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# Quantum Fluctuations Near the Classical Space-Time Singularity

J. V. NARLIKAR

*Tata Institute of Fundamental Research, Bombay-400 005, India*

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*Abstract*

The method of path integration is used to study the effects of quantum fluctuations in the space-time geometry near the classical singularity of general relativity. It is shown that in certain special cases explicit Feynman propagators can be constructed which enable us to evaluate these fluctuations *quantitatively*. The cases discussed are (i) the gravitational collapse of a uniform dust ball, (ii) the Friedmann cosmologies, (iii) the axisymmetric Bianchi type I cosmological model, and (iv) the general anisotropic Bianchi type I cosmological model. In all cases discussed here the quantum uncertainty grows to infinity as the classical space-time singularity is approached. In this wider regime of quantum gravitation non-singular solutions can occur with finite probabilities.

## §(1): *Introduction*

The classical equations of general relativity are derivable from the Hilbert action principle with the action given by

$$S = \frac{c^4}{16\pi G} \int_{\mathcal{V}} R\,(-g)^{1/2} d^4x + S_m \tag{1.1}$$

where $R$ is the scalar curvature and $g$ the determinant of the metric tensor $g_{ik}$ in a given four-dimensional subspace $\mathcal{V}$ of the space-time manifold $\mathfrak{M}$. In (1.1) $S_m$ is the action representing matter in all its forms, free as well as interacting, while $c$ and $G$ are the speed of light and the gravitational constant. The variation $\delta S = 0$ for $g_{ik} \to g_{ik} + \delta g_{ik}$ with the variations vanishing on the boundary $\partial\mathcal{V}$

of $\mathcal{V}$ lead to the familiar Einstein equations:

$$R_{ik} - \tfrac{1}{2} g_{ik} R = - \frac{8\pi G}{c^4} T_{ik} \tag{1.2}$$

Important theorems [1] proved during the 1960's by Penrose, Hawking, and Geroch have established that with the usual assumptions of conventional physics in $S_m$, the manifolds $\mathfrak{M}$ satisfying (1.2) contain a space-time singularity. The singularity may be of quite a general character, although the type that is usually encountered in the applications of (1.2) to cosmology and astrophysics has matter densities and curvature invariants diverging at the singular points.

It is such singularities that we will be concerned with in this article. If we consider a compact portion of matter close to the singularity, it usually happens that the action $S$ tends to zero. Hence, if we choose the timelike span of $\mathcal{V}$ sufficiently close to the classical singularity it may happen that $S$ is so small as to satisfy the relation

$$S \leq \hbar \tag{1.3}$$

Under such circumstances it is well known that classical physics is usually unreliable and has to be replaced by a quantum mechanical treatment. Accordingly it is to the quantum version of (1.2) that we must turn for a reliable account of the behavior of $\mathfrak{M}$ near the classical space-time singularity. In particular we may ask the following question: Is the singularity an inevitable property of $\mathfrak{M}$ even in the wider scheme of quantum theory? It is this question that we will address ourselves to in this article.

When looking at the quantum version of the classical Einstein equations (1.2), we notice that there are two ways in which quantum ideas enter into the discussion. We may consider the right-hand side as representing the behavior of "other" interactions (except gravity) and quantize it. In that case we learn about the quantum behavior of matter in the presence of gravity which is treated classically. This was the approach initiated by Hawking [2] and it has paid a rich dividend through such conclusions as: "A black hole radiates." In the second approach the attention is focused on the left-hand side of the Einstein equations: gravity is treated quantum mechanically, while the other interactions are classical. This second approach, that of quantum gravity, is complementary to the first. While the true perspective of the quantum version of (1.2) should combine both approaches, the enormity of the problem seems to preclude a solution along these lines at present. In this paper we will consider a limited version of the second approach in the expectation that it is more easily tractable.

## §(2): *Quantum Gravity*

There have been many approaches to quantum gravity [3]. The quantization of the weak field in the flat space background treads familiar territory so clearly established by quantum electrodynamics. It is, however, not useful in the strong fields expected near the space-time singularity. The operator formalism and the $S$-matrix approach become difficult to interpret for strong fields because of the dual nature of $g_{ik}$, as field quantities and as components of the space-time metric. It seems that in the context of the question raised above a more direct approach to the answer is provided by the Feynman path integral.

