \tag{49}$$

which means that if the frequency ω changes very slowly with time, the backreaction is equal to $\langle h\rangle$. For gravity this suggests that the backreaction is T_{ik} if the metric varies adiabatically.

It can also be shown that for the ground state of the oscillator considered above, the dispersion condition (34) will hold if B_I' is zero, and will hold approximately if B_I' is nearly zero. To do that we go back to (39) which defines the ground state and use the phase to compute $\langle\beta'\rangle$ and $\langle\beta'^2\rangle$. We find that

$$\langle\beta'^2\rangle-\langle\beta'\rangle^2=B_I'^2[\langle q^4\rangle-\langle q^2\rangle^2]=B_I'^2/8B_R^2. \tag{50}$$

Thus we find that the adiabatic approximation for B_I is a special case of the approximation of small dispersion. This is interesting because it shows a simple connection between adiabaticity and small dispersion, both of which look natural requirements for defining a semiclassical theory.

The backreaction equation for Q will read

$$M\ddot{Q}=-M\frac{\mathrm{d}V}{\mathrm{d}Q}-\frac{\mathrm{d}}{\mathrm{d}Q}\langle h(q)\rangle. \tag{51}$$

In the present example, we require that the frequency should change adiabatically; this implies the consistency condition that $\dot{Q}$ should be sufficiently small. We may illustrate this in the following way. We write the first integral for (51) as

$$\tfrac{1}{2}M\dot{Q}^2=M\varepsilon-V(Q)-\langle h\rangle. \tag{52}$$

Assume that $V(Q)$ has a local minimum at $Q=0$, that the energy $M\varepsilon$ is sufficiently small and that $Q(t)$ is executing small oscillations around $Q=0$ before the backreaction is switched on. A small $\dot{Q}$ then ensures that

$$\dot{\omega} = \dot{Q}\, \mathrm{d}\omega/\mathrm{d}Q \ll \omega^2(Q). \tag{53}$$

When the backreaction is taken as $\langle h \rangle$, it should not destabilise the minimum of $V(Q)$, because if it did, it would imply a large $\dot{Q}$. Since $\langle h \rangle \simeq \frac{1}{2}\hbar\omega(Q)$, this is clearly a statement as to which $\omega(Q)$ are allowed. For example, if $V(Q)=Q^2$, we are allowed $\omega(Q)=(Q-Q_0)^2$, because it only shifts the minimum of $V(Q)$ to another point off the origin. If $V(Q)$ is of higher order, say $V(Q)=Q^4+Q^3$ (only one minimum), one can construct a sensible $\omega(Q)$ which will create an additional minimum ($\omega = \frac{5}{32}Q^2$ works!). It can then cause tunnelling to the new minimum. Requirements of self-consistency thus constrain the choice of $\omega(Q)$.

4. Semiclassical general relativity: a minisuperspace example

The semiclassical theory developed above can be generalised to the Wheeler-DeWitt equation in a straightforward manner. The limit $M \to \infty$ for the toy model corresponds to the limit $G \to 0$ in the Wheeler-DeWitt equation, where G is Newton's gravitational constant (for a comparison of the toy model with the Wheeler-DeWitt equation see I). It can thus be deduced that semiclassical general relativity with a backreaction can be defined if the dispersion in the metric derivative of the phase of the matter wavefunctional is small. The source term then would be the expectation value $\langle T_{ik} \rangle$.

Here we study the validity of a semiclassical cosmological model whose self-consistent solutions were worked out by us in an earlier paper (Singh and Padmanabhan 1987). The model consists of a free, massless, homogeneous scalar field in a $K=+1$ Robertson-Walker universe, written in the conformal time as

$$\mathrm{d}s^2 = \Omega^2(\tau)\left(\mathrm{d}\tau^2 - \frac{\mathrm{d}r^2}{1-r^2} - r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\Phi^2)\right). \tag{54}$$

The action for the system is

$$A = \int \mathrm{d}t[M(-\dot{\Omega}^2 + \Omega^2) + \pi^2\Omega^2\dot{\phi}^2] \tag{55}$$

where $M = 3\pi/4G$. In this section, a dot denotes a derivative with respect to the conformal time τ.

The classical equations of motion, obtained by varying the action (55) with respect to Ω and ϕ are

$$M\ddot{\Omega} + M\Omega = -\pi^2\Omega\dot{\phi}^2 \tag{56}$$

$$2\pi^2 \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\Omega^2 \frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right) = 0. \tag{57}$$

The Hamiltonian constraint can be obtained by demanding the invariance of the action under time reparametrisation

$$H = M\dot{\Omega}^2 + M\Omega^2 - \pi^2\Omega^2\dot{\phi}^2 = 0. \tag{58}$$

The equation of motion for ϕ can be obtained from (56) and (58), and (56) can be obtained from (57) and (58). For convenience, we shall treat (57) and (58) as the independent equations. These equations can also be written down by starting from the Einstein equations and substituting for the metric and the T_{ik} for the field.

