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Semiclassical approximations for gravity and the issue of backreaction

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Abstract. Semiclassical approximations, which are useful in the study of a quantum system interacting with a classical system, are studied and compared. A toy quantum mechanical model with two degrees of freedom (which mimics the features of gravity interacting with quantum fields) is used for illustration. In particular, we consider the Born–Oppenheimer approximation (BOA) (corresponding to $G \to 0$ at fixed $\hbar$), the effective action approach ($\hbar \to 0$ at fixed G) and their combinations. We show that in the strict BOA limit there is *no* backreaction on gravity. Gravity is described by classical equations and the fields are quantised in that background. In the effective action approach one can obtain a semiclassical description for gravity, if certain stringent requirements are satisfied. In most situations of interest these conditions will *not* be met and the $O(\hbar)$ contribution from gravitons will be comparable to that from quantum fields. We study the system using both the Schrödinger equation and path integrals and indicate the correspondence.

1. Introduction and motivation

The interaction of gravity with other fields can be described at three different levels. The first level is the classical one, in which both the gravitational field (g) and the other fields (f) are treated as classical c-numbers, obeying classical equations. In the second level, one would like to provide a fully quantum mechanical description of both f and g by means of, say, a wavefunction $\psi(g,f)$. We know that $\psi(g,f)$ obeys the Wheeler–DeWitt equation (at least, in a formal sense) but we have no means of solving it. This inability has opened up the third level of description, namely the one in which gravity is treated as a classical c-number object and the fields are quantised in this given background gravitational field ($\bar{g}$). In this level of description, $\bar{g}$ obeys some c-number equation and the fields are described by some wavefunction $\chi(\bar{g},f)$ in a given background. It is this third level, in which gravity is classical but the fields are not, that concerns us in this paper. The description in this level needs determination of $\chi(\bar{g},f)$ and $\bar{g}$. Of these, $\chi(\bar{g},f)$ falls in the domain of 'quantum field theory in curved spacetime'. It is usually determined by the (functional) Schrödinger equation

$$[\mathrm{i}(\partial/\partial t) - H(\bar{g},f)]\chi(\bar{g},f) = 0 \tag{1}$$

where $H(\bar{g},f)$ is the Hamiltonian for the field f in a given background metric $\bar{g}$. (Usually, one uses the equivalent Heisenberg-picture version of (1).) To complete the description, we need a c-number equation for $\bar{g}$. The major question in the study of semiclassical gravity is the determination of the correct form of this equation in a well defined order of approximation.

Several choices for this equation have been suggested in the literature. To the 'lowest order' it seems reasonable to assume Einstein's equation for g. This would require using

$$R_{ik}-\tfrac{1}{2}g_{ik}R=8\pi G\bar{T}_{ik}(\bar{f}) \tag{2}$$

where the RHS is the purely classical object constructed from the, say, *expectation* value of f. Such a description is adequate when both gravity and the quantum field are in a highly-excited coherent near-classical state.

Now suppose we want to study quantum fluctuations in f, $O(\hbar)$, around $\bar{f}$. Can we use (1) to study these fluctuations while retaining (2) for the description of g? That is, can we suppress $O(\hbar)$ effects of gravity compared to $O(\hbar)$ effects of f? To bring this question sharply into focus, consider a situation in which f vanishes classically. (For example, the 'vacuum' state could be one state in which this happens: of course, there are several states in which expectation value of the quantum field vanishes which are not 'vacuum'. This distinction is not very important to what we intend to illustrate.) Does it now make sense to use simultaneously the following equations?

$$\bar{R}_{ik}-\tfrac{1}{2}\bar{g}_{ik}\bar{R}=0 \tag{3}$$

$$[\mathrm{i}(\partial/\partial t)-H(\bar{g},f)]\chi(f,\bar{g})=0. \tag{4}$$

The usual answer is: 'yes'. This allows the study of quantum fields in curved spacetimes. Lapchinsky and Rubakov could also derive (3) and (4) from the full Wheeler-DeWitt equation—in a consistent order of approximation—by extending the earlier work of Gerlach (Lapchinsky *et al* 1979, Gerlach 1969, Banks 1985). Their approximation scheme was effectively an expansion in powers of G and corresponds to a weak-coupling limit. This adds to our confidence in (3) and (4).

Several workers have tried to go beyond (3) and (4). There is a popular belief that—again in some consistent approximation procedure—(3) and (4) may be replaced by (4) and the following equation:

$$R_{ik}-\tfrac{1}{2}g_{ik}R=8\pi G\langle T_{ik}\rangle. \tag{5}$$

In (5) the symbol $\langle T_{ik}\rangle$ stands for a 'suitable quantum average'. Two choices are popular for $\langle T_{ik}\rangle$: (i) the 'in'-'out' matrix element of T_{ik} defined by path integral methods, and (ii) the expectation value of T_{ik} in some quantum state, $\langle\psi|T_{ik}|\psi\rangle$. In particular, $|\psi\rangle$ could be the vacuum state. There was a large industry computing this object $\langle T_{ik}\rangle$ with the hope that it has something to do with quantum gravity. We will rather loosely call this term, $\langle T_{ik}\rangle$, as the 'backreaction'. (There have been, however, some dissenting views; e.g. Horowitz 1981.)

If correct, it must be possible to derive (5) from the Wheeler-DeWitt equation, just as one could obtain (3) and (4). Such an attempt was made recently by Hartle (1986). He used the same approximation method as Lapchinsky etc and obtained equations which seem to suggest a backreaction. However, a re-analysis of his calculation shows (see § 2 of this paper) that (5) *cannot* be obtained by this approximation *alone*, i.e. without additional assumptions. If one tries to force a backreaction in this approximation scheme, the value of $\langle T_{ik}\rangle$ becomes undetermined. The form of the backreaction depends on the extra assumptions one is prepared to make, and it is necessary to examine which assumptions are appropriate. Hartle gets the backreaction term only after making a particular choice for the phase of the wavefunction. It is necessary to understand the physical meaning of this choice. We address ourselves to these questions in this paper.

The plan of the paper is as follows. In § 2 we introduce a toy quantum mechanical model with two degrees of freedom, Q and q, and one parameter, M, which determines the strength of Q-q coupling. In § 2.1 we review the WKB limit for Q in the absence of q. In § 2.2 we study the Born-Oppenheimer approximation (BOA) and show that BOA, by itself, does not lead to a backreaction. In § 2.3 we show the correspondence between our result and the analysis by Hartle. In particular, we illustrate the ambiguity of backreaction and identify the extra choice made by Hartle. In § 2.4 we discuss the connection between the choice made by Hartle and the conservation of energy. We also obtain the conditions for the validity of a semiclassical description with backreaction. In § 3, we study BOA from the point of view of path integrals and reproduce the results of § 2. We also compare the effective action approach and BOA. Section 4 connects the toy model with gravity and § 5 summarises the results.

2. Backreaction and the Schrödinger equation

Fortunately, several of the concepts which are involved in semiclassical approximations can be illustrated using a simple model from quantum mechanics. This model consists of two point 'particles' described by the Lagrangian

$$L=\tfrac{1}{2}M(\dot{Q}^2-V(Q))+\tfrac{1}{2}\dot{q}^2-U(Q,q) \tag{6}$$

where Q and q represent the coordinates of the two particles. As we shall show in § 4, Q is analogous to gravity and q is analogous to the quantum field. In the 'level one' description both Q and q are classical and the system is described by the following equations:

$$M\ddot{Q}+MV'=-\partial U(Q,q)/\partial Q \tag{7}$$

$$\ddot{q}=-\partial U(Q,q)/\partial q. \tag{8}$$

In 'level two' both Q and q are quantum mechanical and the system is described by the Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2M}\frac{\partial^2\psi}{\partial Q^2}-\frac{\hbar^2}{2}\frac{\partial^2\psi}{\partial q^2}+MV\psi+U\psi \qquad \psi=\psi(Q,q,t). \tag{9}$$

In 'level three' we demand an intermediate description in which Q is described by a c-number equation, while q is described by a wavefunction $\chi(q,\bar{Q})$ in a given background $\bar{Q}$.

We have chosen the parameter M as multiplying both $\dot{Q}^2$ and $V(Q)$ terms. This is done so that the model will tie in with the description of gravity (see § 4; M will correspond to G^{-1} and an expansion in M^{-1} is equivalent to expansion in G). One may be tempted to think of M as something like the mass of the Q particle. While this may be done, it is not very illuminating. A better feel for M can be obtained by writing (6) as

$$L/M=\tfrac{1}{2}\dot{Q}^2-V(Q)+M^{-1}(\tfrac{1}{2}\dot{q}^2-U(Q,q)). \tag{10}$$

Clearly, M^{-1} determines the coupling between Q and q. In particular, the potential felt by Q is

$$V_{\text{total}}=V(Q)+M^{-1}U(Q,q) \tag{11}$$

so that, as $M \to \infty$, the effect of q on Q is suppressed. In what follows, it is useful to keep in mind the following correspondence: $Q \leftrightarrow$ gravity, $q \leftrightarrow$ field and

$$(M \to \infty) \leftrightarrow (G \to 0). \tag{12}$$

We shall study the Q-q system in two steps. In § 2.1 we will ignore q and study the correspondence between classical and quantum descriptions of Q ('pure gravity'). In § 2.2 we will consider the full system.