In this approach a unique geometry, obtained as the solution $\overline{G}$ of (1.2) is replaced by a variety of geometries $\{G\}$ which could be possible with specified end conditions. Treating each geometry as a path (or a "history") a probability amplitude

$$\exp(iS/\hbar) \tag{2.1}$$

($2\pi\hbar$ = Planck's constant) is ascribed to this path, $S$ being computed for that geometry. In the notation of 3-geometries [4], we may specify $\mathcal{V}$ as the region sandwiched between two spacelike hypersurfaces $\Sigma_1$ and $\Sigma_2$ on which are specified the 3-geometries ${}^{(3)}G_1$ and ${}^{(3)}G_2$, respectively. We then ask for the probability amplitude for this to happen and the answer is provided by

$$K\,[{}^{(3)}G_2, \Sigma_2; {}^{(3)}G_1, \Sigma_1] = \int_{\{G\}} \exp\left[\frac{i}{\hbar} S(G)\right] \mathfrak{D}G \tag{2.2}$$

Here $K$ is the quantum mechanical propagator and the right-hand side is a path integral (or a sum over histories).

There are many conceptual and operational difficulties with this approach also, but in the simplified situations to be considered here they can be circumvented. As in many cases of quantum theory of particles [5] the path integral (2.2) can be evaluated by some mathematical trick.

Before proceeding further it is necessary to dispose at the outset of the surface integrals which crop up on evaluating the action. In the classical variational problem one derives (1.2) by assuming that $\delta g_{ik}$ and $\delta g_{ik,l}$ vanish on $\partial\mathcal{V}$. This is somewhat different from the standard variational problem in classical field theory where the variations of the quantities involved and not their derivatives are assumed to vanish on the boundary. The reason for requiring $\delta g_{ik,l} = 0$ on the boundary arises from the fact that the action integrand $R$ contains up to *second* derivatives of $g_{ik}$. However, it is well known that the appearance of the second derivatives in the action does not alter the dynamical character of the resulting field equations (1.2), which continue to be of only second order as in the rest of physics. Hawking and Gibbons [6] have pointed out that the necessity to have $\delta g_{ik,l} = 0$ on $\partial\mathcal{V}$ can be avoided by adding to the Hilbert action a suitable surface integral over $\partial\mathcal{V}$. The action (1.1) is replaced by

$$S = \frac{c^4}{16\pi G} \int_{\mathcal{V}} R\,(-g)^{1/2}\, d^4x + \frac{c^4}{8\pi G} \int_{\partial\mathcal{V}} (\mathcal{K} + \mathcal{L})\,(-h)^{1/2}\, d^3x + S_m \tag{2.3}$$

where $K$ is the second fundamental form for the space-time geometry and $\mathcal{L}$ depends only on the induced metric $h$ on $\partial\mho$. The additional term is the surface integral whose variation for $\delta g_{ik,l} \neq 0$ cancels the corresponding variation of the main Hilbert term. In what follows, we will ignore the surface terms of this type because they do not produce any dynamical effects. The correctness of the computed kernal $K$ is ensured by its "reproducing property":

$$K\,[{}^{(3)}G_3, \Sigma_3; {}^{(3)}G_1\ \Sigma_1] = \int_{\{{}^{(3)}G_2\}} K\,[{}^{(3)}G_3, \Sigma_3; {}^{(3)}G_2, \Sigma_2] \cdot K\,[{}^{(3)}G_2, \Sigma_2; {}^{(3)}G_1, \Sigma_1]\ d{}^{(3)}G_2 \tag{2.4}$$

As De Witt [7] has pointed out, the conformal degrees of freedom provide a particularly simple subset of all possible geometries. Suppose we denote by an overbar the solution of the classical Einstein equations and write a "nonclassical" metric tensor as

$$g_{ik} = (1 + \phi)^2 \bar{g}_{ik} \tag{2.5}$$

where $\phi$ is a scalar function of space-time. Then we get

$$\int_{\mho} R\,(-g)^{1/2} d^4x \sim \int_{\mho} [(1 + \phi)^2 \bar{R} - 6\phi_{;i}\phi^{;i}]\ (-\bar{g})^{1/2}\ d^4x$$

where the indices are raised with the classical metric $\bar{g}_{ik}$. (The $\sim$ sign indicates that the surface terms have been ignored.) If we now confine ourselves to conformal fluctuations only, the variation of geometries is now restricted to histories of $\phi$. Since $\phi$ occurs only up to quadratic powers in $\phi$ and $\phi_{;i}$ the resulting calculations of $K$ are made considerably easier.