The conjugate momentum for ϕ is

$$\pi_\phi = 2\pi^2\Omega^2\dot{\phi} \tag{59}$$

and hence the Hamiltonian $h(\phi)$ is

$$h(\phi) = \pi_\phi^2/4\pi^2\Omega^2. \tag{60}$$

The classical solution to the gravity-scalar system is given by

$$\Omega^2 = \Omega_0^2 \sin[2(\tau+\tau_0)] \tag{61}$$

$$\phi = \tfrac{1}{2}\ln\left[\left|\frac{\tan(\tau+\tau_0)}{\tan\tau_0}\right|\right] + \phi_0. \tag{62}$$

Here Ω_0^2, τ_0, and ϕ_0 are constants. Since Ω^2 is a positive quantity, the argument of the sine function lies in the range $(0, \pi)$. The solution for the conformal factor is a periodic universe. We shall assume that the evolution of the universe can be described by a semiclassical theory, starting at a time $\tau = 0$, which is sufficiently removed from the initial singularity.

The semiclassical system can be derived from the quantum theory in the manner outlined in § 2. It is described by the classical function $\Omega(\tau)$ and a wavefunction $f(\phi, \tau)$ for the scalar field. The semiclassical equations for this system are

$$M\ddot{\Omega} + M\Omega = -\pi^2\Omega\langle f|\dot{\phi}^2|f\rangle \tag{63}$$

$$\mathrm{i}\hbar\frac{\partial f}{\partial\tau} = \hat{h}(\phi)f(\phi,\tau) = -\frac{\hbar^2}{4\pi^2\Omega^2}\frac{\partial^2}{\partial\phi^2}f(\phi,\tau). \tag{64}$$

A self-consistent solution is found if we choose ϕ to be in the general Gaussian state

$$f(\phi,\tau) = N(\tau)\exp[-B(\phi-\bar{\phi})^2 + \mathrm{i}p(\tau)\phi/\hbar + \mathrm{i}\varepsilon(\tau)/\hbar]. \tag{65}$$

Here B is a complex function of τ, while N, p, $\bar{\phi}$ and ε are real functions of τ. Corresponding to the wavefunction in (65) the probability density is given by

$$|f(\phi,\tau)|^2 = N^2(\tau)\exp\left(-\frac{(\phi-\bar{\phi})^2}{2\sigma^2}\right) \tag{66}$$

where

$$\sigma^2 = \frac{1}{2(B+B^*)} \tag{67}$$

is the dispersion of the Gaussian. Thus $\bar{\phi}$ is the mean value for ϕ, while p is the mean momentum in this state.

Substituting the form (65) for f in the Schrödinger equation (64) we get the following equations for the various parameters in the wavefunction:

$$\dot{B} = -\frac{\mathrm{i}\hbar B^2}{\pi^2\Omega^2} \qquad \dot{N} = \frac{\hbar}{2\pi^2\Omega^2}NB_{\mathrm{I}} \qquad -\dot{\varepsilon} = \frac{\hbar^2}{2\pi^2\Omega^2}B_{\mathrm{R}} + \frac{p^2}{4\pi^2\Omega^2} \tag{68}$$
$$p = 2\pi^2\Omega^2\dot{\bar{\phi}} \qquad \dot{p} = 0.$$

We have written B in terms of its real and imaginary parts, $B = B_{\mathrm{R}} + \mathrm{i}B_{\mathrm{I}}$.

The equation for B in (68) may be simplified by defining a new variable χ by the relation:

$$B \equiv -\frac{i\pi^2\Omega^2}{\hbar}\frac{\dot{\chi}}{\chi} \tag{69}$$

which shows that χ satisfies the equation

$$\frac{d}{d\tau}\left(\Omega^2\frac{d\chi}{d\tau}\right)=0. \tag{70}$$

Note that no factor $\hbar$ appears in the equation for χ, which may be solved to get

$$\chi(\tau)=\chi_1+\chi_2 F(\tau) \tag{71}$$

where χ_1 and χ_2 are constants, and

$$F(\tau)=\int_0^\tau d\tau'/\Omega^2(\tau'). \tag{72}$$

Note that $F(\tau)$ is a real function. Using the equation relating χ and B we get that

$$B(\tau)=-\frac{i\pi^2}{\hbar}\frac{1}{\chi+F(\tau)} \tag{73}$$

where $\chi_0=\chi_1/\chi_2$.

