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Vacuum polarization around an Aharonov–Bohm solenoid

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Abstract. The quantisation of a charged scalar field in an externally specified electromagnetic field, described by the vector potential $A_i = \partial_i f$ with $f(t, r, \theta, z) = B\theta$ is discussed. The electromagnetic field is zero everywhere except at the origin; a singular magnetic field (Aharonov–Bohm field) exists at the origin. The vacuum polarization around such a magnetic field is computed and the non-local behaviour is discussed.

Keywords. Aharonov–Bohm; vacuum polarization; effective action.

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1. Introduction

Quantum theory provides us with an interesting example of non-local influence in the form of Aharonov–Bohm effect (Bohm and Aharonov 1959). In the context of non-relativistic quantum mechanics, this effect manifests itself as a phase shift in the wavefunction of a charged particle going around a solenoid containing a magnetic field. This phenomenon is of special interest because the paths of the electron do not penetrate the region containing the magnetic field.

One can state the result in a different manner: The charged particle is interacting with an electromagnetic field which can be represented as a pure gauge (i.e. $A_i = \partial_i f$) everywhere in the spacetime, *except* at one point; at that point the field is singular in such a manner that the total flux around the origin is finite. Except for this singularity, one could have transformed away the electromagnetic potential and one would have expected no physical effect to survive. In other words, we have, in the case of Aharonov–Bohm effect an example of a situation in which a singular field confined to a point producing an interesting physical effect. (Of course, it is not *necessary* that the field should be singular; a finite impenetrable solenoid will also lead to a similar effect. The relevant point is that *even* a singular field will lead to a finite, physical result.)

In this paper, a relativistic generalization of the above effect in the context of a charged scalar field interacting with an external electromagnetic field F_{ik} is studied. To quantise the scalar field in such a given background, we make some particular gauge choice and introduce a vector potential A_i. One expects the final results to be independent of the gauge and depend only on F_{ik}. In particular, if the field vanishes identically everywhere – so that $A_i = \partial_i f$ – we do not expect any nontrivial effects to arise.

It turns out, however, that certain peculiarities arise in the formulation of the quantum theory if the gauge function f becomes singular in some *finite* domain (the results of this analysis, which has interesting implications for quantisation of fields in external *gravitational* fields, is presented elsewhere (see Padmanabhan 1990). It is, therefore, interesting to see what happens when the gauge function becomes ill-defined at a *single point* in space. If we take $f(x^i) = f(t, r, \theta, z) = B\theta$ then the gauge function is well-defined everywhere except at the origin. *This gauge function is precisely the one corresponding to the Aharonov–Bohm effect.* This leads us to investigate the quantum theory of a charged scalar field in this background.

This study reveals some interesting features. The non-local nature of the Aharonov–Bohm effect, of course, persists even in the relativistic case in an interesting manner. We compute below the (regularized) effective lagrangian L_{eff} for a the scalar field in this background and show that there is non-trivial vacuum polarization around the solenoid. This is true in spite of the fact that L_{eff} is a 'local' density (the word 'local' is used in the sense that the effective action can indeed be expressed as a spacetime integral over an effective *lagrangian*; this, of course, will not be the case in general). Though it is a local object its calculation requires specific choice of boundary conditions, which in turn, contains information of global nature.

The main calculation is presented in § 2 and the results are summarized in § 3.

2. Vacuum polarization around a thin solenoid

2.1 *Gauge invariance and the effective action*

Let us consider a charged scalar field Φ interacting with an externally specified electromagnetic field F_{ik}, described by some four-potential A_i. The quantum effects in an external electromagnetic field can be conveniently studied using the concept of effective action $\mathscr{A}_{\mathrm{eff}}$, defined formally as

$$\exp i\mathscr{A}_{\mathrm{eff}} \equiv \exp i \int \mathrm{d}^4 x L_{\mathrm{eff}} \equiv \int \mathscr{D}\Phi \mathscr{D}\Phi^* \exp i\mathscr{A}[\Phi, \Phi^*, A_\mu] \tag{1}$$

where $\mathscr{A}$ is the action describing the interaction of Φ with A_μ.

$$\mathscr{A} = \int \mathrm{d}^4 x \Phi^* [(i\partial - qA)^2 - m^2 + i\varepsilon] \Phi \equiv \int \mathrm{d}^4 x \Phi^* \hat{D} \Phi \tag{2}$$

(The real and imaginary parts of L_{eff}, for example, describe phenomenon related to vacuum polarization and pair creation due to the electromagnetic field). To calculate L_{eff}, we will use the proper time description due to Schwinger (1951). The central quantity in this description is the Kernel:

$$\begin{aligned} K(x', x; s) &= \langle x' | \exp(isD) | x \rangle \\ &= \int \mathscr{D}x^\mu(\tau) \exp -i \int_0^s \mathrm{d}\tau \left[\frac{1}{2}\dot{x}^2 - qA_\mu \dot{x}^\mu + \frac{1}{2}m^2 - i\varepsilon \right] \end{aligned} \tag{3}$$

from which the effective lagrangian L_{eff} and the propagator $G(x', x)$ can be calculated by the relations

$$L_{\text{eff}} = -i\int_0^\infty \frac{ds}{s} K(x, x; s) \tag{4}$$

and

$$G(x', x) = \int_0^\infty ds K(x', x; s). \tag{5}$$

[A notation which is slightly different from the usual one is used. It is more conventional to define the lagrangian with an $(1/4)\dot{x}^2$ rather than with the $(1/2)\dot{x}^2$. I have normalized the lagrangian in this manner for some algebraic convenience later on. The L_{eff}, of course, is unchanged by this normalization while the Green's function will get multiplied by a factor (1/2)].