If $S_m$ denotes a system of particles $a$ with masses $m_a$ and proper times measured by $s_a$,

$$S_m = -\sum_a \int m_a ds_a \tag{2.6}$$

In this case it is easy to see that (1.1) reduces to

$$S \sim \frac{c^4}{16\pi G} \int_{\mho} [(\phi^2 - 1)\bar{R} - 6\phi_{;i}\phi^{;i}]\ (-\bar{g})^{1/2}\ d^4x \tag{2.7}$$

In the following work we will use (2.7) whenever discussing the conformal fluctuations. To simplify writing we will use $c = 1, \hbar = 1, G = 1$. This means we are using as a unit of mass

$$\left(\frac{c\hbar}{G}\right)^{1/2} \sim 2.2 \times 10^{-5} g \tag{2.8}$$

We will discuss in the next section some results of quantum conformal fluctuations, and in the following section we will consider more general types of fluctuations.

## §(3): *Conformal Fluctuations*

Here we will describe three cases: (i) the gravitational collapse of a homogeneous dust ball, (ii) the Friedmann universe, and (iii) the anisotropic Bianchi type I universe. Since the first two cases have been discussed elsewhere in detail [8] we will only mention them briefly.

(i) *The Collapsing Homogeneous Dust Ball.* In the classical description of the gravitational collapse of a dust ball in an otherwise empty space-time, the following line elements describe the external and internal geometries, respectively:

$$d\bar{s}^2 = \left(1 - \frac{2M}{R}\right) dT^2 - \left(1 - \frac{2M}{R}\right)^{-1} dR^2 \,(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.1}$$

$$d\bar{s}^2 = dt^2 - \bar{Q}^2(t)\left[\frac{dr^2}{1 - \alpha r^2} - r^2\,(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{3.2}$$

Here (3.1) is the familiar Schwarzschild solution for the dust ball of mass $M$. The Robertson-Walker line element of (3.2) uses the comoving coordinates $(r, \theta, \phi)$ and the proper line $t$ of the freely falling dust particles. $\alpha$ is a density related parameter and is related to $M$ by

$$2M = \alpha r_b^3 \tag{3.3}$$

where $r_b$ is the "comoving radius" of the dust ball.

The compact submanifold represented by (3.2) develops a singularity as $\bar{Q} \to 0$. Writing $t = 0$ for this singular epoch, it is convenient to define $\tau = (-t)^{1/3}$ so that the singularity is approached as $\tau$ *decreases* to zero. Close to $\tau = 0$,

$$\bar{Q} \sim \left(\frac{3}{2}\right)^{2/3} \alpha^{1/3}\,\tau^2 \tag{3.4}$$

The invariant comoving coordinate volume of the dust ball is denoted by

$$V = \int_0^{r_b} \frac{4\pi r^2\,dr}{(1 - \alpha r^2)^{1/2}} \tag{3.5}$$

The geometries permitted by quantum fluctuations which preserve the homogeneity and spherical symmetries of the dust ball are conformal to (3.2) with $\phi$ a function of $\tau$ only. We therefore ask for the propagator $K$, which gives the probability amplitude to go to the "state" $\phi = \phi_2$ at $\tau = \tau_2$, starting from a "state" $\phi = \phi_1$, at $\tau = \tau_1 > \tau_2$. From the ideas of the preceding section we write this as the path integral

$$K[\phi_2, \tau_2; \phi_1, \tau_1] = \int \exp\left[-\frac{9i}{16\pi}\alpha V \int_{\tau_1}^{\tau_2} \left(\frac{1}{2}\tau^4 \dot{\phi}^2 - \tau^2\phi^2\right) d\tau\right] \mathfrak{D}\phi \quad (3.6)$$