We impose the condition that $B(\tau=0)$ is real, which implies that χ_0 is an imaginary number. Defining

$$\sigma_0^2=\frac{i\hbar}{4\pi^2}\chi_0 \tag{74}$$

we can write $B(\tau)$ as

$$B(\tau)=\frac{\sigma_0^2-(i\hbar/4\pi^2)F(\tau)}{\sigma_0^4+(\hbar/4\pi^2)^2F^2(\tau)}. \tag{75}$$

Thus the dispersion σ^2 evolves as

$$\sigma^2(\tau)=\frac{\sigma_0^2}{4}+\frac{1}{4\sigma_0^2}\left(\frac{\hbar}{4\pi^2}\right)^2F^2(\tau). \tag{76}$$

Since σ_0^2 is the initial value of the dispersion, we choose it to be a positive quantity. The momentum p is a constant, say p_0, and the mean value $\bar{\phi}$ evolves as

$$\bar{\phi}=\bar{\phi}_0+\frac{p_0}{2\pi^2}F(\tau). \tag{77}$$

The expectation value of $\dot{\phi}^2$ in the Gaussian state (65) is

$$\langle f|\dot{\phi}^2|f\rangle=\frac{1}{4\pi^4\Omega^4}\left(p^2+\hbar^2\frac{|B|^2}{B_R}\right). \tag{78}$$

It can be rewritten using the above results as

$$\langle\dot{\phi}^2\rangle=\frac{1}{4\pi^4\Omega^4}\alpha^2 \tag{79}$$

where α^2 is a constant given by

$$\alpha^2=(p_0^2+\hbar^2/\sigma_0^2). \tag{80}$$

Note that the expectation value of $\dot{\phi}^2$ has the form of (classical value+quantum correction).

We now want to find out how the quantum mode ϕ affects the evolution of the conformal factor $\Omega(\tau)$ in (63). This will provide a self-consistent solution of the semiclassical theory. Using the expression for $\langle\dot{\phi}^2\rangle$ from (79) we can write (63) as

$$M\ddot{\Omega}+M\Omega=-\frac{1}{4\pi^2\Omega^3}\alpha^2. \tag{81}$$

We next recall that the classical system obeys the constraint equation (58). At the semiclassical level this equation is modified to

$$H=M\dot{\Omega}^2+M\Omega^2-\pi^2\Omega^2\langle\dot{\phi}^2\rangle-\Omega^4\langle V(\phi)\rangle=0 \tag{82}$$

and can be rewritten as

$$H=M\dot{\Omega}^2+M\Omega^2-\alpha^2/4\pi^2\Omega^2=0. \tag{83}$$

Equations (81) and (83) together give the solution for $\Omega(\tau)$ as

$$\Omega^2=\alpha_0\sin[2(\tau+\tau_0)] \tag{84}$$

where $\alpha_0^2=\alpha^2/4M\pi^2$. This form of the solution is similar to that of the classical solution (61), but the amplitude of oscillation, α_0, gets modified due to quantum corrections. The form of the constant α^2 in (80) shows that the classical contribution p_0^2 to the amplitude is modified by the quantum correction $\hbar^2/\sigma^2$. We can identify the constant Ω_0^2 of (61) with p_0 and write the semiclassical solution for $\Omega(\tau)$ explicitly as

$$\Omega^2=\frac{1}{4M\pi^2}\left(\Omega_0^4+\frac{\hbar^2}{\sigma_0^2}\right)^{1/2}\sin[2(\tau+\tau_0)]. \tag{85}$$

Putting $\hbar=0$ reproduces the classical solution, as expected.

Having found the semiclassical solution for $\Omega^2(\tau)$, we can use this result to compute $F(\tau)$, which was defined in (72). We get

$$F(\tau)=\frac{1}{2\alpha_0}\ln\left[\left|\frac{\tan(\tau+\tau_0)}{\tan\tau_0}\right|\right]. \tag{86}$$

We can then compute the mean evolution of the field from (77). We find that the mean value $\bar{\phi}$ evolves in exactly the same way as the classical solution (62). This is to be expected, because the state chosen is a Gaussian state. We also find that the dispersion σ^2 for the field, defined in (76), evolves as

$$\sigma^2(\tau)=\frac{\sigma_0^2}{4}+\frac{1}{16\alpha_0^2\sigma_0^2}\left(\frac{\hbar}{4\pi^2}\right)^2\left(\ln\left[\left|\frac{\tan(\tau+\tau_0)}{\tan\tau_0}\right|\right]\right)^2. \tag{87}$$

From (84) and (87) we find that at $\tau=\frac{1}{2}\pi-\tau_0$, the conformal factor goes to zero ('big crunch'), and the spread of the field becomes infinite. The blowing up of the spread shows that the semiclassical theory ceases to be valid near the singularity $\Omega=0$.