Since we expect physical processes (like pair creation) to be invariant under the gauge transformation, $A_\mu \to A_\mu + \partial_\mu f$, we expect L_{eff} to depend only on $F_{\mu\nu}$ and not on the gauge chosen to describe the field. It is interesting to see how this result comes about. Under the gauge transformation $A_\mu \to A_\mu + \partial_\mu f$, the term $(qA_\mu \dot{x}^\mu)$ picks up the additional piece $(q\dot{x}^\mu \partial_\mu f)$ and the path integral amplitude is multiplied by the factor

$$P = \exp iq \int_0^s d\tau (\dot{x}^\mu \partial_\mu f). \tag{6}$$

It is usual to 'perform' the integration in the above expression and obtain

$$P = \exp iq \int_0^s d\tau \frac{d}{d\tau} f[x^\mu(\tau)] = \exp iq[f(x') - f(x)]. \tag{7}$$

If this result is valid, then the physics will be gauge invariant. The L_{eff} only involves the coincidence limit $K(x, x; s)$ for which – assuming we can take $f(x') = f(x)$ when $x = x'$ – the factor P is unity. The propagator is only modified by a phase, and it can be shown that amplitudes for physical processes do not change.

The above discussion can be cast in a different form which is more useful in what follows. We may say that the path integral amplitude for a given path $x'(\tau)$, connecting x and x' gets multiplied by a factor

$$F = \exp iq \int_x^{x'} d\tau \dot{x}^\mu A_\mu \tag{8}$$

in the presence of the electromagnetic field. If we change the gauge, F will be further multiplied by P in (6). As long as we can integrate (6) to obtain (7), each amplitude will be multiplied by a factor which is independent of the path (and dependent only on the end points); so the physics will remain unchanged.

I want now discuss situations in which the result (7) cannot be obtained from (6). The derivation leading to (7) fails in two physically interesting cases.

The first one – which provides interesting analogies with phenomena in curved spacetime and accelerated frames – corresponds to the situation in which the function $f(x)$ is singular in some region; this is discussed in (Padmanabhan 1990) and will not be considered here.

The second situation, which is probably more interesting, is the following: Suppose one of the spatial coordinates we are using is periodic but $f(x)$ does not respect this periodicity. For example, we may use the $x^i = (t, r, \theta, z)$ coordinate system and take $f(x^i) = B\theta$[clearly, $f(\theta + 2\pi n) \neq f(\theta)$]. A path which "winds" around the origin in xy-plane n-times will now produce an additional factor

$$F_n \equiv \exp iqB(\theta - \theta' + 2\pi n). \tag{9}$$

The original Kernel could have been written as

$$K_{B=0}(x', x; s) = \sum_{n=-\infty}^{\infty} K_n(x', x; s) \tag{10}$$

where K_n is the Kernel obtained by summing over paths with a given 'winding number' n. The Kernel in the presence of a gauge-function f will be

$$K_B(x', x; s) = \sum_{n=-\infty}^{\infty} K_n(x', x; s) \exp[iqB(\theta - \theta' + 2\pi n)]. \tag{11}$$

This will modify the L_{eff} as well, since the coincidence limit is also affected:

$$K_B(x, x; s) = \sum_{n=-\infty}^{\infty} K_n(x, x; s) \exp(iqB 2\pi n). \tag{12}$$

Notice that the Kernel invariant under the change $\theta' \to \theta' + (2\pi/qB)$. Thus θ now has a periodicity of $(2\pi/qB)$. It is clear that the propagator $G(x', x)$ will also exhibit this periodicity.

This above example is essentially a calculation of K (and L_{eff}) for an Aharonov–Bohm potential. It is clear that, even though A_μ appears to be expressible as $(\partial_\mu f)$ locally, we do not have a pure-gauge situation. The ill-defined nature of f at origin leads to a delta-function magnetic field along z-axis (Aharonov–Bohm field). This is most easily seen by noticing that the flux through a path around the origin

$$\oint A_\mu \, \mathrm{d}x^\mu = \int_0^{2\pi} A_\theta \, \mathrm{d}\theta = 2\pi B \tag{13}$$

is non-zero. Paths with different winding numbers cannot be continuously deformed to each other. The gauge function f and A_μ can be written in the Cartesian coordinates as

$$f = B \tan^{-1}\left(\frac{y}{x}\right); \quad A_\mu = B\left(0, -\frac{y}{x^2 + y^2}, +\frac{x}{x^2 + y^2}, 0\right). \tag{14}$$

To investigate the physical effects in this context we can compute L_{eff} and G in this background. This is done in the next section. The interpretation of these quantities requires some comment in the present context. Usually, one computes L_{eff} in order to evaluate the effect of the quantum field (here Φ) on the background (here, the F_{ik}). The interest, therefore, is in obtaining L_{eff} as a *functional* of the background field: $L_{\text{eff}} = L_{\text{eff}}[F_{ik}(x)]$. (This is usually accomplished by some approximation like adiabaticity etc.). There is however, an alternative interpretation to L_{eff}, when treated as a *function* of the coordinates: $L_{\text{eff}} = L_{\text{eff}}[F_{ik}(x)] = L_{\text{eff}}(x)$. It can be shown that $V_{\text{eff}} = -L_{\text{eff}}(x)$, when suitably regularized, represents the change in the energy density of virtual particles in the presence of the external field. This is in fact the original procedure (Heisenberg and Euler 1936) adopted by Euler and Heisenberg to derive the effective lagrangian for the electromagnetic field. [A text book derivation of this result is available in Lifshitz and Pitaevskii 1984. I have given the derivation of this result, for the simpler case of self-interacting scalar field, in Appendix 2 for those readers who just want to see the nature of the argument]. Thus, whenever L_{eff} is real, it gives measure of vacuum polarization due to the external field. This is the interpretation we will adopt.

The Green's function $G(x, x')$, of course, describes all physical processes of the scalar field Φ in the presence of the external field. What is more, the coincidence limit of the Green's function $G(x, x)$ – when suitably regularised – will be a measure of the vacuum fluctuations in $|\Phi(x)|^2$ around the event x^i. This is the second quantity we will compute.

