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On the Maximization of the Central Gravitational Red-Shift of a Schwarzschild Sphere with a Given Surface Gravitational Red-Shift

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Abstract

Following Bondi static, spherically symmetric equilibrium configurations with a core and an envelope have been considered. It has been shown that for any configurations with non-negative pressure and density and with a surface red-shift $z_s \leqslant 4.77$ arbitrarily large central red-shifts z_c are possible in the limiting case of arbitrarily large radius. The effects of imposition of further constraints in the form of a real speed of sound not exceeding the speed of light are also examined. It is seen that for a given limiting sound-to-light-speed ratio $\sqrt{\lambda}$. (i) There exists a limiting surface red-shift $z_s(\lambda) \leqslant 1.71$. (ii) A configuration with $z_s > z_s(\lambda)$ is not possible. (iii) A configuration with $z_s = z_s(\lambda)$ has a unique and finite $z_c = z_c(\lambda)$. (iv) For $z_s < z_s(\lambda)$ arbitrarily large central red-shifts can be obtained for configurations with arbitrarily large radii.

§(1): *Introduction*

In the case of a static, spherically symmetric object in equilibrium it is not possible to obtain very high values of surface gravitational red-shift z_s, i.e., the red-shift of a photon emitted from the surface of the object. The highest possible value is $z_s = 4.77$, and that too for a highly unphysical configuration [1]. But the central gravitational red-shift z_c, i.e., the red-shift of a photon emitted from the center of the object, is higher than z_s; and much higher values of z_c can be obtained even for physically plausible models [2]. This fact has been used to obtain models of the quasistellar objects with central gravitational red-shifts [2]. Given an equation of state, i.e., a relation between the pressure p and the density ρ, it is possible to compute z_s and z_c. Following a suggestion from Professor J. A. Wheeler, we concern ourselves with the following problem:

Given a specified surface red-shift z_s and some relevant physical constraints, which configuration, i.e., what equation of state, would give the largest value of z_c?

To answer this question, we shall first write down the equations of relativistic equilibrium in a particularly convenient form as done by Bondi [1]. We shall then state the extremization problem and give a detailed analysis. The conclusions will be summarized in the last section.

§(2): *Equations of Equilibrium Configurations*

We consider a static, spherically symmetric body of mass M and radius R (in Schwarzschild coordinates) in hydrostatic equilibrium. In geometrized units ($c = 1, G = 1$) the Einstein field equations are given, in the usual notation, by

$$R_{ik} - \tfrac{1}{2} g_{ik} R = -8\pi T_{ik} \tag{2.1}$$

where the energy-momentum tensor $T^i{}_k$ is given by

$$T^i{}_k = 0 \qquad (r > R)$$

and

$$T^i{}_k = \mathrm{diag}\,(-p, -p, -p, \rho) \qquad (r \leqslant R) \tag{2.2}$$

The line element is given by

$$ds^2 = (1 - 2M/r)\, dt^2 - (1 - 2M/r)^{-1}\, dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \qquad (r > R) \tag{2.3}$$

and

$$ds^2 = e^\nu\, dt^2 - e^\lambda\, dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \qquad (r \leqslant R)$$

where p, ρ, ν, and λ are functions of r only. The mass interior to $r_1 (r \leqslant r_1 \leqslant R)$ is

$$m(r_1) = \int_0^{r_1} 4\pi r^2 \rho(r)\, dr \tag{2.4}$$

Following Bondi [1] we introduce two dimensionless variables:

$$u(r) = m(r)/r, \qquad v(r) = 4\pi r^2 p(r) \tag{2.5}$$

The field equations (2.1) are then equivalent to the following equations:

$$r\frac{dv}{dr} = \frac{2(u + v)}{1 - 2u} \tag{2.6}$$

$$r\frac{du}{dr} = \frac{H}{(1 - 2u)(dv/du - \alpha)} \tag{2.7}$$

$$4\pi r^2 \rho = u \frac{dv/du - \beta}{dv/du - \alpha} \tag{2.8}$$

$$dp/dr = -\tfrac{1}{2}(\rho + p)\, dv/dr \tag{2.9}$$

where

$$\alpha = -\frac{(u+v)}{(1-2u)}, \qquad \beta = -\frac{v}{u}\,\frac{(2-5u-v)}{(1-2u)} \tag{2.10}$$