2.1. *Classical limit of the Q system*

In this section we will review the WKB and BOA approximations for the Q mode and will set the stage for what follows. (This corresponds to the 'source free, pure gravity' situation.) When the q mode is ignored the Lagrangian is

$$L = \tfrac{1}{2}M(\dot{Q}^2 - V(Q)). \tag{13}$$

To arrive at the classical equations

$$M(\ddot{Q} + V'(Q)) = 0 \tag{14}$$

from the Schrödinger equation

$$\mathrm{i}\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2M}\frac{\partial^2 \psi}{\partial Q^2} + MV\psi \equiv E\psi \tag{15}$$

we may utilise an expansion in powers of $\hbar$. We write

$$\psi(Q) = A(Q)\exp[\mathrm{i}S(Q)/\hbar] \tag{16}$$

where we take S to be real and A to be complex and assume that $A(Q)$ has an expansion in powers of $\hbar$ as $A_0 + \hbar A_1 + \ldots$. (As we shall see, there is no need to expand *both* A and S.) Substituting (16) into (15) we get

$$\left(\frac{(S')^2}{2M} + MV - E\right) - \frac{\mathrm{i}\hbar}{2M}\left(S'' + \frac{2A'}{A}S'\right) - \frac{\hbar^2}{2M}\frac{A''}{A} = 0. \tag{17}$$

Since we assumed that S is real but A is complex, (17) stands for two real equations for three unknown real functions and hence is under-determined. However, it is usual to demand that equation (17) must be satisfied order by order in $\hbar$, i.e. the coefficients of all $\hbar^n$ should individually vanish. (As we shall soon see, this requirement is not crucial to the semiclassical behaviour of Q.) Since $A \simeq \mathrm{O}(1)$ in the leading term, we can write down $\mathrm{O}(1)$ and $\mathrm{O}(\hbar)$ conditions from (17) immediately:

$$[(S')^2/2M] + MV = E \tag{18}$$

$$A_0'/A_0 = -\tfrac{1}{2}(S''/S'). \tag{19}$$

(To be absolutely precise, we should also expand $E = \varepsilon_0 + \hbar\varepsilon_1 + \ldots$. As the reader can verify, this only changes the phase of the wavefunction in an unimportant manner. Hence, we shall not, hereafter, bother to expand E.) Equation (18) is the Hamilton–Jacobi equation and leads to the classical equations of motion for Q; (19), when integrated, gives

$$A_0 = N/2\sqrt{S'} \tag{20}$$

where N is the normalisation constant. From (20) we see that

$$|\psi|^2\,\mathrm{d}Q = (\text{constant}) \times (1/S')\,\mathrm{d}Q \propto \mathrm{d}Q/v(Q) \tag{21}$$

where $v(Q)$ is the classical velocity. That is, $|\psi|^2\,dQ$ is proportional to the classical time interval spent in the range $(Q, Q+dQ)$. This makes sense and agrees with our intuitive idea of the classical limit.

There is, however, another instructive way of analysing (17). We can say that we are interested in terms only up to $O(\hbar)$, i.e. we are willing to set $\hbar^2$, $\hbar^3$ etc to zero, but do not care how they add up to zero. Thus we demand

$$\left(\frac{(S')^2}{2M}+MV-E\right)-\frac{i\hbar}{2M}\left(S''+\frac{2A_0'}{A_0}S'\right)=O(\hbar^2)\simeq 0 \tag{22}$$

without necessarily requiring the individual coefficients of $\hbar^0$ and $\hbar$ to vanish. Since (22) is a single complex equation for two unknowns, S and A_0, of which A_0 is complex, the solution will contain one arbitrary function. For reasons which will be soon clear, we call this function $\hbar U(Q)$ and write (22) as two equations:

$$S'^2/2M+MV-E=-\hbar U(Q) \tag{23}$$

$$(i\hbar/2M)[S''+(2A_0'/A_0)S']=-\hbar U(Q). \tag{24}$$

Equations (23) and (24) are, of course, identical to (22); either one of them can be taken to *define* $U(Q)$ in terms of S and A_0. However, one may be tempted to interpret (23) as a classical equation with quantum corrections to $O(\hbar)$. (We could not have done this if we demanded coefficients of $\hbar^n$ to vanish individually. But since we are not demanding that now, the $O(1)$ and $O(\hbar)$ terms can mix.) Equation (23) is equivalent to classical motion in an effective potential

$$V_{\text{eff}}=MV+\hbar U(Q) \tag{25}$$

which is suggestive of a quantum correction to classical motion. This is quite surprising because no such correction existed in (18) or (19). Clearly, whether we demand coefficients of $\hbar^0$, $\hbar$ etc to vanish individually or demand vanishing up to $O(\hbar^2)$, should not change the physics!

In fact it does not. The backreaction of quantum fluctuations via $\hbar U$ in (23) is *completely spurious*. To see this, let us solve (24). It can be easily integrated to give

$$A_0(Q)=\frac{1}{\sqrt{S'}}\exp\left(iM\int\frac{U(Q)}{S'}\,dQ\right) \tag{26}$$

so that the wavefunction is

$$\psi(Q)=\frac{1}{\sqrt{S'}}\exp\left(\frac{iS}{\hbar}+iM\int\frac{U}{S'}\,dQ\right) \tag{27}$$

we see that the classical limit is no longer governed by S but by the total phase of ψ:

$$\frac{S_{\text{eff}}}{\hbar}\equiv\frac{S}{\hbar}+\int\frac{MU}{S'}\,dQ. \tag{28}$$

It is easy to find the equation satisfied by S_{eff}. We have

$$\frac{(S'_{\text{eff}})^2}{2M}=\frac{1}{2M}\left(S'+\frac{\hbar MU}{S'}\right)^2=\frac{(S')^2}{2M}+\hbar U+O(\hbar^2) \tag{29}$$

or, on using (23) for $(S')^2$,

$$(S'_{\text{eff}})^2/2M=E-MV-\hbar U+\hbar U=E-MV \tag{30}$$

which is just the classical Hamilton-Jacobi equation. In other words, the effect of $\hbar U$ in the equation for S is completely cancelled by the extra phase in A, provided we correctly use the total phase of ψ to determine the classical limit. (Notice that the phase has to be real to give meaningful classical equations; this rules out the possibility of writing $(S')^{-1/2}$ as $\exp[(i/\hbar)(\frac{1}{2}i\hbar \ln S')]$ and treating $\frac{1}{2}i\hbar \ln S'$ as part of the phase!)

We have gone through the above analysis of (23) and (24) in detail because a similar situation arises in the Q–q system. The lesson learnt here will be of use in settling a similar ambiguity in the Q–q system.

In the $\hbar$ expansion we are studying, the genuine quantum corrections appear only in $O(\hbar^2)$. This can be seen by separating ψ into a *real* amplitude and phase:

$$\psi(Q) = R(Q)\exp[i(\theta(Q)/\hbar)] \tag{31}$$

and substituting into the Schrödinger equation. We now take R and θ to be real. The Schrödinger equation is equivalent to the pair of real equations

$$\frac{(\theta')^2}{2M} + MV - E = \frac{\hbar^2}{2M}\frac{R''}{R} \tag{32}$$

$$R\theta'' + 2R'\theta' = 0. \tag{33}$$

Integrating (33) (we assume $\theta' \neq 0$) and substituting into (32) we get

$$\frac{(\theta')^2}{2M} + MV - E = \frac{\hbar^2}{2M}\sqrt{\theta'}\frac{d^2}{dQ^2}\left(\frac{1}{\sqrt{\theta'}}\right). \tag{34}$$

This equation can be solved iteratively as a power series in $\hbar^2$, i.e. by taking $\theta = \theta_0 + \hbar^2\theta_1 + \ldots$. To $O(\hbar^2)$ we get

$$\theta = \theta_0 + \hbar^2\theta_1 + O(\hbar^4) \tag{35}$$

$$\frac{(\theta_0')^2}{2M} + MV - E = 0 \tag{36}$$

$$\theta_1' = \frac{1}{2\sqrt{\theta_0'}}\frac{d^2}{dQ^2}\left(\frac{1}{\sqrt{\theta_0'}}\right). \tag{37}$$

Thus the lowest-order corrections to classical motion are $O(\hbar^2)$ and *not* $O(\hbar)$, if we stick to an $\hbar$ expansion.

