

HOW EMPTY MUST EMPTY SPACE BE ?

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Abstract

We propose a generalised definition of empty space in general relativity which is characterised by vanishing of gravitational charge density $R_{ik}u^i u^k = 0$ instead of $R_{ik} = 0$. It can be argued that so long as there exists energy distribution outside the empty space region, the generalised definition seems to be more appropriate for its description.

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By empty space it is generally understood as the region free of any kind of energy distribution. A region of space could be empty but there could exist energy distribution outside it or the entire space (i.e. the whole of Universe) is empty. These two situations can in principle be different. We wish to argue in the following that a truthful description of the former situation would ask for a generalised definition of empty space. By which we shall mean the space from which all removable matter-energy is removed, however distribution exterior to the empty region may cause non-zero stress-energy inside. In spherically symmetric case this energy distribution could be identified with the radial cosmic string or arising out of a non-zero but constant gravitational potential.

In the Newtonian theory, gravitational field of a mass point is described by the well-known Laplace equation $\nabla^2\phi = 0$ which has the general solution $\phi = k - M/r$. The constant k defines a pure gauge and hence can be chosen freely. In the space occupied by matter, the field is given by $\nabla^2\phi = 4\pi\rho_c$ where ρ_c is the gravitational charge density (we term it so rather than matter density with an eye on the general relativistic (GR) situation where the two denote different things). The empty space is defined by $\rho_c = 0$.

In GR, the empty space is defined by $R_{ik} = 0$ and the corresponding general solution is the well-known Schwarzschild solution. This solution successfully incorporates the Newtonian field and makes predictions for the perihelion shift of mercury, the bending of light and the gravitational red shift, which have been observationally verified. That is the Schwarzschild solution has a sound physical base supported by the observations and there is no obvious justification for any further investigation in this direction. However we would like to ask the question what happens if we define empty space in the Newtonian spirit by $\rho_c = 0$ rather

than $R_{ik} = 0$? In GR ρ_c is, as indicated above, different from matter or energy density. It is in fact given by $R_{ik}u^i u^k = -4\pi\rho_c$, where u^i is the timelike 4-velocity. For a perfect fluid distribution $\rho_c = \rho + 3p$. It is ρ_c that acts as active charge density and produces gravitational attraction.

It is remarkable to note that non-vacuous energy-momentum tensor T_{ik} can in fact give rise to $\rho_c = 0$. Such a prescription exists for geometric cosmic string [1,2] given as

$$T_{ik} = \rho(u_i u_k - w_i w_k) \quad (1)$$

where $u_i u^i = 1 = -w_i w^i$, $u^i w_i = 0$ and ρ is the string energy density which is equal to the string tension. This will imply

$$R_{ik} = -8\pi\rho(u_i u_k - g_{ik} - w_i w_k) \quad (2)$$

satisfying $R_{ik}u^i u^k = 0$. Here u^i is the timelike unit 4-velocity while w^i is the spacelike unit vector defining the string direction.

The important point to note is that for the non-vacuous energy distribution (1), gravitational charge density $\rho_c = 0$; i.e. R_{ik} need not vanish for space to be free of ρ_c and in turn free of gravitational attraction. This means eqn.(2) should be analogous to empty space in the spirit of the Newtonian definition characterised by vanishing of ρ_c .

The aim of this communication is to examine how good is the eqn.(2) as an alternative definition of empty space? The observational support to $R_{ik} = 0$ stems entirely from the Schwarzschild solution and hence it is pertinent to find the analogous radially symmetric solution of eqn.(2) for the field of a static mass point.

For its viability, it should on one hand agree with the Newtonian field in the first approximation and with the Schwarzschild solution on the other to conform to the observationally verified predictions. Since $R_{ik} = 0$ will always be a solution of eqn.(2) for $\rho = 0$, the Schwarzschild solution will hence be included in the general solution of (2) as a special case. The difference between the two will essentially arise in their asymptotic behaviour. The Schwarzschild solution is asymptotically flat while the other will not be, i.e. the asymptotic spacetime is non-flat.