and

$$H = 2v - (u^2 + 6uv + v^2) \tag{2.11}$$

The boundary conditions are given by

$$u = 0, \qquad v = 0 \tag{2.12a}$$

at the origin, and

$$v = 0, \qquad u = u_s, \qquad -\lambda_s = +\nu_s = \ln(1 - 2u_s) \tag{2.12b}$$

at the surface. Given an equation of state, the above equations determine v as a function of u. Various properties of the interior solution on the (u, v) plane have been discussed in [1]. We summarize the ones relevant here (see Figure 1).

(A) $H = 2v - (u^2 + 6uv + v^2) = 0$ is a hyperbola. Its branch passing through the origin divides the interior into two distinct parts, viz., the core ($H \geqslant 0$) and the envelope ($H < 0$). The hyperbola has the following properties: Along the hyperbola $H = 0$, $\alpha = \beta$, and

$$\frac{v}{u} = \frac{p}{\rho} = \left(\frac{1}{u} - 3\right) \pm \sqrt{\left(\frac{1}{u} - 3\right)^2 - 1} \tag{2.13}$$

The upper sign (+) is for the upper part ($v > \frac{1}{4}$) of the branch of the hyperbola shown in Figure 1, while the lower sign (-) is for the lower part ($v \leqslant \frac{1}{4}$) of the same branch. Hence p/ρ increases along the hyperbola and for the upper part of this branch $p/\rho > 1$. The hyperbola has a vertical tangent at the point $T(\frac{1}{4}, \frac{1}{4})$. We shall denote values of the various physical parameters at the center, the core-envelope interface, and the surface by the subscripts c, b, and s, respectively.

(B) The integral of $dv/du = \alpha$ is a family of parabolas

$$v = \sqrt{A(1 - 2u)} - (1 - u) \tag{2.14}$$

The parameter A is determined once we specify the point $(u_s, 0)$ on the parabola:

$$A = (1 - u_s)^2/(1 - 2u_s) \tag{2.15}$$

This particular parabola intersects the hyperbola $H = 0$ at a point (u_p, v_p) where

$$u_p = [u_s/4(1 - 2u_s)]\,[2\sqrt{2} - (2\sqrt{2} + 3)u_s] \tag{2.16}$$

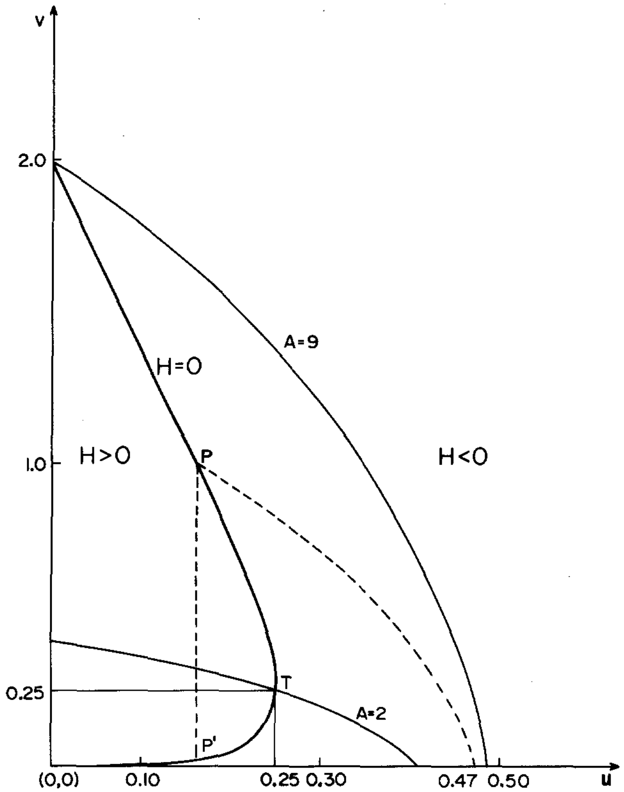

Fig. 1. The (u, v) plane showing the hyperbola $(H = 0)$, the core $(H > 0)$, the envelope $(H < 0)$, and two parabolas with $A = 2$ and $A = 9$. A representative curve which maximizes for a given surface red-shift 3.226, the central red-shift in a particular case, drawn partly in dashed lines, is also shown.