There is another approximation which is possible in solving (17). We may seek a power series expansion in M instead of in $\hbar$. Let us expand both S and A as a power series in M. Since classical action is $O(M)$ we set $S = M\sigma_0 + \sigma_1 + M^{-1}\sigma_2 + \ldots$ and take $A = A_0 + M^{-1}A_1 + \ldots$. But we see that in

$$\psi = (A_0 + M^{-1}A_1 + \ldots)\exp[(i/\hbar)(M\sigma_0 + \sigma_1 + M^{-1}\sigma_2 + \ldots)] \tag{38}$$

we can expand $\exp[(i/\hbar)M^{-1}\sigma_2]$ as $[1 + (i/\hbar)(\sigma_2/M) + \ldots]$ and regroup the terms to obtain

$$\psi = \exp[(i/\hbar)M\sigma_0]\chi(1 + M^{-1}f + \ldots) \simeq \chi\exp[(i/\hbar)M\sigma_0] + O(M^{-1}) \tag{39}$$

where $\chi = A_0\exp[(i/\hbar)\sigma_1]$ and $f = [(A_1/A_0) + (i\sigma_2/\hbar)]$ etc. (This shows that there is no real need to expand S in the first place.) Equation (17) becomes (taking $E = M\varepsilon = O(M)$)

$$M[\tfrac{1}{2}(\sigma_0')^2 + V - \varepsilon] - \tfrac{1}{2}i\hbar[\sigma_0'' + (2\chi'/\chi)\sigma_0'] + O(M^{-1}) = 0. \tag{40}$$

In the limit of $M \to \infty$ we can ignore $O(M^{-1})$ etc terms. The first two terms give equations *identical* to (18) and (19). This fact, of course, has to be expected. Dividing the Schrödinger equation (15) by M we see that it depends only on the ratio $(\hbar/M)$. (Equivalently, the path integral amplitude $\exp(iA/\hbar)$ depends only on $(M/\hbar)$.) Therefore, *in this model*, the $M \to \infty$ limit is identical to the $\hbar \to 0$ limit. As we will see later, these limits describe different physical situations for the full Q-q system.

Even here it is necessary to ensure that our expansion respects the invariance of the Schrödinger equation under the simultaneous scaling $M \to \alpha M$, $\hbar \to \alpha\hbar$. This implies, for example, that σ_1 is $O(\hbar)$ while it is $O(M^0 = 1)$ and similarly for higher corrections.

As in the case of the $\hbar$ expansion, here also we can demand the validity of (40) without asking for the individual coefficients of M^n to vanish. The reader can easily verify that this would have led to a situation similar to that described by (23) and (24), with no new physics.

2.2. Born–Oppenheimer approximation

For the full system described by the Lagrangian of (6) we can work out two kinds of approximations. We can attempt a power series expansion in $\hbar$ or in M. Since L in (6) does *not* scale out as $(M/\hbar)$ (in contrast to the L of (13)) these two approximations are *inequivalent*, and describe different physical situations.

In a $\hbar$-expansion, the $O(1)$ term will give the classical equation of motion for both Q and q, i.e. equations (7) and (8). The $O(\hbar)$ term will give a determinant factor analogous to the $(S')^{-1/2}$ term in (20). This expansion does not give a dichotomous result with Q classical and q quantum mechanical. Instead, the $\hbar$-expansion treats the quantum nature of both Q and q on the same footing. Since our primary interest is not in this situation we will first consider the M^{-1} expansion.

We start with the full Schrödinger equation:

$$-\frac{\hbar^2}{2M}\frac{\partial^2\psi}{\partial Q^2} - \frac{\hbar^2}{2}\frac{\partial^2\psi}{\partial q^2} + MV\psi + U\psi = E\psi \tag{41}$$

and seek a power series solution in M^{-1}. As in (38), we write

$$\psi = (A_0 + M^{-1}A_1 + \ldots)\exp[(i/\hbar)(M\sigma_0 + \sigma_1 + M^{-1}\sigma_2 + \ldots)] \tag{42}$$

and combine terms which are of the same order in M. This can easily be done by pulling out A_0 and expanding $\exp[(i/\hbar)(M^{-1}\sigma_2 + \ldots)]$ in a Taylor series. We get

$$\begin{aligned}\psi &= \exp[(i/\hbar)M\sigma_0]\{A_0\exp[(i/\hbar)\sigma_1]\}[1 + M^{-1}(A_1/A_0) + \ldots] \\ &\quad\times[1 + M^{-1}(i\sigma_2/\hbar) + \ldots] \\ &= \exp[(i/\hbar)M\sigma_0]\chi(1 + M^{-1}B_1 + \ldots).\end{aligned} \tag{43}$$

Here χ is $O(M^0) = O(1)$, and B_n involves (A_n/A_0) and terms of order M^{-n} arising from the exponential. This calculation shows that it is redundant to expand *both* S and A in a power series in M. It is enough to consider

$$\psi = B\exp[(i/\hbar)M\sigma_0] \tag{44}$$

where $B = \chi + M^{-1}B_1 + M^{-2}B_2 + \ldots$ etc. We will take $\sigma_0 = \sigma_0(Q)$ and $\chi = \chi(Q, q)$, $B_i = B_i(Q, q)$. We can also assume—without any loss of generality—that σ_0 is real and B is complex. Substituting (44) into (41) and retaining the leading-order terms we get

$$M(\tfrac{1}{2}\sigma_0'^2+V-\varepsilon)\chi-\left(\tfrac{1}{2}\hbar^2\frac{\partial^2\chi}{\partial q^2}-U\chi+\mathrm{i}\hbar\sigma_0'\chi'+\tfrac{1}{2}\mathrm{i}\hbar\sigma_0''\chi\right)=0. \tag{45}$$

(The prime denotes differentiation with respect to Q; as we said before, we have assumed E to be $O(M)$, i.e. $E=M\varepsilon$, rather than expand it; this only changes the phase in an insignificant manner.)

Equation (45) is one complex equation—or two real equations—for three unknown real functions: σ_0 and the real and imaginary parts of χ; it is under-determined in the same manner as (17) or (40). We tackle this indeterminacy in the same manner: we demand the coefficients of each power of M to vanish individually. To $O(M)$ and $O(1)$ we obtain

$$\tfrac{1}{2}\sigma_0'^2+V-\varepsilon=0 \tag{46}$$

$$\mathrm{i}\hbar\sigma_0'\chi'+\tfrac{1}{2}\mathrm{i}\hbar\sigma_0''\chi=-\tfrac{1}{2}\hbar^2\frac{\partial^2\chi}{\partial q^2}+U\chi. \tag{47}$$

In (47), we note that

$$\begin{aligned}\mathrm{i}\hbar\sigma_0'\chi'+\tfrac{1}{2}\mathrm{i}\hbar\sigma_0''\chi&=\mathrm{i}\hbar\sigma_0'\chi\left(\frac{\partial}{\partial Q}\ln(\sqrt{\sigma_0'}\chi)\right)\\&=\mathrm{i}\hbar\sqrt{\sigma_0'}\,\frac{\partial f}{\partial Q}\end{aligned} \tag{48}$$

where we have defined $f\equiv\sqrt{\sigma_0'}\chi$, or, equivalently

$$\chi(Q,q)=(1/\sqrt{\sigma_0'(Q)})f(Q,q). \tag{49}$$

Using (48) and (49) in (47) we get

$$\mathrm{i}\hbar\sigma_0'\frac{\partial f}{\partial Q}=-\tfrac{1}{2}\hbar^2\frac{\partial^2 f}{\partial q^2}+U(Q,q)f. \tag{50}$$

Equation (50) can be written in a more suggestive fashion. We note that (46) determines a classical trajectory $Q_c(t)$ satisfying

$$\tfrac{1}{2}(\mathrm{d}Q_c/\mathrm{d}t)^2=\varepsilon-V(Q_c) \tag{51}$$

so that, in (50), $U(Q_c,q)=U(t,q)$ and

$$\mathrm{i}\hbar\sigma_0'(Q_c)\frac{\partial}{\partial Q}=\mathrm{i}\hbar\frac{\partial}{\partial t}. \tag{52}$$

Therefore (50) can be written as

$$\mathrm{i}\hbar\frac{\partial f}{\partial t}=-\tfrac{1}{2}\hbar^2\frac{\partial^2 f}{\partial q^2}+U(Q_c(t),q)f. \tag{53}$$

The message of (53) and (46) is loud and clear: to $O(1)$, Q is determined by the classical equations (with q ignored) and the wavefunction $\chi(q,Q_c)$ represents a quantum theory in a given curved background $Q_c(t)$ via (49) and (53). To this order, the wavefunction $\psi(Q,q)$ has the following interpretation:

$$\psi(Q,q)=\begin{Bmatrix}\text{amplitude}\\ \text{for } Q,\, q\end{Bmatrix}$$
$$=\begin{Bmatrix}\text{WKB amplitude for}\\ Q,\text{ ignoring } q\end{Bmatrix}\times\begin{Bmatrix}\text{amplitude for } q\\ \text{given a } Q\end{Bmatrix}$$
$$=\frac{1}{\sqrt{\sigma_0'(Q)}}\exp[(\mathrm{i}/\hbar)(M\sigma_0(Q))]f(q,Q). \qquad (54)$$

Thus to $O(M^{-1})$ we may say that there is no backreaction.

A crucial ambiguity rears its head here. There is no question that (54) correctly describes the *wavefunction* to O(1). But getting the classical equations from a given wavefunction is (unfortunately) non-trivial. In arriving at (51) from (44) we have to use the concept of 'stationary phase' for the wavefunction. We have assumed that it is only the $(M\sigma_0/\hbar)$ term that contributes. This is certainly correct to $O(M)$ but what do we do if we want the classical equations to O(1)?