This can be evaluated exactly since the integrand is quadratic in $\phi$ and $\dot{\phi}$. The standard path integral computation techniques then give

$$K[\phi_2, \tau_2; \phi_1, \tau_1] = \left[\frac{9iV\alpha\tau_1{}^2\tau_2{}^2}{32\pi^2(\tau_1 - \tau_2)}\right]^{1/2} \exp\left\{\frac{9iV\alpha}{32\pi(\tau_1 - \tau_2)}\right.$$
$$\left.\cdot\left[\tau_1{}^3(\tau_1 - 2\tau_2)\phi_1{}^2 + 2\tau_1{}^2\tau_2{}^2\phi_1\phi_2 + (\tau_2 - 2\tau_1)\tau_2{}^3\phi_2{}^2\right]\right\} \quad (3.7)$$

If we apply (3.7) to an initial state in the form of a wave packet with dispersion $\Delta_1$ centered on the classical solution

$$\psi(\phi_1, \tau_1) = (2\pi\Delta_1^2)^{-1/4} \exp\left(-\frac{\phi_1{}^2}{4\Delta_1^2}\right) \quad (3.8)$$

it is seen that the wave packet at $\tau = \tau_2$ remains centered on the classical solution but its dispersion increases rapidly to

$$\Delta_2 = \left[1 + \left(\frac{9V\alpha\Delta_1{}^2}{8\pi}\tau_1{}^3\right)^2\right]^{1/2} \frac{8\pi}{9V\alpha\tau_1\Delta_1}\tau_2^{-2} \quad (3.9)$$

*Thus at the classical singularity the quantum uncertainty diverges*. This uncertainty includes nonsingular states with finite probability. For such nonsingular states the scale factor behaves as

$$Q \sim \bar{Q}\tau_2^{-2} \sim \text{constant} \quad (3.10)$$

We may interpret this as the quantum gravitational tunneling effect.

It is worth pointing out that the above calculation is confined to the interior of the dust ball and therefore does *not* cover the entire space-time. Thus it would be possible, in principle, to extend the surfaces $\Sigma_1$, $\Sigma_2$ ($t = t_1$ and $t = t_2$) beyond the compact region $r \leqslant r_b$, into the external empty space-time. In Figure 1 we have drawn the Penrose diagram for the classical collapse situation to illustrate this.

Notice that in Figure 1 our discussion is confined to the shaded region as it approaches the space-time singularity. It is possible to extend it along the dotted lines leading towards the space-like infinity $I^0$. We can look upon this extension as a compact system in contact with an external environment. In the present discussion we are interested in the quantum behavior of the compact system only, and so we imagine that the external degrees of freedom have been inte-

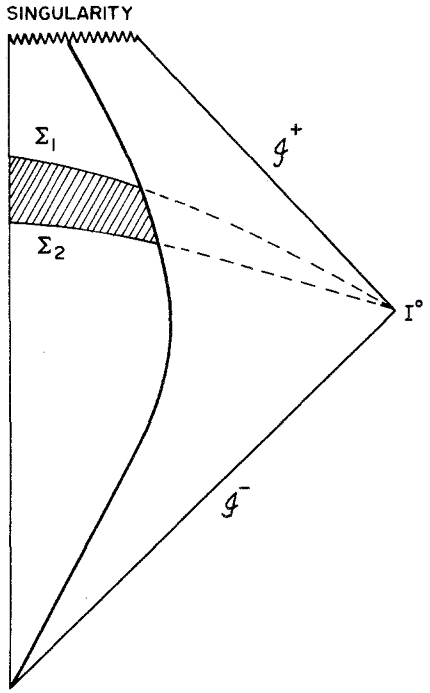


Fig. 1. The Penrose diagram for the collapsing dust ball. The boundary of the ball is shown by the thick line. The surfaces $\Sigma_1$ and $\Sigma_2$ are given by $t = t_1$ and $t = t_2$, respectively. In principle one should extend the $\Sigma$ surfaces outside the dust ball as shown. However, as explained in the text the interest of the present problem is limited to the shaded region.

grated out. This procedure is no different from that adopted elsewhere in quantum mechanics (cf. [5]) for examples of this procedure). Thus the probability calculations for a possible nonsingular final state using (3.7) refer to the final state as seen by an observer interior to the dust ball.