In the region $u > 0, v > 0$ a parabola with a higher A lies above a parabola with a lower A. From (2.7) it is seen that along a parabola u changes but r is constant. Hence a parabola represents a thin mass shell of infinite volume density but finite surface density.

(C) Any equilibrium configuration is characterized by a $v(u)$, curve and in order that r be finite and increases outwards, the tangent to the representative curve at any point should be above, along, or below the tangent to the parabola drawn at that point according as $H \gtreqless 0$. This follows from (2.7).

(D) As we assume $p \geqslant 0, \rho \geqslant 0$ we require that $v \geqslant 0, u \geqslant 0$ and that dv/du not lie in the range between α and β.

(E) From (2.7) and (2.8) it follows that a discontinuity of dv/du results in a discontinuity of ρ (and not of r). In particular, a discontinuous increase (decrease) in dv/du in the core results in a decrease (increase) of ρ in the outward direction.

§(3): *Maximization of the Central Red-Shift*

(A) *The Variational Problem.* The red-shift of a photon emitted at r and received at a great distance is given by

$$z = e^{-\nu/2} - 1 \tag{3.1}$$

Hence the surface and the central red-shifts are given by

$$z_s = e^{-\nu_s/2} - 1 = (1 - 2u_s)^{-1/2} - 1$$

and

$$z_c = e^{-\nu_c/2} - 1 \tag{3.2}$$

respectively.

Maximization of z_c for a fixed z_s means minimization of ν_c for a fixed ν_s (i.e., for a fixed u_s). From (2.6) and (2.7) ν_c can be expressed as a functional, viz.,

$$\nu_c = -2 \int_0^{u_s} \frac{(u+v)\,[dv/du + (u+v)/(1-2u)]}{2v - (u^2 + 6uv + v^2)}\, du + \nu_s \tag{3.3}$$

The problem is to minimize ν_c in (3.3) for fixed end points (0, 0) and $(u_s, 0)$ subject to the constraints discussed in Section 2 as well as any others imposed in addition. As the integrand is linear in dv/du an Euler-Lagrange extremal curve $v(u)$ does not exist. Hence of all the curves $v(u)$ we have to choose the one on the boundary of the admissible domain.

From (3.3) we get, however,

$$\delta\nu_c = \int_0^{u_s} \frac{2}{H^2}\,(2v + 3u^2 + 2uv + 3v^2)\,\delta v du \tag{3.4}$$

which implies that $\delta\nu_c > 0$ for $\delta v > 0$.

Hence of all the admissible curves that pass through (0, 0) and $(u_s, 0)$ we have to choose the lowest possible curve in the (u, v) plane to get the largest z_c.

Now Section 2(C) implies that going inwards along any representative curve the A values of the parabolae (2.14) must be nondecreasing in the envelope and must be nonincreasing in the core. Hence the constraint on the representative curve starting from $(u_s, 0)$ is that it cannot lie below the parabola through $(u_s, 0)$ in the envelope and if it intersects the hyperbola $H = 0$ at (u_b, v_b) it cannot lie above the parabola through (u_b, v_b) in the core.

The largest z_c will hence be obtained by following the parabola through $(u_s, 0)$ in the envelope intersecting $H = 0$ at $P(u_p, v_p)$ and then taking the lowest allowed curve below the parabola in the core. The envelope therefore describes a state of infinite density but finite pressure. In practice, of course, this state is not achieved, but a state of very high density, e.g., as in a neutron star, may come close to it.