Since any complex $f(q, Q)$ can always be separated into a real amplitude and real phase as

$$f(q,Q)=a(q,Q)\exp(\mathrm{i}b(q,Q))$$

the total phase of ψ to O(1) is $(M\sigma_0/\hbar)+b(q,Q)$. This phase is useless for a semiclassical description of Q because it involves the quantum variable q explicitly, except when $b(q, Q)$ is independent of q. Substituting the expression for f into (50) and equating real and imaginary parts, we see that a and b satisfy the equations

$$-\sigma_0'(a'/a)=\frac{\hbar}{2}\left[\frac{\partial^2 b}{\partial q^2}+\frac{2}{a}\left(\frac{\partial a}{\partial q}\right)\left(\frac{\partial b}{\partial q}\right)\right]$$
$$-\hbar\sigma_0' b'=\frac{\hbar^2}{2}\left[\left(\frac{\partial b}{\partial q}\right)^2-\frac{1}{a}\frac{\partial^2 a}{\partial q^2}\right]+U(Q,q).$$

Thus if b is independent of q, a should be independent of Q; what is more, we must have (if $(\partial b/\partial q)=0$)

$$-\hbar\sigma_0' b'\equiv F(Q)=-\frac{\hbar^2}{2}\frac{1}{a(q)}\frac{\mathrm{d}^2 a(q)}{\mathrm{d}q^2}+U(Q,q).$$

This cannot be satisfied in general because of the Q dependence of $U(Q, q)$. Thus if we insist that the *total* phase of ψ should be used for the stationary limit, then in general there is no semiclassical behaviour for Q to O(1).

We are therefore left with three choices. (i) To determine the semiclassical behaviour by $(M\sigma_0/\hbar)$. Then there is no backreaction to O(1). (ii) To stick to only those states which will—at least, approximately—produce a $b(Q, q)$ which is independent of q: $b(Q,q)\simeq b(Q)$. (iii) To put in a phase by hand, i.e. to write f as $(f\,\mathrm{e}^{-\mathrm{i}R})\,\mathrm{e}^{\mathrm{i}R(Q)}$ and combine R with $(M\sigma_0/\hbar)$ to determine the semiclassical limit. The trouble, of course, is that we need some extra input to determine R. What is more, if f satisfies the Schrödinger equation $(f\,\mathrm{e}^{-\mathrm{i}R})$ will not. We will examine this last choice in detail in the next section.

Lastly, it is worth stressing the difference between the $\hbar$ expansion and the M^{-1} expansion in the present context. In a $\hbar$ expansion the leading-order (classical) equation would have been

$$\tfrac{1}{2}M\left[\left(\frac{\partial\sigma}{\partial Q}\right)^2+V-\varepsilon\right]+\left[\frac{1}{2}\left(\frac{\partial\sigma}{\partial q}\right)^2+U\right]=0 \qquad (55)$$

which will give the classical equations of motion (7) and (8) for *both* Q and q. On the other hand, in a M^{-1} expansion Q is not influenced by q *even classically*; note that the term $\frac{1}{2}(\partial\sigma/\partial q)^2 + U$ is O(1) while the other term is O(M). An analogy might make this point clear. Consider the Earth-Moon (Q-q) system moving in space. To the lowest order, the motion of Earth is determined by the Sun's attraction on Earth (the $V'(Q)$ term). We solve for Earth's trajectory $Q_c(t)$. The potential felt by the Moon due to Earth will now be $U(Q_c(t), q)$. We can now work out the Moon's trajectory in space under the influence of $U(Q_c, q)$. (If we want to be fanciful, we can write a Schrödinger equation for the Moon in Earth's potential; this will just be (53).) The perturbation of $Q_c(t)$ due to the Moon is a higher-order effect.

The crucial role played by M in $MV(Q)$ is now clear. In the classical equation for Q there are two force terms, $MV'(Q)$ and $U'(Q, q)$. As $M\to\infty$, the first term dominates.

2.3. *Phase ambiguity and the backreaction*

The results of the previous subsection agree with those derived by Lapchinsky and Rubakov and that of Banks. Translated into the context of gravity, they correspond to equations (3) and (4). We will now compare our analysis with that of Hartle. To do this, we will rewrite our wavefunction in (54) differently. We introduce a *real* phase $R(Q)$ 'by hand' and write

$$\psi = (1/\sqrt{\sigma_0'})f(q, Q)\exp[(i/\hbar)M\sigma_0]$$
$$= f(q, Q)\exp(-iR)\exp(iR)\exp(-\tfrac{1}{2}\ln\sigma_0')\exp[(i/\hbar)M\sigma_0]. \tag{56}$$

Next we regroup the terms and define two complex functions σ_1 and χ_H:

$$\psi \equiv (f\,e^{-iR})\exp[(i/\hbar)(M\sigma_0+\hbar\sigma_1)] \equiv \chi_H \exp[(i/\hbar)(M\sigma_0+\hbar\sigma_1)]. \tag{57}$$

The explicit definitions of χ_H and σ_1 are

$$\chi_H(Q, q) \equiv f(Q, q)\,e^{-iR(Q)} \tag{58}$$

$$\sigma_1(Q) \equiv R + \tfrac{1}{2}i\ln\sigma_0'. \tag{59}$$

Note that $(\chi_H\,e^{i\sigma_1})$ is just $f(\sigma_0')^{-1/2}$. Using (50) for f it is easy to verify that χ_H satisfies the equation

$$i\hbar\sigma_0'\frac{\partial\chi_H}{\partial Q} = \left(-\frac{\hbar}{2}\frac{\partial^2}{\partial q^2} + U + \hbar\sigma_0' R'(Q)\right)\chi_H = i\hbar\frac{\partial\chi_H}{\partial t}. \tag{60}$$

Similarly, using (46) for σ_0, we can write down the equation satisfied by the real part of the phase of ψ: $M\sigma_0 + \hbar R \equiv \sigma$. We get to O(1):

$$(\sigma')^2/2M = E - MV + \hbar\sigma_0' R' + O(M^{-1}). \tag{61}$$

Since the separation into R and χ_H of f is entirely arbitrary, there is no way in which our equations can determine R and χ_H separately. They only fix the combination $(\chi_H\,e^{iR})$. We may, in fact, choose R as we please. We now make one particular choice. We put

$$-\hbar\sigma_0' R'(Q) = \int dq\,\chi_H^*\left(-\frac{1}{2}\frac{\partial^2}{\partial q^2} + U\right)\chi_H \equiv \mathscr{E}(Q). \tag{62}$$

Then (60) and (61) will become

$$\mathrm{i}\hbar\sigma_0' \frac{\partial \chi_{\mathrm{H}}}{\partial Q} = \mathrm{i}\hbar \frac{\partial \chi_{\mathrm{H}}}{\partial t} = \left(-\frac{\hbar^2}{2} \frac{\partial^2}{\partial q^2} + U - \mathscr{E} \right) \chi_{\mathrm{H}} \tag{63}$$

and

$$(\sigma')^2/2M + MV - E + \mathscr{E} = 0. \tag{64}$$

Equations (63) and (64) are identical to Hartle's equations (4.32) and (4.33) in Hartle (1986). Though $\mathscr{E}$ in (64) has the resemblance of a backreaction, our derivation clearly shows that this term is completely spurious. In particular the choice in (62) is *completely arbitrary* as far as BOA is concerned. The theory cannot, and does not, determine R. In fact, $R = 0$ is a perfectly valid choice and will make χ_{H} equal to the f.

The fact that R cannot be determined by theory, but has to be chosen by hand, should be stressed. To see this point more clearly, multiply (60) by χ_{H}^* and integrate over q. We get

$$\mathrm{i}\hbar(\chi_{\mathrm{H}}, \dot{\chi}_{\mathrm{H}}) = \mathscr{E} + \hbar\sigma_0' R'(\chi_{\mathrm{H}}, \chi_{\mathrm{H}}) \tag{65}$$

where we have defined the 'inner product'

$$(A, B) = \int \mathrm{d}q \, A^*(Q, q) B(Q, q). \tag{66}$$

We may assume that $(\chi_{\mathrm{H}}, \chi_{\mathrm{H}}) = 1$. Since our 'Hamiltonian' in (60) is Hermitian, probability is conserved, and we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\chi_{\mathrm{H}}, \chi_{\mathrm{H}}) = \sigma_0' \frac{\mathrm{d}}{\mathrm{d}Q}(\chi_{\mathrm{H}}, \chi_{\mathrm{H}}) = 0 \tag{67}$$

which implies that

$$\mathrm{Im}[\mathrm{i}\hbar(\chi_{\mathrm{H}}, \dot{\chi}_{\mathrm{H}})] = 0. \tag{68}$$

In other words, all terms in (65) are *real* and none of them can be dropped. Hartle's choice in (62) is equivalent to demanding

$$\int \mathrm{d}q \, \chi_{\mathrm{H}}^* \frac{\partial \chi_{\mathrm{H}}}{\partial t} = 0. \tag{69}$$

This is a restriction on χ_{H}, which has to be invoked as a separate assumption.

(Unfortunately, the discussion in Hartle (1986) creates the impression that (62) can be derived; i.e. (4.31) of Hartle (1986) follows from (4.26) or (4.28). This is not true. Recently Halliwell has also emphasised the fact that (62) is one among many choices; see Halliwell (1987).)

Thus, in general, for an arbitrary χ_{H} it is not useful to think of a backreaction. It makes much more sense to use f and σ_0. As we said before, the only situation in which the separation of f as χ and R is *natural* is when the phase of f depends only on Q. Then f can be put as $\chi_{\mathrm{H}} \, \mathrm{e}^{\mathrm{i}R(Q)}$ with *real* χ_{H}. If χ_{H} is real $(\chi_{\mathrm{H}}, \dot{\chi}_{\mathrm{H}})$ automatically vanishes. This requirement is suspiciously close to demanding χ_{H} to be some kind of 'ground state' (remember that ground-state wavefunctions are real). In fact, we will soon see how the ground-state requirement is connected with backreaction.