(ii) *The Friedmann Universe.* Since the above internal solution is applicable to Friedmann cosmologies also, we may use the above results near the Friedmann singularity by changing $t$ to $-t$, and suitably redefining $\tau$.

For the closed universe $V$ may represent the coordinate volume of the entire universe. For the open models $V$ may be restricted to a compact portion. However, in (3.9) if we let $V \to \infty$ we get a finite answer

$$\Delta_2 = \frac{\tau_1^{\,2}}{\tau_2^{\,2}} \Delta_1 \tag{3.11}$$

Thus $\Delta_2 \to \infty$ as $\tau_2 \to 0$. It is, however, intriguing to note that (3.11) does not involve $\hbar$, although the intermediate steps do! It is not clear whether (3.11) can be derived from classical or semiclassical arguments alone.

It has been pointed out elsewhere [8] that the divergent uncertainty near the big bang singularity confirms an earlier conjecture [9] that many of the quantum cosmologies could be devoid of particle horizons. With the technique described above it is possible to show that the quantum mechanical probability of obtaining horizonless cosmological models is finite.

(iii) *Anisotropic Bianchi Type I Cosmology*. The classical solution in this case is given by the line element [1]

$$d\bar{s}^2 = dt^2 - A_1{}^2(t)\,(dx')^2 - A_2{}^2(t)\,(dx^2)^2 - A_3{}^2(t)\,(dx^3)^2 \tag{3.12}$$

where, for $\mu = 1, 2, 3$,

$$A_\mu(t) = \bar{Q}\left(\frac{t^{2/3}}{\bar{Q}}\right)^{2\sin\alpha_\mu}$$

$$\alpha_\mu = \alpha + \frac{2\pi}{3}(\mu - 1), \qquad \alpha = \text{constant} \tag{3.13}$$

and $\bar{Q}$ is given by

$$\bar{Q}^3 = \tfrac{9}{2}Mt\,(t + \Sigma), \qquad M, \Sigma \text{ constants} > 0 \tag{3.14}$$

The universe is dust filled and has density given by

$$\rho = \frac{3M}{4\pi\bar{Q}^3} \tag{3.15}$$

If we are to consider the quantum fluctuations of this model, within the geometries of type I, we should consider $A_\mu(t)$ to fluctuate. In this section we will limit ourselves to the conformal degree of freedom only, which essentially affects $\bar{Q}(t)$ only. In the next section we shall consider the remaining degrees of freedom separately.

Using (2.7) and noting that for (3.12)

$$\bar{R} = \frac{4}{3t\,(t + \Sigma)} \tag{3.16}$$

we get for a comoving 3-space of coordinate volume $V$ between the times $t_2 \leqslant t \leqslant t_1$

$$S \sim \frac{VM}{16\pi}\int_{t_1}^{t_2} [6\,(\phi^2 - 1) - 27t\,(t + \Sigma)\,\dot{\phi}^2]\,dt \tag{3.17}$$

As before we will "approach" the singularity be letting $t_2 \to 0$ so that the model is of a contracting universe. Thus we will be interested in the behavior of $K$ as $t_2 \to 0$. We will assume in the following calculation that $t_2 < t_1 \ll \Sigma$.

We now evaluate under these conditions

$$K[\phi_2, t_2; \phi_1, t_1] = \int \exp\left\{\frac{3iVM}{8\pi}\int_{t_1}^{t_2} [\phi^2 - \tfrac{9}{2}t(t+\Sigma)\dot{\phi}^2]\, dt\right\} \mathfrak{D}\phi \quad (3.18)$$