2.2 *Calculation of* L_{eff} *and* G

To obtain the effective lagrangian and the Green's function we have to compute the path integral

$$K(x^i, x^{i'}; s) = \int \mathscr{D}x \exp i \int_0^s d\tau\, L \tag{15}$$

with

$$L = -\frac{1}{2}\dot{x}_i \dot{x}^i + qA_\mu \dot{x}^\mu - \frac{m^2}{2} + i\varepsilon. \tag{16}$$

where the background field is described by the potential $A_\mu = (0, -\mathbf{A})$ and

$$\mathbf{A} = B\left(-\frac{y}{x^2+y^2}, +\frac{x}{x^2+y^2}, 0\right) = B\nabla\left(\tan^{-1}\frac{y}{x}\right) = \nabla(B\theta). \tag{17}$$

In the Kernel the path-integrations over $z(\tau)$ and $t(\tau)$ can be easily performed. We are thus left with the expression

$$K(x^i, x^{i'}; s) = \left(\frac{1}{2\pi s}\right)\exp\left[-\frac{i}{2s}((t-t')^2 - (z-z')^2) - \frac{is}{2}(m^2 - i\varepsilon)\right]\mathscr{G} \tag{18}$$

where

$$\mathscr{G}(x, x'; y, y'; s) = \int \mathscr{D}x\mathscr{D}y \exp\left[\frac{i}{2}\int_0^s d\tau(\dot{x}^2 + \dot{y}^2) - iq\int_0^s \mathbf{A}\cdot\dot{\mathbf{x}}\, d\tau\right]. \tag{19}$$

This Kernel is evaluated in Appendix 1. The result can be expressed as

$$\mathscr{G} = \sum_{\text{all}\, n} \mathscr{G}_n \equiv \sum_{\text{all}\, n} \exp[-iqB(\phi - 2\pi n)] \int_{-\infty}^{+\infty} d\eta \exp[i\eta(\phi - 2\pi n)] \times \left(\frac{1}{2\pi i s}\right) I_{|\eta|}\left(\frac{rr'}{is}\right) \exp\frac{i}{2s}(r^2 + r'^2). \tag{20}$$

where $\phi = \theta - \theta'$ and $I_\mu(z)$ is the modified Bessel function. The propagator can be found from our Kernel by the relation

$$G(x^i, x^{i'}) = \int_0^\infty \mathrm{d}s K(x, x'; s) = \sum_n G_n. \tag{21}$$

It turns out that this integral is fairly complicated when $m \neq 0$, but simplifies considerably in $m \to 0$ limit. In fact the result for $m \neq 0$ can be easily reconstructed from the $m = 0$ case by the following trick. Let $\tilde{K}_m(t, t; x, x'; y, y'; p; s)$ be the Fourier transform of the exact Kernel with respect to $(z - z')$. It is obvious from (18) that p and m appear in $\tilde{K}_m$ only in the combination $(p^2 + m^2)$. Therefore, $\tilde{K}_m(p) = \tilde{K}_{m=0}\cdot(\sqrt{p^2 + m^2})$. So, if we know the Kernel in the $m = 0$ limit we can always construct the Kernel for $m \neq 0$ by quadrature; we first Fourier transform $K_{m=0}(x, x')$ in $(z, -z')$ to obtain $\tilde{K}_{m=0}(\cdots, q)$ and then construct $\tilde{K}_m(\cdots, p)$ by setting $p = \sqrt{q^2 + m^2}$ and finally Fourier transforming back in p. This analysis also shows that the results are analytic in m. The calculations are messy and the final result can be expressed in terms of integrals over hypergeometric functions. Instead of pursuing this case which is not physically illuminating, we will confine ourselves to a toy model with $m = 0$. The integral for G_n can be evaluated in closed form in the $m \to 0$ limit. We get

$$\begin{aligned} G_n &= \exp[-iqB(\phi - 2\pi n)] \int_0^\infty \frac{\mathrm{d}s}{(2\pi i s)^2} \exp(i/2s) l^2 \\ &\quad \times \int_{-\infty}^{+\infty} \mathrm{d}\eta \exp[-i\eta(\phi - 2\pi n)] I_{|\eta|}\left(\frac{rr'}{is}\right) \exp\frac{i}{2s}(r^2 + r'^2) \\ &= \exp[-iqB(\phi - 2\pi n)] \int_{-\infty}^{+\infty} \mathrm{d}\eta \exp[-i\eta(\phi - 2\pi n)] \\ &\quad \times \int_0^\infty \frac{\mathrm{d}s}{(2\pi i s)^2} I_{|\eta|}\left(\frac{rr'}{is}\right) \exp\frac{i}{2s} R^2 \end{aligned} \tag{22}$$

where $R^2 \equiv l^2 + r^2 + r'^2 = (z - z')^2 - (t - t')^2 + r^2 + r'^2$. The integral over s gives, on substituting $s^{-1} = x$,

$$\int_0^\infty \mathrm{d}x \exp[(i/2)R^2 x] I_{|\eta|}\left(\frac{rr'}{i}x\right) = \frac{i}{rr'} \frac{\exp(-\rho|\eta|)}{\sinh\rho} \tag{23}$$

where

$$\cosh\rho = \frac{R^2}{2rr'} = \frac{1}{2rr'}(r^2 + r'^2 + (z - z')^2 - (t - t')^2). \tag{24}$$

Thus

$$\begin{aligned} G_n &= \exp[iqB(\phi - 2\pi n)] \int_{-\infty}^{+\infty} \mathrm{d}\eta \exp[-i\eta(\phi - 2\pi n)] \exp(-\rho|\eta|) \\ &\quad \times \left(\frac{1}{4\pi^2}\right)\cdot \frac{1}{rr'} \frac{1}{\sinh\rho} \\ &= \frac{\exp[iqB(\phi - 2\pi n)]}{(4\pi^2) rr' \sinh\rho} \cdot \frac{2\rho}{(\phi - 2\pi n)^2 + \rho^2}. \end{aligned} \tag{25}$$