We emphasise the main conclusion of this and the previous subsection: BOA, i.e. expansion in powers of M^{-1}, cannot lead by itself to an unambiguous backreaction. Backreaction requires some crucial extra assumptions which we shall now discuss.

2.4. Backreaction from 'energy conservation'

In this subsection we shall suggest a possible route to obtain the backreaction. In order to motivate this idea, we shall first rephrase equations (62) and (64) and suggest an interpretation for $\mathscr{E}$ which appears in them. This is done in equations (70)-(74) below. Using this purely as an analogy, we then develop a concept of backreaction based on energy conservation. As we have already stressed, we do need some extra assumption to obtain a backreaction. This section merely suggests one such possible route and hence should be treated as tentative.

With the choice made in (62) our equations to $O(M^{-1})$ are

$$i\hbar\sigma_0' \frac{\partial \chi_H}{\partial Q} = i\hbar \frac{\partial \chi_H}{\partial t} = \left(-\frac{\hbar^2}{2}\frac{\partial^2}{\partial q^2} + U(t, q) - \mathscr{E}(t)\right)\chi_H \tag{70}$$

$$(\sigma'^2/2M) + MV - E + \mathscr{E} = 0. \tag{71}$$

We want to understand the physical meaning of (62); we will do this by deriving (70) and (71) by a different route. We begin by noticing that $\mathscr{E}(t)$ can be eliminated from (70) by defining $\rho(t, q)$:

$$\rho(t, q) = \chi_H(t, q) \exp\left(-\frac{i}{\hbar}\int \mathscr{E}(t)\, dt\right). \tag{72}$$

But on using (52) and (62), we see that

$$-\frac{i}{\hbar}\int \mathscr{E}(t)\, dt = -\frac{i}{\hbar}\int \left(-\hbar\frac{dR}{dt}\right) dt = iR.$$

Therefore ρ is just $\chi_H \exp(iR)$, which is nothing but our old friend f (58). Either by substituting (72) into (70), or by the knowledge that $\rho = f$, we can conclude that ρ satisfies the Schrödinger equation

$$i\hbar\frac{\partial \rho}{\partial t} = -\frac{\hbar^2}{2}\frac{\partial^2 \rho}{\partial q^2} + U(t, q)\rho. \tag{73}$$

We have just gone around in one complete circle! But the main idea of this excursion is to note that H has the same 'expectation value' in both the 'states' χ_H and ρ, in the sense of

$$\int_{-\infty}^{+\infty} dq\, \chi_H^*\left(-\frac{1}{2}\frac{\partial^2}{\partial q^2} + U\right)\chi_H \equiv \mathscr{E}(t) = \int_{-\infty}^{+\infty} dq\, \rho^*\left(-\frac{1}{2}\frac{\partial^2}{\partial q^2} + U\right)\rho. \tag{74}$$

One has to be careful in interpreting the above equation. The function ρ satisfies the Schrödinger equation with the Hamiltonian H. Therefore $(\rho, H\rho)$ can be rigorously interpreted as the expectation value of energy in the state ρ. On the other hand $(\chi_H, H\chi_H)$ has no simple interpretation because χ_H does not satisfy the Schrödinger equation with H as the Hamiltonian. Therefore, it is easier to work with ρ than with χ_H.

After these preliminaries let us now consider a state in which Q is described by the WKB wavefunction and q is described by $\rho(Q, q)$. We have to obtain some kind of 'classical trajectory' out of this quantum state. As we saw before, the phase ambiguity creates a problem in obtaining a unique trajectory by the usual stationary phase method. We will now look at an alternative prescription.

We ask: what is the total energy of the Q–q system to $O(M^{-1})$? Since the total system is in an energy eigenstate (see (41)), this is just E. But suppose the evolution of Q is nearly classical and described by *some* trajectory $\bar{Q}(t)$ yet to be determined. Then we expect a contribution

$$H_0 = \tfrac{1}{2}M\dot{\bar{Q}}^2 + MV(\bar{Q}) \tag{75}$$

from $\bar{Q}$ to the total energy. The contribution from q is $\langle\rho|H|\rho\rangle$. Since the 'total energy' is E we can write

$$E = H_0 + \langle\rho|H|\rho\rangle = H_0 + \mathscr{E} = \tfrac{1}{2}M\dot{\bar{Q}}^2 + MV(\bar{Q}) + \mathscr{E}(\bar{Q}(t)). \tag{76}$$

This defines for us a classical trajectory $\bar{Q}(t)$. Since E is a constant we can differentiate (76) with respect to t and get

$$M\ddot{\bar{Q}} + MV' = -\frac{\partial\mathscr{E}}{\partial Q} = -\frac{\partial}{\partial Q}\langle\rho|H|\rho\rangle = -\frac{\partial}{\partial Q}\langle\chi_{\rm H}|H|\chi_{\rm H}\rangle. \tag{77}$$

We can now use (77) and (73) together to define our semiclassical limit with backreaction. Let us forget how we reached here and just ask what these two equations mean. (In particular, let us treat t in (73) as just an independent variable and not the variable defined by σ_0 in the first half of (70).) Equation (73) gives a wavefunction $\rho(q, t; U)$. Using this one can determine, in principle, the expectation value $\mathscr{E}$ and hence write down (77). On solving this, one obtains the classical trajectory $\bar{Q}(t)$ which, in turn, fixes $U(q, t) = U(q, \bar{Q}(t))$ of equation (73). Of course, all this has to be done self-consistently.

In doing this, we have changed the meaning of t considerably. This point should be stressed. In the original set (63) and (64) (which is the same as (70) and (71)), t was related to $\bar{Q}$, determined by σ_0. That is, Q and t are related by the classical trajectory defined via the zeroth-order Hamilton–Jacobi equation. In the set of equations (73) and (77) the functional form of $\bar{Q}$ is different. This is because now $\bar{Q}$ is determined by (77) which is equivalent to the dynamics by a modified Hamilton–Jacobi function. Thus the prescription of backreaction suggested here is quite different in spite of superficial similarity. (In fact the symbol t in (73) is not the same as the symbol used in (70) and defined via σ_0. To be precise, we should have used a different notation.) In arriving at the last equality in (77) we have used (74). Thus the extra assumption which has gone into obtaining the backreaction is the conservation of $H_0 + \mathscr{E}$. It is not easy to see this point using χ, but it is quite clear when we use ρ.

Determining $\bar{Q}$ by demanding energy conservation is quite different from *just* using BOA. This is why BOA cannot give unambiguous backreaction.

Now that we have identified the extra ingredient which has gone into the backreaction equation, we can ask about the limits of validity of this equation. This is best done using the path integral approach, which we will take up in § 3. Nevertheless, some basic points can be seen from the structure of $\langle\rho|H|\rho\rangle$. Let us look at the contribution to $\langle\rho|H|\rho\rangle$ from the potential

$$\langle U\rangle = \int_{-\infty}^{+\infty} \rho^* U(Q, q)\rho \, \mathrm{d}q. \tag{78}$$

To the lowest order in $\hbar$ this will just be $U(Q, \langle q\rangle)$ where $\langle q\rangle$ is the expectation value of q in this state. This contribution is independent of $\hbar$ and is purely classical; this term contributes the RHS of (7). Up to the next higher order the contribution can be computed by 'effective potential' methods (Stevenson 1984). It is most likely to be

$$\langle U\rangle \simeq U(Q,\langle q\rangle)+\tfrac{1}{2}\hbar\left(\frac{\partial^2 U}{\partial q^2}\right)^{1/2}\bigg|_{q=\langle q\rangle}+\mathrm{O}(\hbar^2) \tag{79}$$

where we have assumed $U''(Q,q)$ to be positive near $q=\langle q\rangle$. This is the first non-trivial quantum mechanical backreaction which we are interested in.

The troubles begin exactly here. In writing the expression for energy of the Q component, we have settled for a classical value, i.e. a O(1) contribution in $\hbar$. It is *a priori* inconsistent to take an $\mathrm{O}(\hbar)$ contribution from q but ignore an $\mathrm{O}(\hbar)$ contribution from $MV(Q)$. The latter—by the same argument—is likely to be

$$\tfrac{1}{2}\hbar(V'')^{1/2}|_{Q=Q_c}. \tag{80}$$

Thus if we are interested in $\mathrm{O}(\hbar)$ corrections and want to use 'conservation of energy' then we must take into account both (79) and (80). In a field theory (which is what we are ultimately interested in) (79) and (80) will be the one-loop contributions to the effective potential. In other words, we expect the one-loop contributions from gravitational self-interaction, represented by (80), to contribute along with one-loop corrections from fields. Since we do not know how to do one-loop quantum gravity, we have to ask when we can ignore (80) compared to (79). In our model we need to satisfy

$$V''(Q_c)\ll\left(\frac{\partial^2 U}{\partial q^2}\right)_{Q=Q_c}\qquad q=\langle q\rangle. \tag{81}$$

As we will discuss in § 4, there is no simple reason for gravity to respect a condition like (81). This is equivalent to saying that, in general, we cannot have a consistent semiclassical backreaction for gravity. This is such a strong and disturbing statement that one should be careful to check all other alternatives before making it. There is nothing sacred about expressions (79) and (80). They are derived under certain assumptions which have been questioned (Stevenson 1984). Probably it is not correct to use such expressions when Q is in a highly-excited near-classical state. We hope to investigate this matter in greater detail using minisuperspace models.