To obtain $K$ we first obtain the solution $\phi = \phi_c(t)$ which makes the integral (3.17) stationary with respect to $\phi \to \phi_c + \delta\phi$ and which satisfies the boundary conditions

$$\phi_c(t_1) = \phi_1, \qquad \phi_c(t_2) = \phi_2 \quad (3.19)$$

It is easy to verify that $\phi_c$ satisfies the differential equation

$$\frac{d}{dt}[t(t+\Sigma)\dot{\phi}_c] + \frac{2}{9}\phi_c = 0 \quad (3.20)$$

which has the general solution expressible in terms of the Legendre functions of fractional order:

$$Q_c(t) = \alpha P_{-1/3}(\lambda) + \beta Q_{-1/3}(\lambda) \quad (3.21)$$

with $\lambda = 1 + 2t/\Sigma$. The constants $\alpha$ and $\beta$ can be determined with the help of the boundary conditions (3.19) and we may then write

$$K[\phi_2, t_2; \phi_1, t_1] = f(t_2, t_1) \exp\left\{\frac{3iVM}{8\pi}\int_{t_1}^{t_2}\left[\phi_c{}^2 - \frac{9}{2}t(t+\Sigma)\dot{\phi}_c{}^2\right] dt\right\} \quad (3.22)$$

The function $f(t_2, t_1)$ is obtained from the calculation of the Van Vleck determinant [10]. Although this exact calculation is in principle possible, it is very cumbersome and not really necessary since we are interested only in the time interval close to $t = 0$ ($t \ll \Sigma$). We will accordingly consider only this limiting situation.

Noting that [11] for $\lambda \approx 1$,

$$P_n(\lambda) \sim 1 - \frac{n(n+1)}{2}(1-\lambda), \qquad Q_n(\lambda) \sim -\frac{1}{2}\ln\left(\frac{\lambda - 1}{2}\right) \quad (3.23)$$

we approximate $\phi_c(t)$ by the simple expression

$$\phi_c(t) \sim A + B\ln t \quad (3.24)$$

where $A$ and $B$ are determined with the help of (3.19):

$$A = \frac{\phi_1 \ln t_2 - \phi_2 \ln t_1}{\ln t_2 - \ln t_1}, \qquad B = \frac{\phi_2 - \phi_1}{\ln t_2 - \ln t_1} \quad (3.25)$$

A straightforward computation then shows that in this limit

$$K[\phi_2, t_2; \phi_1, t_1] = f(t_2, t_1)\exp F \quad (3.26)$$

where, with $\eta = \ln (t_1/t_2)$

$$F = \frac{27iVM\Sigma}{16\pi\eta} (\phi_2 - \phi_1)^2 \tag{3.27}$$

Also, in this approximation $f(t_2, t_1)$ is given by

$$f(t_2, t_1) = \left(\frac{27VM\Sigma}{16i\pi^2\eta}\right)^{1/2} \tag{3.28}$$

As in the spherically symmetric case we can now work out what happens to a wave packet $\psi(\phi_1, t_1)$ given by the right-hand side of (3.8) as the propagator given by (3.26)-(3.28) acts on it. A tedious but simple calculation shows that at $t_2$ the wave packet continues to be centered on $\phi_2 = 0$ but has a dispersion given by

$$\Delta_2 = \frac{4\pi}{27M\Sigma V\Delta_1} \ln \frac{t_1}{t_2} \tag{3.29}$$

Thus $\Delta_2$ diverges logarithmically as $t_2 \to 0+$.

In the spherically symmetric case the divergence in $\Delta_2$ was quadratic in $\tau_2^{-1}$ (or as $t_2^{-2/3}$) as $\tau_2(t_2)$ tends to the singular epoch $\tau_2 = 0$, $t_2 = 0$. The *rate* of divergence is *reduced* in the anisotropic case to the logarithmic form.

## §(4): *Anisotropic Fluctuations*

The Bianchi type I model described above is characterized by two properties: shear and expansion. Of these the quantum fluctuations in the expansion function $Q(t)$ are dealt with by the conformal technique. To deal with shear fluctuations we have to use a different technique.

Following the transformations used by Misner [12] we define

$$A_1 = e^{\xi+\zeta+3^{1/2}\eta}, \quad A_2 = e^{\xi+\zeta-3^{1/2}\eta}, \quad A_3 = e^{\xi-2\zeta}, \quad Q = e^{\xi} \tag{4.1}$$

Thus the fluctuations considered in Section 3 relate to $\xi$, while $\eta$ and $\zeta$ were held constant, i.e., at their classical functional forms. We will now consider the fluctuations in $\eta$ and $\zeta$, while $\xi$ is kept at its classical value.