As a check, consider the $B=0$ limit. Then the Green's function is given by

$$\begin{aligned}
G &= \frac{1}{4\pi^2}\cdot\frac{1}{rr'}\frac{1}{\sinh\rho}\cdot\sum_{\text{all}\,n}\frac{2\rho}{(\phi-2\pi n)^2+\rho^2} \\
&= \frac{1}{4\pi^2}\frac{1}{rr'}\cdot\frac{1}{\sinh\rho}\cdot\frac{\sinh\rho}{(\cosh\rho-\cos\phi)} \\
&= \frac{1}{4\pi^2}\frac{1}{rr'}\cdot\frac{2rr'}{|\mathbf{r}-\mathbf{r}'|^2+(z-z')^2-(t-t')^2} \\
&= \frac{1}{2\pi^2}\cdot\frac{1}{|\mathbf{x}-\mathbf{x}'|^2-(t-t')^2}
\end{aligned} \tag{26}$$

which is the standard massless propagator except for a factor 2, which is due to our operator being $\frac{1}{2}(\Box^2-m^2)$ rather than $(\Box^2-m^2)$.

We are interested in the coincidence limit $G(x,x)$ of the Green's function which is measure of the field fluctuations $\langle\Phi^2\rangle$. To isolate the divergences and regularize this expression we will take this limit in two steps. Let us first consider the sum over all n for the case $\phi=0$; i.e. $\theta'=\theta$. Then,

$$G=\sum_n\frac{\exp(2\pi iqBn)}{(4\pi^2)(rr'\sinh\rho)}\left[\frac{2\rho}{\rho^2+(2\pi n)^2}\right]. \tag{27}$$

If we now decompose the quantity qB into an integer and fractional part, it is clear that only the fractional produces a non-trivial effect. So we can ignore the integral part of the qB and just retain the fractional part. In what follows, I will assume that this has been done and denote by the same symbol qB the fractional part; $0<|qB|<1$. Since the sum extends over both positive and negative n, it is also obvious that only the cosine part of the exponential contributes. Therefore, the sign of qB does not matter. Calculating the sum using standard formulas we get

$$\begin{aligned}
G &= \sum_n\frac{\exp(2\pi iqBn)}{(4\pi^2)(rr'\sinh\rho)}\left[\frac{2\rho}{\rho^2+(2\pi n)^2}\right]. \\
&= \frac{1}{8\pi^3}\frac{1}{rr'\sinh\rho}\sum_n\frac{2(\rho/2\pi)}{(\rho/2\pi)^2+(n)^2}\cos(2\pi qBn) \\
&= \frac{1}{8\pi^3}\frac{1}{rr'\sinh\rho}\cdot 2\pi\cdot\frac{\cosh\frac{\rho}{2\pi}(\pi-2\pi|qB|)}{\sinh(\rho/2)} \\
&= \frac{1}{4\pi^2}\cdot\frac{1}{rr'\sinh\rho}\cdot\frac{\cosh\frac{\rho}{2}(1-2|qB|)}{\sinh\rho/2} \\
&= G_{B=0}\cdot\frac{\cosh\frac{\rho}{2}(1-2|qB|)}{\cosh\frac{\rho}{2}}.
\end{aligned} \tag{28}$$

The second factor signals non-trivial effects due to B. We now set $r=r'$ ($\equiv r$, some finite value) and let $l^2=(z-z')^2-(t-t')^2$ go to zero. In this limit,

$$\cosh\rho = 1+\frac{l^2}{2r^2}; \quad \rho\approx\frac{l}{r} \quad (\text{for } l\ll r) \tag{29}$$

giving

$$\begin{aligned}\lim_{x\to x'} G &\cong \frac{1}{2\pi^2}\cdot\frac{1}{l^2}\cdot\frac{\cosh\frac{l}{2r}(1-2|qB|)}{\cosh\frac{l}{2r}}\\ &\approx \frac{1}{2\pi^2}\frac{1}{l^2}\left[1+\frac{1}{8}\frac{l^2}{r^2}((1-2|qB|)^2-1)\right],\\ &= \frac{1}{2\pi^2}\frac{1}{l^2}+\frac{1}{4\pi^2}\left(\frac{1}{r^2}\right)(4|qB|)(|qB|-1).\end{aligned} \tag{30}$$

All the divergence is isolated in the first term which is precisely the coincidence limit in the absence of the field. It seems natural to regularize $G(x,x)$ by subtracting this term. Then we get

$$G_{\text{reg}}(r) = -\frac{1}{\pi^2}\left(\frac{1}{r^2}\right)(|qB|)(1-|qB|) \tag{31}$$

which is finite at all $r\neq 0$ and falls as r^{-2} from the location of the field. It should be remembered that the qB in the above expressions refer to the fractional part of this quantity. If this fractional part is zero, such a magnetic field does not lead to any effect. This is a usual feature seen in Aharonov–Bohm effect. In fact, here the effect is maximum at $qB=(1/2)$, that is when the fractional part is largest.