Lastly, we would also like to mention that a classical trajectory obtained by using energy conservation does not really make sense, if the individual energies of Q and q modes fluctuate violently. So, in a way, our prescription has to be supplemented by some kind of 'adiabaticity' requirement as well. All these make the concept of backreaction very dubious and of limited validity.

We will now turn to the semiclassical approximation schemes based on path integrals.

3. Path integrals and backreaction

The backreaction is usually derived in a couple of lines using path integrals. We start with the full path integral (PI):

$$K=\int \mathscr{D}Q\,\mathscr{D}q\,\exp\{(\mathrm{i}/\hbar)[A_0(Q)+A_1(Q,q)]\} \tag{82}$$

and perform the q integration, obtaining

$$K=\int \mathscr{D}Q\,\exp\{(\mathrm{i}/\hbar)[A_0(Q)+A_{\mathrm{eff}}(Q)]\} \tag{83}$$

where we have defined

$$\exp(\mathrm{i}A_{\mathrm{eff}}(Q)/\hbar)=\int \mathscr{D}q\,\exp(\mathrm{i}A_1(Q,q)/\hbar). \tag{84}$$

We now do the Q integration in the saddle-point approximation (SPA), determining Q as a solution to

$$\frac{\delta A_0}{\delta Q}=-\frac{\delta A_{\mathrm{eff}}}{\delta Q}=\frac{\int \mathscr{D}q(\delta A_1/\delta Q)\exp(\mathrm{i}A_1(Q,q)/\hbar)}{\int \mathscr{D}q\,\exp(\mathrm{i}A_1/\hbar)}. \tag{85}$$

The right-hand side is the backreaction. This is, in fact, the most familiar derivation. It is therefore necessary to look at the derivation of (85) closely and compare it with our results in previous sections. We shall now address ourselves to this task.

3.1. Born–Oppenheimer approximation of the path integral

We shall first show that the expansion in powers of M^{-1} in the path integral leads to exactly the same result as in § 2.2, namely that there is no unique backreaction. In §§ 3.2 and 3.3 we will discuss under what circumstances a backreaction can be obtained.

Consider the path integral in (82) with the boundary conditions $Q=Q_1$, $q=q_1$ at $t=t_1$ and $Q=Q_2$, $q=q_2$ at $t=t_2$. The action for our system is

$$A=A_0(Q)+A_1(Q,q)=\int \mathrm{d}t\,M(\tfrac{1}{2}\dot{Q}^2-V(Q))+\int \mathrm{d}t(\tfrac{1}{2}\dot{q}^2-U(Q,q)). \tag{86}$$

Performing the q integration in (82), we get

$$K(Q_2q_2t_2;\,Q_1q_1t_1)=\int \mathscr{D}Q\,\exp\left(\frac{\mathrm{i}}{\hbar}M\int \mathrm{d}t(\tfrac{1}{2}\dot{Q}^2-V(Q))\right)\mathscr{G}(q_2t_2;\,q_1t_1|Q). \tag{87}$$

We want to evaluate this expression in the limit $M\to\infty$. The phase of $(\mathscr{G}\,\mathrm{e}^{\mathrm{i}A_0/\hbar})$ oscillates rapidly as $M\to\infty$, while the amplitude $\mathscr{G}$ (which is of $\mathrm{O}(1)$) varies slowly. Clearly, a saddle-point approximation will pick the solution $Q=\bar{Q}$ so that

$$\left[\frac{\delta A_0}{\delta Q}\right]_{Q=\bar{Q}}=0 \qquad \text{i.e.} \qquad \ddot{\bar{Q}}+V'(\bar{Q})=0. \tag{88}$$

In this $(M\to\infty)$ limit, the kernel is

$$\begin{aligned}K(Q_2,q_2,t_2;\,Q_1,q_1,t_1)&=N\left[\frac{\partial^2\bar{A}_0}{\partial Q_1\,\partial Q_2}\right]^{-1/2}\exp[(\mathrm{i}/\hbar)\bar{A}_0(\bar{Q})]\mathscr{G}(q_2t_2;\,q_1t_1|\bar{Q})\\ &=\{\text{saddle point propagator for } Q \text{ ignoring } q\}\\ &\quad\times\{\text{amplitude to go from } (q_1t_1) \text{ to } (q_2t_2) \text{ in a background } \bar{Q}\}.\end{aligned} \tag{89}$$

This result corresponds to equations (50) and (54) of § 2.2. The propagator $\mathscr{G}(q_2t_2;\,q_1t_1|Q)$ is equivalent to the Schrödinger equation in a background potential $U(Q,q)$; it can be shown that: (i) the saddle-point kernel for Q is exactly the one we will obtain from the WKB wavefunctions, and (ii) this kernel propagates the WKB wavefunctions to WKB wavefunctions. (This is a standard result; e.g. Holstein and Swift (1982).)

The phase ambiguity encountered in § 2.3 also has a simple interpretation in terms of kernels. Suppose we rewrite (89) introducing $1 = e^{iR}\,e^{-iR}$. If we redefine $\mathscr{G}$ as $(\mathscr{G}\,e^{-iR}) = \mathscr{G}'$ and combine $(iA_0/\hbar)$ and iR we will reach the situation described by (60) and (61): $\mathscr{G}'$ will satisfy a modified Schrödinger equation and the iR term will give a backreaction to A_0. It is quite clear that this term is spurious unless supplemented by some other argument.

3.2. *When does* PI *give backreaction?*

The natural temptation to write the PI in (87) as

$$\exp\{(i/\hbar)[A_0(Q) - i\hbar \ln \mathscr{G}(q_2 t_2, q_1 t_1 | Q)]\} \tag{90}$$

and do a saddle-point integration on the whole phase must be resisted if we are interested in the strict $M \to \infty$ limit. To see this point clearly, consider the ordinary integral

$$I = \int_{-\infty}^{+\infty} dx\, g(x)\, e^{iMf(x)} \tag{91}$$

in the limit of $M \to \infty$. We fix the saddle point $x = x_0$ by obtaining the stationary point for the phase, at which

$$f'(x_0) = 0 \tag{92}$$

and evaluate the integral as

$$I \simeq g(x_0)\, e^{iMf(x_0)} \left(\frac{2\pi i}{f''(x_0)} \right)^{1/2}. \tag{93}$$

Instead, if we write I as

$$I = \int_{-\infty}^{+\infty} dx \exp\{i[Mf(x) - i \ln g(x)]\} \tag{94}$$

and treat the whole phase as stationary we would have obtained

$$I \simeq g(x_1)\, e^{iMf(x_1)} \left(\frac{2\pi i}{f''(x_1) - (i/M)(g'/g)'x_1} \right)^{1/2} \tag{95}$$

where x_1 is the solution to

$$f'(x_1) = \frac{i}{M} \frac{g'(x_1)}{g(x_1)}. \tag{96}$$

The right-hand side of (96) represents the backreaction of g on f. If we are interested in the strict $M \to \infty$ limit, (93) is the correct result and (96) is wrong. The fact that the phase is $O(M)$ while the amplitude is $O(1)$ uniquely fixes the form of the exponential. For the same reason (87) is correct and (90) is wrong.

Let us now consider a different integral:

$$I_1 = \int_{-\infty}^{+\infty} dx\, g(x, \varepsilon) \exp[i(M/\varepsilon) f(x)] \tag{97}$$

where we have introduced an extra parameter ε. Suppose we want to evaluate I_1 in the limit of $\varepsilon \to 0$. To do this we need to know the behaviour of $g(x, \varepsilon)$ as $\varepsilon \to 0$.

Suppose g is analytic near $\varepsilon = 0$. Then we can again calculate I_1 by saddle point. Since the $\varepsilon \to 0$ limit is identical to the $M \to \infty$ limit, as far as the phase is concerned, we get the same saddle point as in (92).

A more interesting case is when $g(x, \varepsilon)$ has the expansion

$$g(x, \varepsilon) = \exp[\mathrm{i} g_0(x)/\varepsilon](1 + \varepsilon g_1(x) + \ldots) \tag{98}$$

near $\varepsilon = 0$. Equivalently

$$\ln[g(x, \varepsilon)] = \mathrm{i} g_0(x)/\varepsilon + \mathrm{O}(\varepsilon). \tag{99}$$

So that, to the leading order, ln g is $\mathrm{O}(\varepsilon^{-1})$. Here we *must* write I_1 as

$$I_1 = \int_{-\infty}^{+\infty} \mathrm{d}x \exp[(\mathrm{i}/\varepsilon)(Mf(x) + g_0(x))] \tag{100}$$

and the saddle point (for $\varepsilon \to 0$) is at $x = x_1$ where

$$f'(x_1) = -\frac{1}{M} g_0'(x_1) = -\frac{\mathrm{i}\varepsilon}{M} \frac{g'(x_1)}{g(x_1)}. \tag{101}$$

This is identical to (96) when $\varepsilon = 1$, and represents a backreaction. Note that this evaluation is correct in the $\varepsilon \to 0$ limit, irrespective of the value of M. Similarly (93) is correct in the $M \to \infty$ limit irrespective of the value of ε. These two correspond to different approximations.