(B) *Maximum Central Red-Shift.* We can now find the maximum central red-shift for a given surface red-shift. The only assumption we make to begin with is that the pressure and the density are non-negative throughout. We note that for $u_s > 0.4106$ $(z_s > 1.364)$ the parabola through $(u_s, 0)$ intersects the hyperbola $H = 0$ above the turning point $T(\frac{1}{4}, \frac{1}{4})$ and that for $u_s > 0.485$ $(z_s > 4.77)$ the point of intersection lies on the negative side of the v axis. Hence two cases arise as follows:

(I) $$u_s \leqslant 0.4106 \qquad (z_s \leqslant 1.364)$$

In this case, the parabola through $(u_s, 0)$ will intersect the hyperbola at or below the point T. Hence the representative curve should be the parabola through $(u_s, 0)$ intersecting the hyperbola $H = 0$ at $P(u_p, v_p)$ and then the hyperbola which is the lowest curve in the core till the origin. We cannot follow $H = 0$ strictly. Actually we take a curve $H = \epsilon f(u)$, where ϵ is a small positive number and $f(u)$ is any non-negative function of u which satisfies $f(0) = 0$, $f(u_p) = 0$, and $df(0)/du > 0$. For a particular choice of $f(u)$ we can evaluate z_c from (3.3) and the radius r_s given by

$$\ln\left(\tfrac{4}{3}\pi \rho_c r_s^2\right) = \ln u_p + \int_0^{u_p} \left[2\frac{(1-2u)(dv/du - \alpha)}{H} - \frac{1}{u}\right] du \tag{3.5}$$

We have investigated the behavior of z_c and r_s by choosing

$$\begin{aligned} f(u) &= u/u_1 && (u < u_1) \\ &= 1 && (u_1 \leqslant u < u_2) \\ &= \frac{(u_p - u)}{(u_p - u_2)} && (u_2 \leqslant u \leqslant u_p) \end{aligned} \tag{3.6}$$

For a given ϵ, u_2 is determined by matching the slope with that of the parabola at P, and we have chosen $u_1 \approx \epsilon$. It is seen that as $\epsilon \to 0$ both z_c and r_s diverge. Hence for any $z_s \leqslant 1.364$ z_c can be made arbitrarily large by making the radius of the thin mass shell suitably large.

(II) $$0.4106 < u_s \leqslant 0.485 \; (1.364 < z_s \leqslant 4.77)$$

In this case the point of intersection of the parabola through $(u_s, 0)$ with the hyperbola will be above the point T, for $u_s = 0.485$ the point of intersection being on the v axis at $v = 2$. Hence for $u_s < 0.485$ the representative curve should be the parabola through $(u_s, 0)$ intersecting the hyperbola $H = 0$ at $P(u_p, v_p)$, then the ordinate through P intersecting the hyperbola at $P'(u_p, v_p')$ below the point T, and finally a curve $H = \epsilon f(u)$ as discussed above till the origin (cf. Figure 1). Again in practice the ordinate cannot be drawn exactly from P. Strictly, we should continue the parabola slightly beyond P, say up to $H = \eta$, go along the ordinate from $H = \eta$ and do a similar matching near P'. In this case also we shall end up with infinite z_c and r_s as $\epsilon \to 0$. For $u_s = 0.485$ we have to follow the v axis, which means $u = 0$, $\rho = 0$ and hence an empty core. Assuming $u_s = 0.485 - \eta$ it is seen that z_c and r_s diverge as $1/\eta$ and $1/\sqrt{\eta}$, respectively. Note that for $u_s > 0.485$ the point of intersection of the parabola with the hyperbola is on the negative side of the v axis, giving a negative density core. Hence $z_s = 4.77$ is the maximum surface red-shift that can be achieved for a non-negative pressure and density configuration.

It is clear that in the cases discussed above we encounter highly unphysical situations such as a core of zero density but finite pressure. Also, following an ordinate implies $v/u = p/\rho$ and hence for the part of the ordinate above the point T, $p > \rho$. Hence we should impose additional constraints so as to make the solutions physically more plausible. In the next section we discuss one such constraint and the consequences thereof.

(C) *An Additional Constraint.* In order to avoid the situations discussed in Section 3(B) above we impose a constraint that the speed of sound should everywhere be real and not exceed the speed of light. This can be put in the form

$$0 \leqslant dp/d\rho \leqslant \lambda \qquad (0 < \lambda \leqslant 1) \tag{3.7}$$

The lower limit implies that the density shall not increase outwards, since $dp/dr < 0$.

Consider the envelope first.