The above semiclassical solution was written under the assumption that the average energy equations are valid in the $G\to 0$ limit of the Wheeler-DeWitt equation. We did not impose the constraint that the dispersion in the derivative of the phase should be small. We now show that if this constraint is imposed, we can find an explicit criterion for the validity of the average energy equations.

We shall check if the leading dispersion condition (34) is satisfied for the model we have considered here. We return to the wavefunction for the Gaussian state, given by (65), and note that the phase β in this state is given by

$$\beta(\phi, \tau) = -\hbar B_I(\tau)(\phi - \bar{\phi})^2 + p\phi + \varepsilon(\tau). \tag{88}$$

We can convert the derivatives of β with respect to the conformal factor Ω into derivatives with respect to the conformal time τ with the help of the general prescription that we gave while deriving the semiclassical limit. Thus the condition (34) may be written as

$$\langle\dot{\beta}\rangle^2 = \langle\dot{\beta}^2\rangle. \tag{89}$$

Using the expression (88) for the phase, and the equations of motion (68) we get that

$$\langle\dot{\beta}\rangle = -\hbar\dot{B}_I\langle(\phi - \bar{\phi})^2\rangle + G(\tau) \tag{90}$$

and

$$\langle\dot{\beta}^2\rangle = \hbar^2\dot{B}_I^2\langle(\phi - \bar{\phi})^4\rangle + G^2(\tau) - 2\hbar\dot{B}_I G(\tau)\langle(\phi - \bar{\phi})^2\rangle + 4\hbar^2 B_I^2\dot{\bar{\phi}}^2\langle(\phi - \bar{\phi})^2\rangle \tag{91}$$

where

$$G(\tau) = -\frac{\hbar^2}{2\pi^2\Omega^2} B_R - \frac{p^2}{4\pi^2\Omega^2}. \tag{92}$$

Hence

$$\langle\dot{\beta}^2\rangle - \langle\dot{\beta}\rangle^2 = \hbar^2\dot{B}_I^2[\langle(\phi - \bar{\phi})^4\rangle - \langle(\phi - \bar{\phi})^2\rangle^2] + 4\hbar^2 B_I^2\dot{\bar{\phi}}^2\langle(\phi - \bar{\phi})^2\rangle. \tag{93}$$

The right-hand side can be evaluated in the Gaussian state (65) and we get that

$$\langle\dot{\beta}^2\rangle - \langle\dot{\beta}\rangle^2 = \hbar^2\frac{\dot{B}_I^2}{8B_R^2} + \frac{2\hbar^2 B_I^2}{\Omega^4 B_R}p^2. \tag{94}$$

Now, in general $\dot{B}_I$ and B_I are non-zero, and hence the dispersion condition does not hold exactly. In fact, as we said before, except in special states like stationary states, the dispersion condition will not hold exactly, but only approximately. Thus if we demand that the semiclassical Wigner function for the conformal factor and its conjugate momentum should be peaked around a classical trajectory, we find that the expectation value of T_{ik} in a Gaussian state can be used as a source only if further restrictions are imposed on the parameters of the state.

To check when the semiclassical approximation is valid, we proceed as follows. First, we use the solution for $B(\tau)$ from (75) to write the two terms on the right side in (94) explicitly, as functions of time. We get

$$D_1(\tau) \equiv \hbar^2\frac{\dot{B}_I^2}{8B_R^2} = \frac{\hbar^4}{8\Omega^4\sigma_0^4}\left(\frac{\sigma_0^4 - (\hbar/4\pi^2)^2F^2(\tau)}{\sigma_0^4 + (\hbar/4\pi^2)^2F^2(\tau)}\right)^2 \tag{95}$$

and

$$D_2(\tau) \equiv \frac{2\hbar^2 B_I^2}{\Omega^4 B_R}p^2 = \frac{2\hbar^2 p^2}{\Omega^4\sigma_0^2}\left(\frac{(\hbar/4\pi^2)^2F^2}{\sigma_0^4 + (\hbar/4\pi^2)F^2}\right). \tag{96}$$

