

LETTER TO THE EDITOR

Conformal invariance, gravity and massive gauge theories

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Abstract. A gauge theory is constructed maintaining the invariance of the action of massless scalar fields under local conformal transformations. The theory leads to gravity in a natural fashion and also allows for the gauge field to be massive.

The action for a massless, complex scalar field $\varphi(x)$ can be taken to be

$$A = \frac{1}{2} \int (\partial^i \varphi^*) (\partial_i \varphi) \sqrt{-g} d^4x. \tag{1}$$

This action is invariant under the *global* (1) phase transformations

$$\varphi \rightarrow (\exp(i\alpha))\varphi \quad \varphi^* \rightarrow (\exp(-i\alpha))\varphi^* \tag{2}$$

where α is a real constant. In the spirit of gauge field formalism [1], we may demand the invariance of our theory under the *local* transformations:

$$\varphi(x) \rightarrow (\exp(i\alpha(x)))\varphi \quad \varphi^*(x) \rightarrow (\exp(-i\alpha(x)))\varphi^*. \tag{3}$$

It is well known that invariance under (3) can be achieved by introducing a 'gauge field' $A_i(x)$ through the covariant derivative

$$D_i \equiv \partial_i - iqA_i(x) \tag{4}$$

which replaces the ordinary derivative ∂_i in the action. The modified action

$$A \equiv \frac{1}{2} \int (D^i \varphi)^* (D_i \varphi) \sqrt{-g} d^4x \tag{5}$$

is invariant under (3), provided A_i transforms as

$$A_i \rightarrow A_i - (1/q)\partial_i\alpha. \tag{6}$$

The action in (5), however, does not contain a 'kinetic energy' term for $A_i(x)$. This can be taken care of by adding to (5) the simplest 'kinetic energy' term that is invariant under (6). We thus arrive at the complete action:

$$A \equiv \frac{1}{2} \int (D^i \varphi)^* (D_i \varphi) \sqrt{-g} d^4x - \frac{1}{4} \int F_{ik} F^{ik} \sqrt{-g} d^4x \tag{7}$$

with

$$F_{ik} = \partial_i A_k - \partial_k A_i. \tag{8}$$

The original action in (1) is also invariant under the following transformations:

$$g_{ik} \rightarrow (\exp(2\lambda))g_{ik} \quad g^{ik} \rightarrow (\exp(-2\lambda))g^{ik} \quad \phi \rightarrow (\exp(-\lambda))\phi \quad (9)$$

with λ being some *real* constant. From the standpoint of gauge fields one would like to extend the transformations in (9) to *local* transformations with $\lambda = \lambda(x)$. Usually, this is done in the literature [2] by modifying the action in (1) to

$$A \equiv \frac{1}{2} \int (\partial_i \varphi^* \partial^i \varphi - \frac{1}{6} R \varphi^* \varphi) \sqrt{-g} \, d^4x \quad (10)$$

where R is the scalar curvature associated with the metric tensor g_{ik} . Clearly, the form in (10) is very different from what one would have expected in a local gauge theory. In a gauge theory, one would have expected the introduction of a gauge field $A_i(x)$ which would ensure invariance under (9)!

We shall now develop such a model theory. Surprisingly enough, this theory leads to (10) only in a special limiting case.

We first note that the complex nature of φ is not essential in (9). Hence, without loss of generality we shall assume $\varphi(x)$ to be real. Let us modify the action in (1) to

$$A \equiv \frac{1}{2} \int (\varphi_i \varphi^i) \sqrt{-g} \, d^4x \quad (11)$$

where $\varphi_i = \nabla_i \varphi$ and

$$\nabla_i \equiv (\partial_i - eA_i). \quad (12)$$

In (12) e is a real constant and $A_i(x)$ is a gauge field. Using the analogy of electromagnetism it is easy to verify that $A_{(4)}$ is invariant under (9) if A_i transforms as