Following the parabola in the envelope is equivalent to following a curve

$$p = k(\rho - \rho_0) \tag{3.8}$$

with $k \to 0$. This is because from (3.8) and (2.8) we get

$$k\left[1 - \left(\frac{4\pi r^2 \rho_0}{u}\right)\frac{\alpha - dv/du}{\beta - dv/du}\right] = \frac{v}{u}\left[\frac{\alpha - dv/du}{\beta - dv/du}\right] \tag{3.9}$$

and as $dv/du \to \alpha$, $k \to 0$. Hence in the envelope following the parabola amounts to having $dp/d\rho \to 0$ and (3.7) is satisfied. Hence even if (3.7) is imposed, for a given z_s we can follow the parabola in the envelope arbitrarily closely as in Section 3(B) and intersect the hyperbola close to the point P.

In the core we get from (3.7)

$$p \leqslant \lambda\rho - C \tag{3.10}$$

where $C \geqslant 0$ since the density does not increase outwards across $H = 0$. Before proceeding further it is essential that we investigate the properties of a $p = k\rho$ curve in the core.

A $p = k\rho$ curve starting from the origin outwards intersects the hyperbola $H = 0$ many times at well-defined points and ultimately spirals into a point (u_k, v_k) on $H = 0$ below the first point of intersection where

$$u_k = \frac{2k}{1 + 6k + k^2}, \quad v_k = \frac{2k^2}{1 + 6k + k^2} \tag{3.11}$$

From (2.8) along a $p = k\rho$ curve

$$\frac{dv}{du} = \frac{2k - (5k + 1)u - (k + 1)v}{(1 - 2u)(v - ku)}\, v \tag{3.12}$$

From (3.12) it is seen that dv/du is an increasing function of k in the core. Hence at any point in the core going inwards along two k curves the one with higher k will be the lower one. Also putting $H = 0$ in (3.12) we obtain $dv/du = \alpha$, the slope of the parabola. Hence at any point on $H = 0$ we can match a $p = k\rho$ curve with any k to the parabola at that point. From above it follows that for a given point $P(u_p, v_p)$ on $H = 0$ we can go inwards along various $p = k\rho$ curves. For a critical value of k, say $k = k_c$, the curve passes through the origin. For $k < k_c$ the k curve will be above the k_c curve intersecting the positive v axis. Also the point P defines a $k_0 = v_p/u_p$ where k_0 is that value of k for which a $p = k\rho$ curve starting from the origin spirals into the point P. As $k_0 > k_c$ for $k_0 > k > k_c$ the k curve from P will lie below the k_c curve. If $k > k_0$ the corresponding sprial point is above the point P and hence the k curve should go upwards (see Figure 2).

Let us now take up (3.10). From (3.10) and (2.8) we get

$$\frac{dv}{du}(v - \lambda u + v_0) \leqslant \alpha(v + v_0) - \lambda u \beta \tag{3.13}$$

where

$$v_0 = C/4\pi r^2 \geqslant 0$$

If

$$(v - \lambda u) > 0$$

$$\frac{dv}{du} \leqslant \frac{\alpha(v + v_0) - \lambda u \beta}{v - \lambda u + v_0} = f(u, v, \lambda, v_0) \text{ (say)} \tag{3.14}$$

Using $\alpha > \beta$ in $H > 0$ it can be checked that

$$\partial f/\partial v_0 < 0 \quad \text{and} \quad \partial f/\partial \lambda > 0$$