As we shall demonstrate in the following that eqn.(2) for the situation under consideration reduces to the single equation $\nabla^2\phi = 0$, the usual Laplace equation of the Newtonian theory. This obviously has the general solution $\phi = k - M/r$ where k and M are constants of integration. The amazing fact is that $\phi = k = \text{const} \neq 0$ solution of (2) gives rise to a non-flat spacetime which will define the asymptotic limit for the general solution. On the other hand Einstein's vacuum equation $R_{ik} = 0$ reduces to two equations, one of which is the above Laplace equation and the other is its first integral which determines the free constant, $k = 0$. The corresponding spacetime in that case is the well-known Schwarzschild. That is, $\phi = 0, -M/r$ are the two solutions, common to both the Newtonian and GR theories, respectively representing flat and Schwarzschild spacetimes, while $\phi = k \neq 0$ is only a solution of the former but not of the latter. That is the non-zero constant k in the solution of the Laplace equation is the distinguishing feature between $R_{ik} = 0$ and eqn.(2) in this situation.

Consider the general spherically symmetric metric in the curvature coordinates,

$$ds^2 = A dt^2 - B dr^2 - r^2(d\theta^2 + \sin^2\theta d\psi^2) \quad (3)$$

where A and B are functions of r and t . We take $u = \sqrt{A}dt$ and $w = \sqrt{B}dr$, then eqn.(2) implies $R_{01} = 0$, $R_0^0 = R_1^1 = 0$ and $R_2^2 = R_3^3 \neq 0$ of which $R_{01} = 0$ and $R_0^0 = R_1^1$ lead to $A = B^{-1} = 1 + 2\phi(r)$, say (the multiplicative constant in $A = \text{const.}B^{-1}$ can always be absorbed by redefining the time coordinate). Note that the Birkhoff's theorem continues to hold good implying staticity of the field. Then we have

$$R_0^0 = -\nabla^2 \phi = -\frac{1}{r}(r\phi)'' = 0 \quad (4)$$

and

$$R_2^2 = R_3^3 = -\frac{2}{r^2}(r\phi)' \quad (5)$$

where a prime denotes differentiation w.r.t. r . Hence the general solution of eqn.(2) is the solution of (4), which means $\phi = k - M/r$ and $R_2^2 = -2k/r^2 \neq 0$ unless $k = 0$. For $R_{ik} = 0$ we should in addition to (4) have $R_2^2 = 0$ which fixes $k = 0$. This is how Einstein's vacuum equation does not permit arbitrariness in choosing zero of the function ϕ . Since it satisfies the Laplace equation in conformity with the classical physics, I would tend to call it gravitational potential. It may be noted that ϕ satisfies the same Laplace equation in both Newtonian as well as GR theories, where the former provides freedom to choose k while the latter does not. It is in this sense we would like to say that GR defines the absolute zero of ϕ , which is arbitrary in the Newtonian theory [3].

The general solution of eqn.(2) will thus be given by the metric

$$ds^2 = (1 + 2k - 2M/r)dt^2 - (1 + 2k - 2M/r)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\psi^2) \quad (6)$$

with $4\pi\rho = -k/r^2 \geq 0$ for $k \leq 0$. The metric differs from the Schwarzschild's only in appearance of the constant k . Note that the constant $1 + 2k$ cannot be absorbed by coordinate transformations even when $M = 0$, it could only be shifted from g_{11} to g_{22} and g_{33} while in g_{00} it could be absorbed. It is clear that the event horizon will now be defined by $r = 2M/(1 + 2k)$. The orbital motion will also require the scaling of $M \rightarrow M/(1 + 2k)$ and hence the metric (6) will agree with the observed effects provided k is small enough so as not to cause any disagreement. That is the observed values will put an upper bound on k . The surface gravity will be given, as in the Schwarzschild case, by M/r^2 evaluated at the horizon, which will read as $(1 + 2k)^2/4M$. It does not however adhere to the same scaling of M . Since $k < 0$, it will mean an increasing trend in all the observable effects and a decreasing trend in surface gravity.