$$A_i \rightarrow A_i - (1/e) \partial_i \lambda. \quad (13)$$

We have

$$\begin{aligned} \nabla'_i \varphi' &\equiv (\partial_i - eA'_i) \phi' = (\partial_i - eA_i + \partial_i \lambda) e^{-\lambda} \varphi \\ &= e^{-\lambda} (\partial_i - eA_i) \phi = e^{-\lambda} (\nabla_i \varphi) \end{aligned} \quad (14)$$

so that

$$\begin{aligned} \sqrt{-g'} g'^{ik} \nabla'_i \phi' \nabla'_k \phi' &= e^{4\lambda} \sqrt{-g} e^{-2\lambda} g^{ik} e^{-\lambda} (\nabla_i \varphi) e^{-\lambda} (\nabla_k \varphi) \\ &= \sqrt{-g} g^{ik} \nabla_i \varphi \nabla_k \varphi \end{aligned} \quad (15)$$

which demonstrates the invariance of the action. One can generalise the above 'prescription' to arbitrary matter fields. A matter field with 'dimension' β is assumed to transform under (9) as $\psi \rightarrow (\exp(\beta\lambda))\psi$. For such a field, the covariant derivative is taken to be $\nabla_i = \partial_i + e\beta A_i$.

The action in (11) is still lacking in the kinetic term for $A_i(x)$. This can be easily remedied by modifying the action to the form

$$A \equiv \frac{1}{2} \int (\nabla_i \varphi) (\nabla^i \varphi) \sqrt{-g} \, d^4x + \alpha \int F_{ik} F^{ik} \sqrt{-g} \, d^4x \quad (16)$$

where α is a coupling constant and F_{ik} is given by

$$F_{ik} = \partial_i A_k - \partial_k A_i. \quad (17)$$

The action in (16) is invariant under the local conformal transformations just as the action in (7) is invariant under local phase transformation. At this stage, the status of action in (10)—which is also invariant under conformal transformations—is not clear. In particular, it is not known how (16) and (10) are related to each other. We shall now look more closely into this relationship.

We may begin by noticing that the action in (16) depends on $\varphi(x)$, $A_i(x)$ and $g_{ik}(x)$, but lacks the kinetic term for $g_{ik}(x)$. Naturally, one should construct from g_{ik} a suitable kinetic term which is invariant under (9). The usual choice

$$A \equiv \frac{1}{16\pi} \int R\sqrt{-g} d^4x \tag{18}$$

is inadmissible because of the lack of conformal invariance. A conformally invariant action can be constructed by proceeding in a step by step manner. We begin by noting that the conformally covariant derivative of $g_{ik}(x)$ is

$$\nabla_i g_{km} = \partial_i g_{km} + 2eA_i g_{km}. \tag{19}$$

Using this, we can form conformally invariant (zero-dimension) objects as follows:

$$\bar{\Gamma}^i_{jk} \equiv \frac{1}{2}g^{il}(-\nabla_l g_{jk} + \nabla_k g_{lj} + \nabla_j g_{lk}) \tag{20}$$

$$\bar{R}^i_{jk} \equiv \partial_l \bar{\Gamma}^i_{jk} - \partial_k \bar{\Gamma}^i_{jl} - \bar{\Gamma}^i_{mk} \bar{\Gamma}^m_{jl} + \bar{\Gamma}^i_{ml} \bar{\Gamma}^m_{jk} \tag{21}$$

$$\bar{R}_{jl} \equiv \bar{R}^i_{jil}. \tag{22}$$

The scalar curvature of dimension 2 can be formed from (22):

$$\bar{R} = \bar{R}_{ik} g^{ik} = R + \frac{6e}{\sqrt{-g}} \partial_i(\sqrt{-g} A^i) + 6e^2 A_i A^i. \tag{23}$$