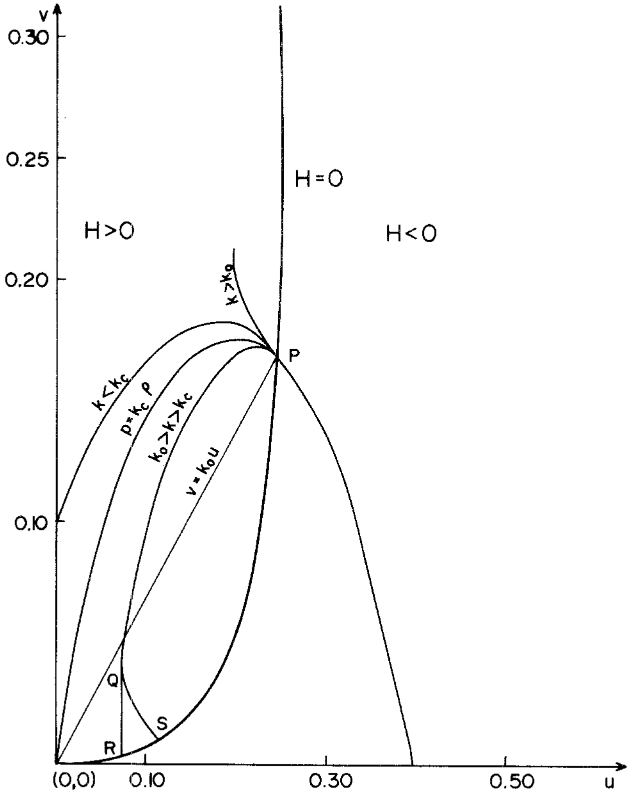

Fig. 2. The (u, v) plane showing the envelope with a parabola through the point $(u_s, 0)$ intersecting the hyperbola $H = 0$ at $P(u_p, k_0u_p)$ and the core with various $p = k\rho$ isothermal curves starting from P. The points Q and S are the points of intersection of a $p = \lambda\rho$ curve with $k_0 > \lambda > k_c$, with the straight line $v = \lambda u$, and with the hyperbola $H = 0$, respectively. Along the vertical QR, $u = 4\pi r^2\rho$ is a constant. (The k curves are not to scale.)

Hence of all curves satisfying (3.10) the lowest will be the curve $p = \lambda\rho$ which we choose as the limiting curve. If $(v - \lambda u) < 0$ we can make use of the freedom of choosing v_0 to make $(v - \lambda u + v_0) > 0$. Wherever permitted a curve dropping vertically downwards should be our first choice. Along such a curve v_0 is therefore chosen such that $(v - \lambda u + v_0) \to 0+$ so that $f \to +\infty$ in (3.14). Note that going down vertically implies $u = 4\pi r^2\rho$ is a constant so that ρ increases inwards.

We can now discuss the core solution starting from the point $P(u_p, v_p)$ on $H = 0$ arrived at by following the parabola from $(u_s, 0)$. The point P defines a

k_c and a k_0 as explained above. As before we discuss two cases, the first one for P lying below or on the turning point $T(\frac{1}{4}, \frac{1}{4})$ the second one for P lying above it.

(I) $u_s \leqslant 0.4106 \quad (z_s \leqslant 1.364)$

In this case the point P lies below or on the point T. Various cases arise depending on the value of λ as follows.

(i) $\lambda < k_c$

In this case $(v - \lambda u) > 0$ and hence the lowest curve will be the $p = \lambda\rho$ curve, which will intersect the v axis above the origin and the origin can be reached only by following the v axis. This implies a zero-density finite-pressure configuration and also an increase in density outwards at the point of intersection of the v axis with the $p = \lambda\rho$ curve [cf. Section 2(E)], both of which are contrary to our constraint. Hence no solution is possible. Physically this means that the given surface red-shift is too high to be compatible with the given upper limit to the stiffness of the equation of state.

(ii) $\lambda = k_c$

In this case also $(v - \lambda u) > 0$, hence the lowest curve will be the $p = \lambda\rho = k_c\rho$ curve, which will pass through the origin. Hence we shall get a fixed isothermal core and a well-defined value for the central red-shift z_c.

(iii) $k_c < \lambda < k_0$

As the $v = \lambda u$ straight line is below the straight line $v = k_0 u$ we have $v > \lambda u$ to start with. Hence we start with a $p = \lambda\rho$ curve. This will intersect $v = \lambda u$ say at a point Q above the hyperbola $H = 0$ along a vertical. We now keep on choosing v_0 such that $(v - \lambda u + v_0) \to 0 +$ and follow the vertical downwards till we reach the hyperbola say at R (see Figure 2). From R we keep on adjusting v_0 so as to go along the hyperbola till the origin. [Of course by going along the hyperbola $H = 0$ we mean going along $H = \epsilon f(u)$ etc. as explained in Section 3(B) (I).] Consequently we can have infinitely large central red-shift z_c for infinitely large radius r_s as $\epsilon \to 0$.