Let us next consider the effective lagrangian in this background field. The quantity

$$L_{\text{eff}} = -i\int_0^\infty \frac{ds}{s}K(x,x;s) \tag{32}$$

can be computed from G by the following trick. We first take the limits $\mathbf{r}\to\mathbf{r}'$ and $t\to t'$ in $K(x,x';s)$ but retaining $l^2=(z-z')^2$ as non-zero. Note that K depends only on l^2. From the definitions of G, L_{eff} and the expression (18), it is clear that

$$\frac{\partial K}{\partial l^2} = +\frac{i}{2}\frac{K}{s}. \tag{33}$$

Therefore,

$$\begin{aligned}L_{\text{eff}} &= -2\int_0^\infty ds\frac{iK(x,x;s)}{2s} = -\lim_{x\to x'}\frac{\partial}{\partial l^2}\left\{2\int_0^\infty ds K(x,x';s)\right\}\\ &= -\lim_{x\to x'}\left[2\cdot\frac{\partial G(x,x')}{\partial l^2}\right].\end{aligned} \tag{34}$$

Thus the coincidence limit of the derivative of the Green's function gives us the L_{eff}.

We have the expression

$$G(l^2) = \frac{1}{4\pi^2}\frac{1}{r^2}\frac{1}{\sinh\rho}\frac{\cosh\frac{\rho}{2}(1-2qB)}{\sinh\rho/2}$$

$$\cosh\rho = 1 + \frac{l^2}{2r^2} = 1 + 2\sinh^2\frac{\rho}{2} \tag{35}$$

Or, equivalently,

$$\sinh\frac{\rho}{2} = \frac{l}{2r} \tag{36}$$

using this, we can rewrite $G(l^2)$ as

$$G(l^2) = \frac{1}{2\pi^2}\cdot\frac{1}{l^2}\cdot\left(\frac{\cosh\frac{\rho}{2}(1-2qB)}{\cosh\frac{\rho}{2}}\right) \equiv \frac{1}{2\pi^2}\cdot\frac{1}{l^2}\cdot\left(\frac{\cosh\frac{\rho}{2}A}{\cosh\frac{\rho}{2}}\right). \tag{37}$$

The derivative has the structure

$$\frac{\partial G}{\partial l^2} = -\frac{1}{2\pi^2}\frac{1}{l^4}\frac{\cosh\frac{\rho}{2}A}{\cosh\frac{\rho}{2}} + \frac{\partial}{\partial l^2}\left(\frac{\cosh\frac{\rho}{2}A}{\cosh\frac{\rho}{2}}\right)\cdot\frac{1}{2\pi^2 l^2}. \tag{38}$$

We are, of course, only interested inthe limit $l\to 0$ and $\rho\to 0$ of this expression. Let

$$\frac{\cosh\frac{\rho}{2}A}{\cosh\frac{\rho}{2}} \cong 1 + a_1 l^2 + a_2 l^4 + \mathcal{O}(l^6). \tag{39}$$

Then

$$\begin{aligned}\frac{\partial G}{\partial l^2} &= -\frac{1}{2\pi^2}\frac{1}{l^4}(1 + a_1 l^2 + a_2 l^4) + \frac{1}{2\pi^2 l^2}(a_1 + 2a_2 l^2 + \mathcal{O}(l^4))\\ &= -\frac{1}{2\pi}\frac{1}{l^4} - \frac{a_1}{2\pi^2}\frac{1}{l^2} - \frac{1}{2\pi^2}a_2 + \frac{a_1}{2\pi^2 l^2} + \frac{a_2}{\pi^2}\\ &= -\frac{1}{2\pi^2}\frac{1}{l^4} + \frac{a_2}{2\pi^2}.\end{aligned} \tag{40}$$

Computing a_2 from the Taylor expansion, we get

$$\frac{\partial G}{\partial l^2} \cong -\frac{1}{2\pi^2}\frac{1}{l^4} + \frac{1}{2\pi^2}\cdot\frac{1}{384r^4}[A^4 - 10A^2 + 9]. \tag{41}$$

The first term is exactly what one would have obtained in the absence of the field.

Therefore, it seems reasonable to regularize $L_{\text{eff}} = -2(\partial G/\partial l^2)$ by subtracting the first term. This gives

$$\begin{aligned}(L_{\text{eff}})_{\text{reg}} &= -\frac{1}{\pi^2}\cdot\frac{1}{384r^4}[A^4 - 10A^2 + 9]\\ &= \frac{1}{24\pi^2}\cdot\frac{|qB|}{r^4}(1-q^2B^2)(2-|qB|)\\ &= \frac{1}{12\pi^2}\frac{|qB|}{r^4}(1-q^2B^2)\left(1-\frac{1}{2}|qB|\right).\end{aligned} \tag{42}$$

3. Conclusions

The specific functional forms which we have derived above may not be of much significance because they will be very much model dependent. But the really interesting feature is the following. The Aharonov–Bohm field produces a non-trivial vacuum polarization around it. This polarization exists in regions where the field is absent.

The reason for this effect, of course, is topological. The presence of the singular magnetic field at the origin forces one to divide the paths into inequivalent classes labelled by a winding number; the phases contributed by the paths depend on this winding number.

Lastly, it is interesting to observe that fluxes which satisfy the condition $qB = n$ where n is some integer, lead to no effects at all. Such a result is well-known in the non-relativistic versions of the Aharonov–Bohm effect; our analysis shows that it carries over to the relativistic situation as well.

Appendix 1

The Kernel needed in the text is identical to the non-relativistic propagator for a charged particle in the Aharonov–Bohm field. This has been calculated previously by several authors in different contexts (Edwards and Gulyaev 1964, 1967). The procedure is as follows:

Consider the path integral for a Newtonian particle in a given vector potential:

$$K(\mathbf{x}, \mathbf{x}'; t) = \int \mathscr{D}x(t) \exp i \int_0^t \mathrm{d}t \left[\frac{1}{2}m\dot{\mathbf{x}}^2 + q\mathbf{A}\cdot\dot{\mathbf{x}}\right]. \tag{A1.1}$$

We take $\mathbf{A}$ to be

$$\mathbf{A} = B\left(-\frac{y}{x^2+y^2}, +\frac{x}{x^2+y^2}, 0\right) = B\nabla\left(\tan^{-1}\frac{y}{x}\right) = \nabla(\theta B). \tag{A1.2}$$