In (23) we have expressed \bar{R} in terms of the basic fields g_{ik} and A_i ; R is the conventional scalar curvature constructed from g_{ik} . Since R is of dimension 2, $\varphi^2 \bar{R}$ is a convenient candidate for the Lagrangian. Assuming that the field equations for gravity are of second order, the most general conformally invariant action has the form:

$$A = \int L\sqrt{-g} d^4x \tag{24}$$

$$L = -\frac{\alpha_1}{12} \varphi^2 \bar{R} + \frac{\dot{\alpha}_2}{2} (\nabla_i \varphi)(\nabla_k \varphi) g^{ik} - \frac{\alpha_3}{4} F_{ik} F^{ik} - \frac{\lambda}{4!} \varphi^4 \tag{25}$$

where $\alpha_1, \alpha_2, \alpha_3$, are constants. We have added a $\lambda\varphi^4$ term which is also conformally invariant.

The action in (24) and (25) represent the most general conformally invariant modification of our original action (1). Using (23), we can rewrite this action as

$$A \equiv \int \sqrt{-g} d^4x \left(-\frac{\alpha_1}{12} \varphi^2 R + \frac{\alpha_2}{2} \partial_i \varphi \partial^i \varphi + \frac{1}{2} e^2 (\alpha_2 - \alpha_1) \varphi^2 A_i A^i - \frac{1}{2} e (\alpha_2 - \alpha_1) A^i \partial_i \varphi^2 - \frac{1}{4} F_{ik} F^{ik} - \frac{\lambda}{4!} \varphi^4 \right). \tag{26}$$

In arriving at (26) we have used freedom in the overall multiplicative factor to set $\alpha_3 = 1$. We have also omitted a total 4-divergence.

The action in (26) is quite rich in structure. As it stands there is no way of fixing the constants $\alpha_1, \alpha_2, \lambda$. One very *special* choice corresponds to $\alpha_1 = \alpha_2 = \alpha$ (say). Then we get

$$A = \int d^4x \sqrt{-g} \left(\frac{1}{2}\alpha(\varphi_i\varphi^i - \frac{1}{6}R\varphi^2) - \frac{1}{4}F_{ik}F^{ik} - \frac{\lambda}{4!}\varphi^4 \right). \quad (27)$$

The vector field A_i , which was originally introduced to maintain the conformal invariance, has completely disappeared from the φ - g sector! We thus recover the usual $\frac{1}{6}R\varphi^2$ coupling.

It should, however, be kept in mind that $\alpha_1 = \alpha_2$ is a very special choice. In other words the $\frac{1}{6}R\varphi^2$ term is in no way mandatory to maintain conformal invariance. In fact, the action in (26) can be cast in a much more interesting form using the freedom of conformal invariance. We know that the field equations obtained from (26) are conformally invariant. Hence if $(\varphi(x), g_{ik}(x), A_i(x))$ is a solution then so is $(\exp(-\lambda(x))\varphi(x), (\exp 2\lambda(x))g_{ik}(x), A_i - (1/e)\partial_i\lambda)$. Using this fact we can set $\varphi = \text{constant}$. Taking $\alpha_1 = -1$ and

$$\varphi^2 = 3/4\pi G \quad \varphi_i = 0 \quad (28)$$

the action becomes

$$A = \int \sqrt{-g} d^4x \left[\frac{1}{16\pi G} R + \frac{e^2(1+\alpha_2)}{2} A_i A^i - \frac{1}{4} F_{ik} F^{ik} - \frac{\lambda}{4!} \left(\frac{3}{4\pi G} \right)^2 \right]. \quad (29)$$

Notice that this action corresponds to gravity interacting with a massive vector field. In other words, the choice of a particular gauge for φ gives mass to the vector gauge field.

The above discussion has been purely classical. Quantum field theoretic considerations can add more structure to the theory. In particular, radiative corrections may provide a natural mechanism for realising (28). The quantum version of the above model is under investigation.

References

- [1] Huang K 1982 *Quarks, Leptons and Gauge Fields* (Singapore: World Scientific)
- [2] Hoyle F and Narlikar J V 1974 *Action at a Distance in Physics and Cosmology* (San Francisco: Freeman) pp 27-8