The most remarkable feature of eqn.(2), in contrast to $R_{ik} = 0$, is that metric (6) does not reduce to flat space when $M = 0$. Since $4\pi\rho = -k/r^2 \neq 0$ always (whether $M = 0$ or not) unless $k = 0$, hence it will be non-flat so long as $k \neq 0$. Its curvature is given by the sole non-zero component of the Riemann tensor [4], $R^2^3_{23} = k/r^2$ (this component remains invariant for coordinate transformations that preserve spherical symmetry, i.e. it is an invariant of spherical symmetry). It is clear that free particles will have flat spacetime trajectories indicating absence of the Newtonian gravity. The curvature of space at any given r is proportional to the curvature of the sphere of that radius. The curvature of sphere can only be perceived by observing the convergence of free particles trajectories at the poles and so would be the case for parallelly propagating geodesics at a fixed r (note that particles do not "fall" radially) for the spacetime (6) with $M = 0$. In view of being free of the Newtonian gravity and having all but one, which is invariant for spherical symmetry, curvature

non-zero, it could justifiably termed as the “minimally” curved spacetime. It is however curious that gravitational acceleration is zero but tidal acceleration is not. This indicates that it is purely a relativistic effect. This spacetime differs from the flat spacetime only in the context of convergence of free particles trajectories, as on the surface of a sphere.

Let us now address some general remarks on how good can eqn.(2) be an alternative definition of empty space ?

1. To create an empty space we should remove matter-energy from certain region of space. What does this mean: $\rho_c = 0$ or $R_{ik} = 0$? Operationally the former is achievable by removing all ponderable matter and energy yet R_{ik} need not be zero. The basic question is : Can we allow for the string energy distribution (1) in empty space or R_{ik} must necessarily vanish ? The former seems to conform to the Machian picture. Ultimately only observations will decide how empty must empty space be ?

Consider a sphere made up of steel, say. Carve the entire metal out and spread it uniformly over the surface, so as to create a thin spherical shell of $r > 2M/(1 + 2k)$. The question is, what equation will describe the hollow interior of the shell, eqn.(2) or $R_{ik} = 0$? If the former, then the interior will be described by the metric (6) with $M = 0$ and the exterior by $M \neq 0$ and the two will match continuously at the boundary. In the case of the latter, the interior is flat spacetime and the exterior the Schwarzschild and the two will not match continuously at the boundary. Such a construct is not allowed in GR, unless the boundary moves facilitating the matching.

2. The spacetime (6) describing field of a mass point is not asymptotically flat

instead it can be said to be asymptotically “minimally” curved in the sense defined above. Note that in contrast to GR, flat spacetime is not a solution of the generalised empty space eqn.(2). This is very important from the Machian point of view which relates the locally measured inertial properties with the distribution in the rest of the Universe. Here we would not like to enter into a discussion on Mach’s Principle, which we would leave for a separate paper, except to say that these features of eqn.(2) are in conformity with the Machian definition of empty space. We may thus say that eqn.(2) defines the Machian (vacuum) empty space of which the metric (6) is the general solution for spherical symmetry. It may be termed as the Machian Schwarzschild solution. Its asymptotic limit ($M = 0$) will be the analogue of flat spacetime and is termed the Machian “minimally” curved (free) spacetime from which all the removable matter-energy is removed.

3. In the Machian picture, the field of a mass point can be envisioned as a mass point sitting in a spherical cavity of $r > 2M/(1 + 2k)$ and if the rest of the Universe has homogeneous and isotropic distribution outside the cavity whose imprint on the field inside will manifest through the constant k . Since the exterior distribution is isotropic and homogeneous about the cavity, it cannot produce gravitational force inside but will instead produce a constant non-zero potential which is denoted by k . In this picture $k = 0$ will correspond to removal of the rest of the Universe as is the case for the Schwarzschild solution. So long as there exists non-vacuous rest of the Universe k should not vanish. Then what we can achieve as analogue of flat spacetime will be the Machian free spacetime, characterised by $\rho_c = 0$, indicating absence of all removable matter-energy in a region of space.