Using the solution for $F(\tau)$ from (86) we note that as τ goes from $\tau = 0$ (starting point of the evolution) to $\tau = \pi/2 - \tau_0$ (the final singularity), the function

$$T_1(\tau) = \left(\frac{\sigma_0^4 - (\hbar/4\pi^2)^2F^2(\tau)}{\sigma_0^4 + (\hbar/4\pi^2)^2F^2(\tau)}\right)^2 \tag{97}$$

starts at unity, decreases, and finally becomes unity again. In the same range for τ the function

$$T_2(\tau)=\left(\frac{(\hbar/4\pi^2)^2F^2}{\sigma_0^4+(\hbar/4\pi^2)F^2}\right) \tag{98}$$

goes from zero to unity. Thus both $T_1(\tau)$ and $T_2(\tau)$ are bounded from above. This implies that

$$D_1(\tau)\leqslant\frac{1}{8\Omega^4}\left(\frac{\hbar}{\sigma_0}\right)^4 \tag{99}$$

and

$$D_2(\tau)\leqslant\frac{2p^2}{\Omega^4}\left(\frac{\hbar}{\sigma_0}\right)^2. \tag{100}$$

For the validity of the semiclassical approximation, the dispersion in (94) should be small compared with $\langle\dot\beta\rangle^2$. We recall that for a quantum state satisfying the Schrödinger equation we have the relation

$$\langle\dot\beta\rangle=-\langle h\rangle \tag{101}$$

where $\langle h\rangle$ is the expectation value of the Hamiltonian. For the Gaussian state (65) we find that

$$\langle h\rangle=\frac{1}{4\pi\Omega^2}\left(p^2+\hbar^2\frac{B_R^2+B_I^2}{B_R}\right). \tag{102}$$

Thus we require that

$$D_1+D_2\ll\langle h\rangle^2. \tag{103}$$

Using (99), (100) and (102) we find this will be ensured if

$$\frac{1}{8\Omega^4}\left(\frac{\hbar}{\sigma_0}\right)^4+\frac{2}{\Omega^4}\left(\frac{\hbar^2}{\sigma_0^2}\right)p^2\ll\frac{1}{16\pi^2\Omega^4}\left(p^2+\hbar^2\frac{B_R^2+B_I^2}{B_R}\right)^2 \tag{104}$$

and hence if

$$\frac{1}{8}\left(\frac{\hbar}{\sigma_0}\right)^4+2\left(\frac{\hbar^2}{\sigma_0^2}\right)p^2\ll\frac{1}{16\pi^2}\left(p^2+\hbar^2\frac{B_R^2+B_I^2}{B_R}\right)^2. \tag{105}$$

Note that the terms on the left-hand side explicitly depend on $\hbar$, whereas the leading term on the right-hand side is p^2, and independent of $\hbar$. Hence, for a moment let us ignore the $\hbar$-dependent term on the right-hand side. We then find that the requirement

$$\hbar^2/\sigma_0^2\ll p^2 \tag{106}$$

is a necessary and sufficient condition for the inequality in (105) to hold. We can now check that this requirement is consistent with our dropping the $\hbar$-dependent terms on the right-hand side of (105). The term $\hbar^2B_R(\tau)$ starts at $\hbar^2/\sigma_0^2$, decreases monotonically, and becomes zero at the final singularity. It is hence bounded above by $\hbar^2/\sigma_0^2$. The term $\hbar^2B_I^2/B_R$ was examined above, and it is always smaller than $\hbar^2/\sigma_0^2$. Hence, dropping these terms was consistent with the requirement (106).

We find that if the semiclassical approximation is to be valid, the condition (106) must hold. It is a constraint on the spread of the Gaussian state at the initial time at which the state is chosen. This can be interpreted in a nice manner. We saw in equations (79) and (80) that $\hbar^2/\sigma_0^2$ is the quantum correction to the classical momentum p (recall that p is constant). Thus the semiclassical approximation holds if the quantum correction to the classical momentum is small compared with the classical momentum. This matches with our intuitive expectation that if $\langle T_{ik} \rangle$ is to be used as the source, the quantum corrections should be small compared with the classical value of T_{ik}. Since we have arrived at this result starting from the semiclassical Wigner function, it provides an independent support for the use of the Wigner function in the semiclassical theory.

Our calculations show that even though one can find interesting self-consistent solutions to the average energy equations, the validity of those solutions is questionable, unless we are working in a predetermined domain of validity for the semiclassical theory. Outside this domain we must use the full quantum theory of gravity. It is not sufficient to take the limit $G \to 0$ to define the semiclassical limit. An additional condition on the quantum state of the matter field should be satisfied. We believe this could have interesting consequences when the semiclassical theory is applied to the process of black-hole evaporation.

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