The connection between the above example and the Q-q system should be clear. The $\hbar$ plays the role of ε. We do know that $\mathscr{G}(q_2 t_2; q_1 t_1 | Q)$ has an expansion in $\hbar$ with a leading term $\mathrm{O}[\exp(\mathrm{i}/\hbar)F]$ near $\hbar \to 0$. Writing $\mathscr{G}$ as

$$\mathscr{G} \equiv D \exp[(\mathrm{i}/\hbar)\theta] \tag{102}$$

our path integral in (87) becomes

$$K(Q_2 q_2 t_2; Q_1 q_1 t_1) = \int \mathscr{D}Q \exp[(\mathrm{i}/\hbar)(A_0 + \theta)]D. \tag{103}$$

If we evaluate K for the $M \to \infty$ limit, then $(D\, \mathrm{e}^{(\mathrm{i}/\hbar)\theta})$, which is $\mathrm{O}(M^0 = 1)$, acts as an amplitude and we get (89). However, if we evaluate K in the $\hbar \to 0$ limit, we obtain the equations

$$\frac{\delta A_0}{\delta Q} = -\frac{\delta \theta}{\delta Q}. \tag{104}$$

We seem to have got a backreaction in (104) but this is not what we want. The trouble is that this is just the *classical* equation for the Q-q system! Note that θ is independent of $\hbar$, so we have not obtained a *quantum* backreaction. This is precisely the situation we encountered with $\langle U(Q, q)\rangle$ in the previous section.

To get a quantum backreaction we have to calculate higher orders in $\hbar$. Can we not just calculate to $\mathrm{O}(\hbar)$ in (103) and use it in (104)? In fact we can, and it is often done. Since D has an expansion in $\hbar$ as $(D_0 + \hbar D_1 + \ldots)$ we can incorporate $\mathrm{O}(\hbar)$ terms in (103) by a saddle-point integration which includes D. There are, however, two problems with this approach.

The first one is purely technical. As it stands, $\mathscr{G}$ depends on the endpoint q_1 and q_2 of q. We have to do something about this if our backreaction has to be independent of the boundary conditions on q. This is related to the correspondence between expectation values and path integral averages. We will say more about this in the next section.

The more important conceptual point is the following. We are actually trying to evaluate A_{eff} of (84) in a series expansion in $\hbar$ with the leading term being θ. This calculation requires an expansion of q around some classical value and some assumption regarding the smallness of the fluctuations. But if we do a similar analysis on $A_0(Q)$ we will pick up an $O(\hbar)$ correction from quantum fluctuations of Q. It is conceptually inconsistent to ignore these while retaining $O(\hbar)$ terms from q. Since the $O(\hbar)$ effective action contains an effective potential

$$V_{\text{total,eff}} \simeq \tfrac{1}{2}\hbar\left[\left(\frac{\partial^2 V}{\partial Q^2}\right)^{1/2} + \left(\frac{\partial^2 U}{\partial q^2}\right)^{1/2}\right] \tag{105}$$

we recover the same situation as in § 2.4. Thus, the path integral calculations confirm our earlier conclusions.

3.3. PI averages and expectation values

There is one last technical point which has to be clarified while comparing the approaches based on the Schrödinger equation and path integrals. This concerns the correspondence between expectation values and path integral averages.

Consider, for the sake of illustration, just the q degree of freedom described by the action

$$A(q) = \int_{q_1,t_1}^{q_2,t_2} [\tfrac{1}{2}\dot{q}^2 - U(t, q)]\, dt. \tag{106}$$

This action defines the path-integral kernel $K(q_2t_2; q_1t_1)$:

$$K(q_2t_2; q_1t_1) = \int \mathcal{D}q\, e^{iA/\hbar}. \tag{107}$$

We can also use A to define 'path-integral averages' of functionals of $q(t)$:

$$\bar{F}(t; q_2t_2, q_1t_1) \equiv \frac{\int \mathcal{D}q\, F[q(t)]\, e^{iA/\hbar}}{\int \mathcal{D}q\, e^{iA/\hbar}}. \tag{108}$$

In particular, when $F(q(t))$ is a (local) function of q (like $q, q^3 \ldots$ etc), it is easy to show that

$$\bar{F}(t; q_2t_2; q_1t_1) = \frac{\int_{-\infty}^{+\infty} dq\, K(q_2t_2; qt) F(q) K(qt; q_1t_1)}{K(q_2t_2; q_1t_1)}. \tag{109}$$

We want to emphasise that the PI average, $\bar{F}$, is *not* the expectation value $\langle\psi|F|\psi\rangle$ in some state $|\psi\rangle$. This should be obvious in the Heisenberg picture, in which $\bar{F}$ can be written as

$$\bar{F}(t; q_2t_2, q_1t_1) = \frac{\langle q_2, t_2|\hat{F}[\hat{q}(t)]|q_1, t_1\rangle}{\langle q_2t_2|q_1t_1\rangle} \tag{110}$$

where we have defined the eigenstates of the operator $\hat{q}(t)$: $\hat{q}(t_i)|q_i, t_i\rangle = q_i|q_i, t_i\rangle$ with $i = 1, 2$. This is not an expectation value. (In fact the physical meaning of $\bar{F}$ is quite obscure (Brown 1984)). On the other hand, the backreaction in § 2 was defined in terms of expectation values. It is necessary to examine under what conditions (110) becomes an expectation value.

One such situation is well known. If $U(q, t) = U(q)$, so that stationary states exist, then

$$\lim_{t_2\to\infty}\lim_{t_1\to-\infty} K(q_2t_2;\, q_1t_1) \simeq \psi_0(q_2)\psi_0^*(q_1) \tag{111}$$

where $\psi_0(q)$ is the ground-state wavefunction. (The limits in (111) are taken, strictly speaking, in the Euclidean version of the action.) In this case

$$\bar{F}(t;\, +\infty, -\infty) = \int_{-\infty}^{+\infty} \mathrm{d}q\, F(q)|\psi_0(q)|^2 \tag{112}$$

is the ground-state expectation value.

If $U(q, t)$ is approximately stationary, then we can again expect (111) to be valid approximately. In this case the backreaction defined by path integral will match with that defined via the ground-state expectation value.

(Another situation frequently encountered is the one in which $U(t, q)$ is asymptotically static and allows the definition of 'in' and 'out' ground states. In this case

$$K(q_2, +\infty;\, q_1, -\infty) = \langle \mathrm{out}|\mathrm{in}\rangle \tag{113}$$

and

$$F(t;\, +\infty, -\infty) = \frac{\langle \mathrm{out}|\hat{F}[\hat{q}(t)]|\mathrm{in}\rangle}{\langle \mathrm{out}|\mathrm{in}\rangle} \tag{114}$$

Note that, in general, $|\mathrm{out}\rangle \neq |\mathrm{in}\rangle$ and hence this is *not* an expectation value.)

Lastly, we mention a way of modifying the path integral so that PI averages reproduce expectation values exactly. (Unfortunately, it turns out to be not very useful.) The modification of PI is suggested by the following relation:

$$\begin{aligned}
\langle\psi|\hat{F}[\hat{q}(t)]|\psi\rangle &= \int_{-\infty}^{+\infty} \mathrm{d}q_1\, \mathrm{d}q_2 \langle\psi|q_2, t_2\rangle\langle q_2, t_2|\hat{F}[\hat{q}(t)]|q_1, t_1\rangle\langle q_1t_1|\psi\rangle \\
&= \int_{-\infty}^{+\infty} \mathrm{d}q_2\, \psi^*(q_2t_2) \int_{-\infty}^{+\infty} \mathrm{d}q_1\, \psi(q_1t_1) \int \mathscr{D}q\, F[q(t)]\, \mathrm{e}^{\mathrm{i}A/\hbar} \\
&\equiv \int \bar{\mathscr{D}}q\, F[q(t)]\, \mathrm{e}^{\mathrm{i}A/\hbar}.
\end{aligned} \tag{115}$$

In the last step, we have redefined the measure by including the end-point integrations as well:

$$\bar{\mathscr{D}}q = \mathscr{D}q\, \psi^*(q_2, t_2)\psi(q_1t_1)\, \mathrm{d}q_2\, \mathrm{d}q_1 \tag{116}$$

The last expression can be used even when $F[q(t)]$ is a non-local functional. One can now go through the usual motions and define a state-dependent effective action.