It may be noted that the constant non-zero potential ϕ gives rise to the string energy distribution (1) in the spherically symmetric case. That is so long as there exists energy distribution outside the spherical shell, the empty space inside will be described by eqn.(2) rather than $R_{ik} = 0$. This is because the exterior uniform distribution will produce a non-zero constant ϕ inside which can only be sustained by (2) and not by $R_{ik} = 0$. Hence in a realistic situation, where there does occur a non-vacuous rest of the Universe, the generalised empty space equation (2) should be the right equation.

In the spirit of Mach's Principle it can be said that existence of rest of the Universe forbids ϕ to attain the absolute zero, i.e. ϕ can be zero only when there exists no rest of the Universe. Like zero of the absolute temperature, it could be approached but never achieved.

4. As stated above there is no easy physical way to distinguish the Machian empty space (6) from the Schwarzschild because k is expected to be very-very small to be detected through $M/(1+2k)$ with $k \leq 0$, in the perihelion shift and the light bending. On the other hand the Machian free spacetime ((6) with $M = 0$) can only be distinguished from flat spacetime by considering the geodesic deviation (tidal effect) in θ or ψ - direction as $R^2{}^3{}_{23}$ is alone non-zero. The tidal force will diverge as $r \rightarrow 0$; but recall particles experience no radial fall in this spacetime and hence will remain at fixed r as they move along θ or ψ - direction (which will always be at $r > 0$). Hence there is though curvature singularity at $r = 0$, which should make tidal force diverge, but it is not physically detectable. This is a very curious and interesting feature of the Machian free spacetime and hence it is almost as good and benign as the flat spacetime.

5. There is a very strange and exotic kind of energy distribution in (1) which could, apart from cosmic string, be viewed as the counteracting superposition of timelike and spacelike dust and the two seem to counteract each other gravitationally. The conservation equation for (1) will give $u_{i;k}u^k = 0$ and $w_{i;k}w^k = 0$. One can alternatively take the view that given a metric one can always define timelike and spacelike unit vectors, using them and the metric give a prescription for R_{ik} such that $R_{ik}u^i u^k = 0$. This is how one writes eqn.(2) and let that define the generalised (Machian) empty space. It involves only the kinematic quantities g_{ik} , u_i and w_i . One then does not seek physical interpretation of (1).
6. As mentioned earlier the stress tensor (1) was written for geometric string [1,2] and hence the metric (6) has already been considered for string with a mass point [5] as well as for global monopole [6]. Treating this as a generalised vacuum, one should obtain the solutions of eqn.(2) in various situations. All spherically symmetric solutions readily follow for charged mass point as well as cosmological constant and the general solution is : $A = B^{-1} = 1 + 2k - 2M/r + Q^2/r^2 + \Lambda r^2/3$. It would be particularly interesting to find the Machian generalisation of the Kerr solution. We have found some solutions of eqn.(2) with the Kerr-NUT symmetry [7] but the search for a proper Machian Kerr solution is still on. With a view to examine validity of eqn.(2) as the generalised vacuum equation we wish to take up a comprehensive project to find solutions of eqn.(2) in all kinds of situation [8], particularly where there exist $R_{ik} = 0$ solutions.

Finally the most pertinent question is : what are the distinguishing predictions of the generalised empty space eqn.(2)? One if the observations on perihelion shift

and light bending show an upward trend howsoever small would be welcome. The other place to look for is to measure tidal force between two parallelly propagating particles in a freely falling spaceship. I would believe that such precision experiments could be devised in future. The prediction has to be tested passively, only as departure from flat space because we cannot estimate the constant k independently. That is any departure from flat space behaviour will be an indication in favour of the generalised definition. I would rather suspect that this question may remain undecided for a long time to come.

References

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