Ignoring surface terms the action integral now takes the form

$$S \sim -\frac{3V}{8\pi} \int_{t_1}^{t_2} e^{3\bar{\xi}} (\dot{\bar{\xi}}^2 - \dot{\zeta}^2 - \dot{\eta}^2)\, dt \tag{4.2}$$

where $V$ has the same meaning as before.

Here the bar on top of $\xi$ indicates that we are treating it as prescribed classical function. Using the notation of Section 3 we write

$$e^{3\bar{\xi}} = \tfrac{9}{2}Mt\,(t + \Sigma) \tag{4.3}$$

We can simplify (4.2) further by defining a new time coordinate given by

$$\tau = \int e^{-3\bar{\xi}} dt = \frac{2}{9M\Sigma} \ln \frac{t}{t + \Sigma} \tag{4.4}$$

Let $\tau_1 = \tau(t_1)$, $\tau_2 = \tau(t_2)$. Note that as before we wish to "approach" the singularity so that $t$ is decreasing towards $t = 0$ and $\tau$ is decreasing towards $\tau = -\infty$. Thus $\tau_2 < \tau_1$, and we are interested in the limit $\tau_2 \to -\infty$.

Writing $\eta' \equiv d\eta/d\tau$, $\zeta' \equiv d\zeta/d\tau$, we get for the propagator in the usual notation

$$K[\eta_2, \zeta_2, \tau_2; \eta_1, \zeta_1, \tau_1] = \iint \exp\left[\frac{3iV}{8\pi} \int_{\tau_1}^{\tau_2} (\eta'^2 + \zeta'^2)\, d\tau\right] \mathfrak{D}\eta\, \mathfrak{D}\zeta \tag{4.5}$$

The solution of the double path integral is no different from that of a free particle of mass $3V/4\pi$ moving in a plane! We get

$$K[\eta_2, \zeta_2, \tau_2; \eta_1, \zeta_1, \tau_1] = -\frac{8\pi^2(\tau_2 - \tau_1)}{3V} \exp \frac{3iV[(\eta_2 - \eta_1)^2 + (\zeta_2 - \zeta_1)^2]}{8\pi(\tau_2 - \tau_1)} \tag{4.6}$$

As before we apply $K$ to a wave packet of dispersion $\Delta_1$ (in probability) for $\eta$ and $\zeta$ round their classical values, $\bar{\eta}$ and $\bar{\zeta}$, respectively, and moving with the classical "velocity" $(\bar{\eta}', \bar{\zeta}')$.

Elementary calculations show that the wave packet "moves" in the $(\eta, \zeta)$ plane with the constant classical velocity $(\bar{\eta}', \bar{\zeta}')$ and at the same time diffuses so that its dispersion at $\tau = \tau_2$ is $\Delta_2$, where

$$\Delta_2{}^2 = \Delta_1{}^2 + \frac{4\pi^2(\tau_2 - \tau_1)^2}{9V^2\Delta_1{}^2} \tag{4.7}$$

Hence as the classical singularity ($t_2 \to 0+$, $\tau_2 \to -\infty$) is approached $\Delta_2$ diverges as

$$\Delta_2 \sim \frac{4\pi}{27M\Sigma V\Delta_1} |\ln t_2| \tag{4.8}$$

This is exactly the same behavior as noted in (4.1) for the divergence in conformal fluctuations.

The constancy of the velocity of the wave packet in the $(\eta, \zeta)$ plane in the above example, and the stationarity of the wave packet at $\phi = 0$ in the preceding section ensure that the quantum fluctuations preserve the classical solution as the mean in all the cases discussed here. However, the divergence of the fluctuations renders the mean without operational significance.

## §(5): *An Axisymmetric Singularity*

The general Bianchi type I model was discussed in two stages: first by considering fluctuations in the volume expansion $\xi$ for given classical shear terms $\bar{\eta}$

and $\bar{\zeta}$ and next by considering the fluctuations in $\eta$ and $\zeta$ for a fixed classical expansion function $\bar{\xi}$. An exact treatment of the case where all three degrees of freedom $\xi, \eta, \zeta$ are allowed quantum fluctuations is not possible within the present framework. It is of course possible to deal with this case by the WKB approximation for evaluating path integrals [10], but we shall not go into those details here. Even under the more limited assumptions we have already made a case for divergent fluctuations near the classical singularity.