This corresponds to the Aharonov–Bohm field along z-axis with the flux:

$$\oint \mathbf{A}\cdot\mathrm{d}\mathbf{s} = \int_0^{2\pi} A_\theta \,\mathrm{d}\theta = 2\pi B. \tag{A1.3}$$

To calculate the path integral, we evaluate it for paths with winding number n and sum over them with suitable phase. The path integral for paths with winding number n is given by

$$\begin{aligned} K_n(\mathbf{x}, \mathbf{x}'; T) &\equiv \int \mathscr{D}\mathbf{x}\,\delta(\theta - \theta' - 2\pi n) \exp iA \\ &= \int \mathscr{D}\mathbf{x}\,\delta\left[\int_0^T \frac{x\dot{y} - y\dot{x}}{x^2 + y^2} \mathrm{d}t - 2\pi n\right] \exp iA \\ &= \int_{-\infty}^{+\infty} \mathrm{d}\lambda \exp(-2\pi i n \lambda) \int \mathscr{D}\mathbf{x} \exp i \int_0^T \mathrm{d}t \left(L + \lambda \frac{x\dot{y} - y\dot{x}}{x^2 + y^2}\right). \end{aligned} \tag{A1.4}$$

The new, unconstrained, lagrangian is

$$\begin{aligned} L &= \frac{1}{2} m \dot{x}^2 + (qB + \lambda) \frac{x\dot{y} - y\dot{x}}{x^2 + y^2} \\ &= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2) + \nu \dot{\theta} \end{aligned} \tag{A1.5}$$

where $\nu = (qB + \lambda)$: It is clear that the integral over $z(t)$ separates out and poses no problems. Let us evaluate the xy-part of the Kernel. The hamiltonian for the motion in the xy-plane has the form

$$H = \frac{1}{2m} (\mathbf{p} - q\mathbf{A}_{\mathrm{eff}})^2. \tag{A1.6}$$

Therefore, the path integral can be expressed as

$$K_\nu(\mathbf{x}, \mathbf{x}'; T) = \sum_{\mathrm{all}\,E} \psi_E(\mathbf{x}) \psi_E(\mathbf{x}') \exp(-iET) \tag{A1.7}$$

where $\psi_E(\mathbf{x})$ are the eigenfunctions of H with eigenvalue E:

$$H\psi_E = \frac{1}{2m} (-i\nabla - qA_{\mathrm{eff}})^2 \psi_E = E\psi_E. \tag{A1.8}$$

In polar coordinates, this equation becomes

$$\frac{1}{2m}\left[-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \left(-\frac{i}{r}\frac{\partial}{\partial \theta} - \frac{\nu}{r}\right)^2\right]\psi_E = E\psi_E. \tag{A1.9}$$

Taking the wavefunction as $f(r)\exp(-ik\theta)$ $[k = 0, \pm 1, \cdots]$, we get for the radial part the equation:

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}f}{\mathrm{d}r}\right) - \frac{(k+\nu)^2}{r^2} f + a^2 f = 0; \quad a^2 = 2mE. \tag{A1.10}$$

This equation is solved by

$$f = J_{k+\nu}(ar) \tag{A1.11}$$

where $J_\nu(z)$ is the Bessel function properly normalized wavefunctions can be taken to be

$$\psi_E(r,\theta) = \frac{1}{\sqrt{4\pi}} J_{k+\nu}(ar)\exp(-ik\theta). \tag{A1.12}$$

We substitute these functions into (A1.7) and evaluate the 'sum' (which is now an integral) over all E

$$\begin{aligned} K_\nu(\mathbf{x},\mathbf{x}';T) &= \int_0^\infty \mathrm{d}E\exp(-iET)\cdot\frac{1}{4\pi}\cdot J_{k+\nu}(ar)J_{k+\nu}(ar')\exp[-ik(\theta-\theta')] \\ &= \int_0^\infty \frac{\mathrm{d}(a^2)}{2m}\exp(-ia^2 s)\cdot\frac{1}{4\pi}J_{k+\nu}(ar) \\ &\quad\times J_{k+\nu}(ar')\exp[-ik(\theta-\theta')];\quad s=\frac{T}{2m}. \end{aligned} \tag{A1.13}$$

Using the identity

$$\int_0^\infty a\,\mathrm{d}a\exp(-ia^2 s)J_p(ar)J_p(ar') = \left(\frac{1}{2is}\right)\exp\left(-\frac{r^2+r'^2}{4is}\right)I_{|p|}\left(\frac{rr'}{2is}\right) \tag{A1.14}$$

we obtain the result

$$\begin{aligned} K_\nu(\mathbf{x},\mathbf{x}';T) &= \sum_{\text{all }k}\frac{1}{2\pi m}\exp[-ik(\theta-\theta')]\cdot\left(\frac{2m}{2iT}\right) \\ &\quad\times\exp\left(-\frac{r^2+r'^2}{4iT}\cdot 2m\right)I_{|k+\nu|}\left(\frac{rr'}{2iT}2m\right) \\ &= \sum_{\text{all }k}\left(\frac{1}{2\pi iT}\right)\exp[-ik(\theta-\theta')] \\ &\quad\times\exp\left(+\frac{im}{2T}(r^2+r'^2)\right)I_{|k+\nu|}\left(\frac{mrr'}{iT}\right). \end{aligned} \tag{A1.15}$$