(iv) $\lambda = k_0$

As $\lambda \to k_0$ from below the intersection of the $p = \lambda\rho$ curve with the straight line $v = \lambda u$ occurs at progressively higher points and for $\lambda = k_0 (v - \lambda u) = 0$ at the point P itself. Hence in this case we keep on choosing a sufficiently positive v_0 to go along the hyperbola right from P till the origin. Again we can get arbitrarily large z_c for infinite r_s.

(v) $\lambda > k_0$

In this case $(v - \lambda u) < 0$ at the point P itself. So again we keep on adjusting v_0 to go along the hyperbola till the origin with the consequent results.

We illustrate the above cases by taking a numerical example:

$$u_s = 0.352 \qquad (z_s = 0.838)$$

has $P(u_p, v_p) = (0.231, 0.105)$, with $k_0 = 0.4524$ and $k_c = \frac{1}{3}$. Hence for $\lambda < \frac{1}{3}$, say $\lambda = \frac{1}{4}$, we cannot have a core solution. What it physically means is that for a surface red-shift as high as 0.838 we cannot have an equation of state less stiff than $dp/d\rho = \frac{1}{3}$. For $\lambda = \frac{1}{3}$ the core is isothermal $p = \rho/3$ with the consequent central red-shift $z_c = 2.09$. For $0.4524 > \lambda > \frac{1}{3}$ the core which maximizes the central red-shift consists partly of an isothermal $p = \lambda\rho$ and partly of the hyperbola. For $\lambda \geqslant 0.4524$ the maximizing core consists entirely of the hyperbola.

(II) $\qquad 0.4106 < u_s \leqslant 0.432 \qquad (1.364 < z_s \leqslant 1.71)$

In this case as the point P lies above the point T $k_0 > 1$ and as $\lambda \leqslant 1$ we will have $\lambda < k_0$. Hence only the following cases arise:

(i) $\qquad \lambda < k_c \qquad$ (ii) $\lambda = k_c \qquad$ (iii) $k_c < \lambda < k_0$

and for all the cases the analysis is identical with that of the corresponding cases discussed in Section 3(c) (I). Note that in the case (iii) the intersection of the $p = \lambda\rho$ curve with the straight line $v = \lambda u$ cannot be above the point T and hence we will not be required to follow the ordinate above the point T, which would imply $p > \rho$. The upper limit $u_s = 0.432$ $(z_s = 1.71)$ is set by the fact that for $u_s > 0.432$ $(z_s > 1.71)$ we get $k_c > 1$ and no core solution is possible with $\lambda \leqslant 1$. For $u_s = 0.432$ the core has $p = \rho$ with $z_c = 6.35$. Thus $z_s = 1.71$ is the maximum possible surface red-shift of a static, Schwarzschild sphere in equilibrium consistent with causality.

§(4): *Conclusions*

In the well-known Schwarzschild internal solution, arbitrarily large central red-shifts are obtained by letting $z_s \to 2$ from below. In our variational problem we have shown that in the general case arbitrarily large central red-shifts are possible for any surface red-shift $z_s < 4.77$.

However, this general case, like the Schwarzschild internal solution, demands unrealistic equations of state. A further constraint has therefore been imposed in the form of the existence of a speed of sound which is real and which does not exceed a fraction $\sqrt{\lambda}$ $(\leqslant 1)$ of the speed of light. With this restriction we have shown that each λ determines a limiting surface red-shift $z_s(\lambda)$. No configuration with $z_s > z_s(\lambda)$ is possible while for $z_s < z_s(\lambda)$ configurations with arbitrarily large central red-shifts can be constructed. For $z_s = z_s(\lambda)$ the central red-shift has a unique finite limiting value $z_c(\lambda)$. For example, $z_s(\frac{1}{3}) = 0.838$ and $z_c(\frac{1}{3}) = 2.09$ while $z_s(1) = 1.71$ and $z_c(1) = 6.35$.

In all the cases discussed above an indefinitely large central red-shift is obtained at the cost of an indefinitely large radius. In practice there may be constraints imposed by other physical requirements, e.g., when these objects are considered as models of QSO's. Some of these have been discussed in [2] and others will be discussed in a future paper.

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