It is, however, possible to deal exactly with the axisymmetric case of (3.12) with $A_1(t) = A_2(t)$. We describe this case briefly. Define

$$\lambda(t) = \tfrac{1}{2}A_1{}^{1/2}(A_1 + A_3), \qquad \mu(t) = \tfrac{1}{2}A_1{}^{1/2}(A_1 - A_3) \tag{5.1}$$

Then the action functional for the dust universe becomes

$$S \sim -\frac{V}{6\pi}\int_{t_1}^{t_2} (\dot{\lambda}^2 - \dot{\mu}^2)\, dt \tag{5.2}$$

where $V$ has the same interpretation as before. The propagator corresponding to this is

$$K[\lambda_2, \mu_2, t_2; \lambda_1, \mu_1, t_1] = \frac{6\pi^2(t_2 - t_1)}{V} \exp \frac{-iV[(\lambda_2 - \lambda_1)^2 - (\mu_2 - \mu_1)^2]}{6\pi(t_2 - t_1)} \tag{5.3}$$

The corresponding dispersion in probability grows as

$$\Delta_2{}^2 = \Delta_1{}^2 + \frac{9\pi^2}{4V^2\Delta_1{}^2}(t_2 - t_1)^2 \tag{5.4}$$

As in the general case the singularity here is identified with $t_2 \to 0+$. Although $\Delta_2$ does *not* diverge according to (5.4) as $t_2 \to 0$, the departures from classical solutions are in fact divergent. To see this consider the classical solution to the problem in the following form:

$$\bar{A}_1 \propto (t + \Sigma)^{2/3}, \qquad \bar{A}_3 \propto t\,(t + \Sigma)^{-1/3} \tag{5.5}$$

For $\Sigma > 0$ this represents a pancake singularity at $t = 0$. However, in a nonclassical solution $A_3$ will be nonzero and finite at $t = 0$ so that the ratio

$$\frac{A_3}{\bar{A}_3} \sim \frac{1}{t} \longrightarrow \infty$$

as $t \to 0+$. The corresponding fluctuations in the directions perpendicular to the axis of symmetry are finite at the singularity.

## §(6): *Conclusion*

MacCallum [13] has reviewed the quantum cosmological models, especially the homogeneous ones, and has outlined the various conceptual difficulties of quantizing the "full theory." The procedures adopted by different workers in the area are different and are of a piecemeal nature. The expectation is that the approximations and specializations made may still give some indication of the actual situation in the full theory. As MacCallum points out, ". . . two different schemes might lead to inequivalent results" [13]. For example, Misner's approach [12] using canonical quantization seems to indicate that the space-time singularity would not be avoided for Binachi type IX cosmologies by going over to a quantum theory. This appears to be opposite in spirit to the work of this paper, where it is shown that there is a finite probability for having a nonsingular cosmology of Bianchi type I. It is difficult to compare the two conclusions since they come from different approaches: the Hamiltonian (ADM) formalism in Misner's case and the path integral approach of this paper.

The examples presented here make use of the facility of evaluating path ingegrals for quadratic functionals exactly. It has therefore been possible in all cases to compute the exact propagators describing the quantum behavior of space-time geometries. Being exact solutions they can be applied with some confidence in the limiting cases of space-time singularities.

In all cases discussed here the result has been that the classical Einstein solution is the "mean" of the quantum solutions, but that the fluctuations around the mean diverge at the classical singularity. On intuitive grounds this result can be expected as a consequence of the uncertainty principle of quantum theory. At the classical singularity of the type considered here all matter is concentrated at "one point." Quantum mechanically we cannot reconcile such a definitive statement with the uncertainty principle.

We should not therefore expect the final state of gravitational collapse of a compact object to be singular. Rather, as the object shrinks, the alternatives to the classical solution increase in number and there is a finite probability for the object to bypass the singular state altogether. In the same way we need not identify a definite singular state as the beginning of the universe. Even though its present behavior looks "classical," it may have passed through radically different nonclassical phases in its history.

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