Putting this expression into (A1.4) we get

$$\begin{aligned} K_n(\mathbf{x},\mathbf{x}';T) &= \sum_{\text{all }k}\int_{-\infty}^{+\infty}\mathrm{d}\lambda\exp(-2\pi in\lambda)\cdot\frac{1}{(4\pi iT)}\exp[-ik(\theta-\theta')] \\ &\quad\times\exp\left(\frac{im}{2T}(r^2+r'^2)\right)I_{|k+\nu|}\left(\frac{mrr'}{iT}\right) \\ &= \sum_{\text{all }k}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\nu}{(4\pi iT)}\exp[-2\pi in(\nu-qB)]\exp[-ik(\theta-\theta')] \end{aligned}$$

$$\times \exp\left(\frac{im}{2T}(r^2 + r'^2)\right) I_{|k+\nu|}\left(\frac{mrr'}{iT}\right)$$

$$= \sum_{\text{all } k} \int_{-\infty}^{+\infty} \frac{d\eta}{(4\pi iT)} \exp(2\pi qBin) \exp[-2\pi in(\eta - k) - ik(\theta - \theta')]$$

$$\times \exp\left(\frac{im}{2T}(r^2 + r'^2)\right) I_{|\eta|}\left(\frac{mrr'}{iT}\right). \tag{A1.16}$$

We now use the result

$$\sum_{\text{all } k} \exp[-ik(\theta - \theta' - 2\pi n)] = 2\pi \sum_{\text{all } k} \delta(\theta - \theta' - 2\pi n - 2\pi k) \tag{A1.17}$$

to get

$$K_n(\mathbf{x}, \mathbf{x}'; T) = \exp[iqB(\theta - \theta' - 2\pi n)] \int_{-\infty}^{+\infty} \frac{d\eta}{(2\pi iT)}$$

$$\times \exp\left[-i\eta(\theta - \theta' - 2\pi x) + \frac{im}{2T}(r^2 + r'^2)\right] I_{|\eta|}\left(\frac{mrr'}{iT}\right) \tag{A1.18}$$

where we have first used the delta function to replace $2\pi n$ by $(\theta - \theta' + 2\pi k)$ and then renamed k as n. The full Kernel is given by

$$K(\mathbf{x}', \mathbf{x}; T) = \sum_{\text{all } n} (\text{Kernel for winding number } n) = \sum_{\text{all } n} K_n. \tag{A1.19}$$

This is the expression used in the text.

As a check consider the limit of $B = 0$. Then,

$$K(r', r; \theta', \theta; T) = \sum_{\text{all } n} \int_{-\infty}^{+\infty} \frac{d\eta}{(2\pi iT)} \exp\left[-i\eta(\theta - \theta') + \frac{im}{2T}(r^2 + r'^2)\right]$$

$$\times \exp(+i\eta 2\pi n) I_{|\eta|}\left(\frac{mrr'}{iT}\right)$$

$$= \int_{-\infty}^{+\infty} \frac{d\eta}{(2\pi iT)} \exp\left[-i\eta(\theta - \theta') + \frac{im}{2T}(r^2 + r'^2)\right]$$

$$\times 2\pi \cdot \sum_{\text{all } j} \delta(2\pi\eta - 2\pi j) \cdot I_{|\eta|}$$

$$= \sum_{\text{all } j} \frac{1}{(2\pi iT)} \exp\left[-ij(\theta - \theta') + \frac{im}{2T}(r^2 + r'^2)\right] \cdot I_{|j|}\left(\frac{mrr'}{iT}\right) \tag{A1.20}$$

using the identity

$$\sum_{\text{all } j} I_{|j|}(\alpha) \exp(ijx) = \exp(\alpha \cos x) \tag{A1.21}$$

we get

$$K(r, r'; \theta, \theta'; T) = \frac{1}{2\pi iT} \exp\left[\frac{im}{2T}(r^2 + r'^2)\right] \exp\left[-\frac{imrr'}{T} \cos(\theta - \theta')\right]$$

$$= \left(\frac{1}{2\pi i T}\right) \exp \frac{im}{2T}(r^2 + r'^2 - 2rr' \cos(\theta - \theta'))$$

$$= \left(\frac{1}{2\pi i T}\right) \exp \frac{im}{2T}|\mathbf{r} - \mathbf{r}'|^2 \tag{A1.22}$$

which is correct Feynman propagator in two-dimensions.

Appendix 2

Consider a scalar field Φ governed by the action

$$A[\Phi] = \frac{1}{2}\int \mathrm{d}^D x \, \mathrm{d}t \, [\Phi^i \Phi_i - V(\Phi)] \tag{A2.1}$$

where $V(\Phi)$ is a potential describing the self interaction of the scalar field and D is the dimension of space we are working in. [For usual field theory $D = 3$]. Let us suppose that we are interested in studying the effect of quantum fluctuations around some classical solution $\Phi = \Phi_c$. This solution will be taken to be either constant or adiabatically varying so that we can ignore its derivatives. To obtain the effective action, we will expand Φ around Φ_c as

$$\Phi = \Phi_c + \phi \tag{A2.2}$$

and retain terms up to quadratic order in ϕ. The Lagrangian then becomes

$$L_{\text{total}} = L_0(\Phi_c) + \frac{1}{2}(\phi^i \phi_i - m^2 \phi^2) = L_0(\Phi_c) + L_{\text{eff}} \tag{A2.3}$$

where

$$m^2 = V''(\Phi_c). \tag{A2.4}$$

It is convenient, at this stage to switch over to the Euclidean sector. The effect of quantum fluctuations ϕ which we are interested in are contained in the Kernel

$$K \equiv \exp - \int \mathrm{d}x_E \, L_{\text{eff}} = \int \mathcal{D}\phi \exp - \int \mathrm{d}x \phi \hat{D} \phi = (\det \hat{D})^{-1/2} \tag{A2.5}$$

where D is the Euclidean space operator

$$\hat{D} = -\frac{1}{2}(\Box - m^2) \tag{A2.6}$$

in which $\Box$ denotes the $(D + 1)$ dimensional D'lambertian. [containing D-space and 1 Euclidean time]. We will now write this determinant as

$$\det D = \exp [\mathrm{Tr} \ln D] \tag{A2.7}$$

so that the Kernel becomes

$$(\det D)^{-1/2} = \exp -\frac{1}{2} \mathrm{Tr} \ln D = \exp -\frac{1}{2}\int \mathrm{d}x_E \langle x | \ln D | x \rangle \equiv \exp - \int \mathrm{d}x_E \, L_{\text{eff}}. \tag{A2.8}$$