(i) Define

$$\exp[(\mathrm{i}/\hbar)W[J]] = \int \bar{\mathscr{D}}q\, \exp\!\left(\frac{\mathrm{i}}{\hbar}\int Jq\, \mathrm{d}t + (\mathrm{i}/\hbar)A\right). \tag{117}$$

(ii) Define q_c by $\delta W/\delta J(t) = q_c(J, t)$. We see that

$$q_c(J, t) = \frac{\int \bar{\mathscr{D}}q\, q(t)\exp[(\mathrm{i}/\hbar)(A + \int Jq\, \mathrm{d}t)]}{\int \bar{\mathscr{D}}q\, \exp[(\mathrm{i}/\hbar)(A + \int Jq\, \mathrm{d}t)]} \tag{118}$$

and hence

$$q_c(J=0,t)=\frac{\int \bar{\mathscr{D}}q\, q\, e^{(i/\hbar)A}}{\int \bar{\mathscr{D}}q\, e^{(i/\hbar)A}}=\frac{\int_{-\infty}^{+\infty} dq\, q|\psi|^2}{\int_{-\infty}^{+\infty} dq\, |\psi|^2}=\langle\psi|q(t)|\psi\rangle. \tag{119}$$

(iii) Invert the equation $\delta W/\delta J = q_c$ to get $J = J(q_c)$ and define the 'state-dependent effective action'

$$W_{\mathrm{eff}}(q_c,\psi)\equiv W[J(q_c)]-J(q_c)q_c. \tag{120}$$

As usual

$$\frac{\delta W_{\mathrm{eff}}}{\delta q_c}=\frac{\delta W}{\delta J}\frac{\delta J}{\delta q_c}-q_c\frac{\delta J}{\delta q_c}-J=q_c\frac{\delta J}{\delta q_c}-q_c\frac{\delta J}{\delta q_c}-J=-J(q_c). \tag{121}$$

So that the solution to the variational principle $(\delta W_{\mathrm{eff}}/\delta q_c)=0$ implies $J=0$ or, equivalently, from (119):

$$q_c=\langle\psi|q|\psi\rangle. \tag{122}$$

Except for this amusing result, namely that $W_{\mathrm{eff}}(q,\psi)$ allows a variational principle to determine $\langle\psi|q|\psi\rangle$ in *any* state ψ, W_{eff} is a fairly useless object. In particular, its definition and calculation requires an *a priori* specification of a state ψ, i.e. we need to solve the Schrödinger equation. This is impractical as well as going against the spirit of path integrals.

In summary, we may say that there is no simple correspondence between expectation values in arbitrary quantum states and path integral averages. A natural correspondence can be defined only for the expectation values in the ground state. On the other hand, our discussion in § 2.4 did not require any specification of the quantum state $\rho(Q,q)$ for q. Thus we have to conclude that there is some conceptual inequivalence between the backreaction obtained from the path integral and that obtained from the Schrödinger equation.

4. The correspondence with gravity

In this section we shall indicate briefly the correspondence between our toy model and gravity. This correspondence is very well defined in the context of minisuperspace models; for a more general situation, the Wheeler–DeWitt equation is a functional equation and hence the correspondence is, at best, formal.

Consider a four-dimensional metric expressed in terms of the metric for the 3-geometry, lapse and shift functions:

$$ds^2=(N^2-N_\alpha N^\alpha)\,dt^2+2N_\alpha\,dt\,dx^\alpha+g_{\alpha\beta}\,dx^\alpha\,dx^\beta \qquad \alpha,\beta=1,2,3. \tag{123}$$

It is well known (see, e.g., Wheeler 1964, Misner *et al* 1973) that the Einstein action

$$A_E=\frac{1}{16\pi G}\int R\sqrt{-g}\,d^4x \tag{124}$$

contains the time derivatives of $g_{\alpha\beta}$ but not of N or N_α. The variation of A_E with respect to N and N_α will give the constraint equations while the variation with respect to $g_{\alpha\beta}$ gives the dynamical equations. In particular, it is known that the Hamiltonian constraint, obtained by varying N, contains information about the dynamics. In terms of $g_{\alpha\beta}$, N_α and N the action $A_{\rm E}$ has the form

$$A_{\rm E}=\frac{1}{16\pi G}\int {\rm d}t\,{\rm d}^3\boldsymbol{x}(\tfrac{1}{4}\mathscr{G}^{\mu\nu\alpha\beta}\dot{g}_{\mu\nu}\dot{g}_{\alpha\beta}+\sqrt{|^3g|}\,{}^3R)-NH-N_\alpha H^\alpha$$
$$\mathscr{G}_{\mu\nu\alpha\beta}=\tfrac{1}{2}\sqrt{|^3g|}(g_{\mu\alpha}g_{\nu\beta}+g_{\mu\beta}g_{\nu\alpha}-2g_{\mu\nu}g_{\alpha\beta}). \tag{125}$$

In (125) ${}^3R$ is the scalar curvature made out of $g_{\alpha\beta}$, ${}^3g$ is the determinant of $g_{\alpha\beta}$ and the last two terms represent the constraints. The dynamical part of $A_{\rm E}$, which is the first term, can be written in a more suggestive form as

$$A_{\rm E}=\frac{1}{16\pi G}\int {\rm d}t[\tfrac{1}{4}G_{AB}\dot{g}^A\dot{g}^B-V(g^A)] \tag{126}$$

in which we have defined the coordinates in the superspace, g^A, to represent the 3-metric $g^{\alpha\beta}(\boldsymbol{x})$; the summation convention in the superspace includes integration over spatial coordinates as well (for details, see DeWitt 1967, Misner 1972); $V(g^A)$ stands for $(-|^3g|^{1/2}\,{}^3R)$. In a general situation (126) has only a formal significance. But in minisuperspace models, one can actually choose a finite set of gravitational degrees of freedom $g^A(t)$, $(A=1,2,\ldots,n)$ which depend only on t. In such a case, (126) actually represents a dynamical system with n degrees of freedom.

For our purpose, only the structure of $A_{\rm E}$ is important: it has 'kinetic' and 'potential' energy terms, but there is *no* coupling constant between these two terms. The only coupling constant (G^{-1}) appears as an overall multiplicative factor. This is similar to our action for Q discussed in § 2.1. Comparing (126) with the action for our toy model

$$A_Q=M\int {\rm d}t[\tfrac{1}{2}\dot{Q}^2-V(Q)] \tag{127}$$

we see that $V(Q)$ corresponds to $-|^3g|^{1/2}\,{}^3R$ and M corresponds to G^{-1}.

When matter fields are introduced, we have to add the matter action, $A_{\rm M}$, to $A_{\rm E}$. The usual structure for the matter action in a gravitational field will be something like

$$A_{\rm M}\sim\tfrac{1}{2}\int {\rm d}t\,{\rm d}^3x[\mathscr{M}(g)(\partial f)^2+U(g)P(f)]. \tag{128}$$

Equation (128) is *purely schematic*; its detailed form is irrelevant for our discussion, except to the extent that, in general, both the kinetic and potential parts of $A_{\rm M}$ will couple to gravity through $\mathscr{M}(g)$ and $U(g)$.

Let us now see what our study of Q–q system implies for semiclassical gravity.

Subsection 2.1 dealt with 'pure gravity' and showed how the classical and quantum theories based on $A_{\rm E}$ can be linked. We saw in § 2.1 that the relevant parameter indicating a transition to the classical limit is $(\hbar/M)$. For gravity it is $(G\hbar)$. When this parameter goes to zero, either in the $G\to 0$ limit or in the $\hbar\to 0$ limit, we obtain classical gravity from the quantum version. It is only the combination $(G\hbar)$ which matters, and not individual behaviour of G and $\hbar$. This result follows from the fact that G^{-1} multiplies the whole action. Subsection 2.2 was analogous to the situation of gravity interacting with a quantised field. An expansion in $\hbar$ will treat the quantum nature of both gravity and field on an equal footing. On the other hand, an expansion in powers of G (corresponding to expansion in M^{-1}) will treat gravity semiclassically. Our results in § 2 showed that, to leading order (i.e. to $O(G^{-1})$ in the phase), there is no backreaction; to this order, quantum field theory in curved spacetime is valid. The non-trivial backreaction terms which we expect are $O(\hbar/M)$ in our toy model, or equivalently $O(G\hbar)$. In this order, the quantum corrections from the self-interaction of gravity (coming from the ${}^3R$ term which corresponds to $V(Q)$) could be comparable to the backreaction from the quantum fluctuations of the field.

Lastly, we may ask: can we suppress the gravitational contribution compared to that from other fields in $O(\hbar)$? In general, we cannot do this. The potential energy in A_M depends on the spatial gradients (among other things) of the fields. These spatial gradients will produce a scalar 3-curvature, ${}^3R$, which will in turn contribute $O(\hbar)$ corrections to gravity. But it may be possible to achieve this suppression in an *ad hoc* manner by choosing suitable minisuperspace models. This corresponds to suppressing a large class of fluctuations both in gravity and the field. The validity of this procedure is, of course, dubious.

Throughout this paper we have concentrated on the technical questions related to backreaction. There is a basic conceptual issue involved in using any kind of expectation value as a source for gravity. This has to do with the collapse of the wavefunction on measurements and related interpretational issues of quantum theory. We feel that the technical issues discussed here are easier to tackle and should be understood before deeper problems are addressed. This was the spirit in which we have ignored the latter.

5. Outlook

We have performed a detailed study of the semiclassical limit of a Q–q system and have tried to draw analogies with semiclassical gravity. Basically, the analysis casts doubts on the existence of an unambiguous semiclassical description for gravity. We agree that this result is somewhat counterintuitive; but we see no way out of this situation.

It seems feasible to construct *semiclassical* minisuperspace models and study the correspondence between classical cosmology and quantum cosmology in greater detail. In particular, it would be worthwhile to compare the path integral and Schrödinger equation approaches in the context of minisuperspace models. Such a programme is underway, and we hope to make a direct comparison between the $O(\hbar)$ contributions from fields and gravity in the context of minisuperspace models.

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