In arriving at the last expression, we have used some basis vectors $|x>$ to evaluate the trace. We will now use the integral representation for the logarithm,

$$\ln F = \int_0^\infty \frac{ds}{s} \exp(-Fs) \tag{A2.9}$$

to get

$$L_{\text{eff}} = \frac{1}{2}\langle x|\ln D|x\rangle = -\frac{1}{2}\int_0^\infty \frac{ds}{s}\langle x|\exp - sD|x\rangle$$
$$= -\frac{1}{2}\int_0^\infty \frac{ds}{s} K(x, x; s). \tag{A2.10}$$

Where the quantity

$$K(x, y; s) = \langle x|\exp(-sD)|y\rangle \tag{A2.11}$$

is just the Euclidean path integral Kernel for a *quantum mechanical* particle with the hamiltonian D.

In this particular case, the hamiltonian is

$$h = D = \frac{1}{2}(-\Box + m^2) = -\frac{1}{2}\left(\frac{d^2}{d\tau^2} + \nabla^2\right) + \frac{1}{2}m^2. \tag{A2.12}$$

We can evaluate this Kernel easily. The lagrangian corresponding to this hamiltonian is

$$l = +\frac{1}{2}\left(\left(\frac{d\tau}{ds}\right)^2 + \left|\frac{d\mathbf{x}}{ds}\right|^2\right) - \frac{1}{2}m^2 \tag{A2.13}$$

which represents a free particle in $(D+1)$ dimensional space with a constant background potential $(m^2/2)$. [Note that m^2 is treated as a constant in the adiabatic limit]. Therefore

$$K(x, x; s) = \left(\frac{1}{2\pi s}\right)^{(D+1)/2} \exp -\frac{1}{2}m^2 s. \tag{A2.14}$$

We thus get the expression for the effective lagrangian to be

$$L_{\text{eff}} = -\frac{1}{2}\int_0^\infty \frac{ds}{s}\left(\frac{1}{2\pi s}\right)^{(D+1)/2} \cdot \exp\left(-\frac{1}{2}m^2 s\right). \tag{A2.15}$$

This is the conventional expression. We will now show that the same result can be obtained from the calculation of energy density of virtual particles.

The ground state energy in the present context is that of a scalar field theory with mass m.

$$E_0 = \frac{1}{2}\int \frac{d^D k}{(2\pi)^D}(k^2 + m^2)^{1/2} \tag{A2.16}$$

We begin our manipulations by calculating

$$\frac{\partial E_0}{\partial m^2} = \frac{1}{4}\int \frac{d^D k}{(2\pi)^D} \frac{1}{(\mathbf{k}^2 + m^2)^{1/2}}$$

$$=\frac{1}{4}\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\cdot\frac{2}{\sqrt{\pi}}\int_0^\infty \mathrm{d}\lambda\exp[-\lambda^2(\mathbf{k}^2+m^2)]$$

$$=\frac{1}{4}\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\int_0^\infty\frac{\mathrm{d}s}{(2\pi s)^{1/2}}\exp\left[-\frac{1}{2}s(k^2+m^2)\right]. \qquad \text{(A2.17)}$$

We will now use a little trick to eliminate the $s^{-1/2}$ factor. We introduce a variable p and rewrite this factor as another integral

$$\frac{1}{(2\pi s)^{1/2}}=\int_{-\infty}^{+\infty}\frac{\mathrm{d}p}{2\pi}\exp\left(-\frac{1}{2}sp^2\right). \qquad \text{(A2.18)}$$

Then we get

$$\frac{\partial E_0}{\partial m^2}=\frac{1}{2}\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\int_0^\infty \mathrm{d}s\int_{-\infty}^{+\infty}\frac{\mathrm{d}p}{2\pi}\exp\left[-\frac{1}{2}s(\mathbf{k}^2+p^2+m^2)\right]. \qquad \text{(A2.19)}$$

We can now combine the **k** and p integrations into a $(D+1)$-dimensional integration over the vector $\mathbf{q}=(\mathbf{k},p)$. Then

$$\frac{\partial E_0}{\partial m^2}=\frac{1}{4}\int\frac{\mathrm{d}^{D+1}q}{(2\pi)^{D+1}}\int_0^\infty \mathrm{d}s\exp\left[-\frac{1}{2}(\mathbf{q}^2+m^2)s\right]$$

$$=\frac{1}{4}\int_0^\infty \mathrm{d}s\exp\left(-\frac{1}{2}sm^2\right)\int\frac{\mathrm{d}^{D+1}q}{(2\pi)^{D+1}}\exp\left(-\frac{1}{2}s\mathbf{q}^2\right)$$

$$=\frac{1}{4}\int_0^\infty\frac{\mathrm{d}s}{(2\pi s)^{1/2(D+1)}}\exp\left(-\frac{1}{2}m^2s\right). \qquad \text{(A2.20)}$$

Integrating this expression with respect to m^2, we get

$$E_0=-\frac{1}{2}\int_0^\infty\frac{\mathrm{d}s}{s(2\pi s)^{1/2(D+1)}}\exp\left(-\frac{1}{2}m^2s\right)=L_{\text{eff}}. \qquad \text{(A2.21)}$$

which is the required result.

We have omitted an integration constant which is independent of m^2. As it stands (A2.21) is also divergent at $s=0$; however, in this form the divergences are easy to isolate and handle by renormalization [In Lorentzian space, $L_{\text{eff}}=-E_0$; but the L_{eff} in Euclidean sector differ by a sign from that in Lorentzian space, so that $L_{\text{eff}}=E_0$].

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