

# On Some Transonic Aspects of General Relativistic Spherical Accretion onto Schwarzschild Black Holes

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## ABSTRACT

The equations governing general relativistic, spherically symmetric, hydrodynamic accretion of polytropic fluid onto black holes are solved in Schwarzschild metric to investigate some of the transonic properties of the flow. Only stationary solutions are discussed. For such accretion, it has been shown that real physical sonic points may form *even for flow with  $\gamma < \frac{4}{3}$  or  $\gamma > \frac{5}{3}$* . Behaviour of some flow variables in the close vicinity of the event horizon are studied as a function of specific energy and polytropic index of the flow.

**Key words:** accretion, accretion discs – black hole physics – general relativity – hydrodynamics

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## 1 INTRODUCTION

Investigation of accretion processes onto celestial objects was initiated by Hoyle & Littleton (1939) by computing the rate at which pressure-less matter would be captured by a moving star. Subsequently, theory of stationary, spherically symmetric and transonic hydrodynamic accretion of adiabatic fluid onto a gravitating astrophysical object at rest was formulated in a seminal paper by Bondi (1952) using purely Newtonian potential and by including the pressure effect of the accreting material. Later on, Michel (1972) discussed fully general relativistic polytropic accretion on to a Schwarzschild black hole by formulating the governing equations for steady spherical flow of perfect fluid in Schwarzschild metric. Following Michel's relativistic generalization of Bondi's treatment, Begelman (1978) discussed some aspects of the critical points of the flow for such an accretion. Spherical accretion and wind in general relativity have also been considered using equations of state other than the polytropic one and by incorporating various radiative processes (Shapiro, 1973a,b, Blumenthal & Mathews

1976, Brinkmann 1980). Recently Malec (1999) provided the solution for general relativistic spherical accretion with and without back reaction and showed that relativistic effects enhance mass accretion when back reaction is neglected.

It is to be noted that one very important issue in understanding the flow profile for accretion onto gravitating astrophysical objects is the 'transonicity' of the flow. Let the instantaneous dynamical velocity and the local acoustic velocity of a compressible fluid moving along a space curve parameterized by  $r$  be  $u(r)$  and  $a(r)$  respectively. Local Mach number  $M(r)$  of the fluid can then be defined as the ratio of the dynamical flow velocity to its sound speed, i.e.,  $M(r) = \left[ \frac{u(r)}{a(r)} \right]$ . The flow will be locally subsonic or supersonic according to  $M(r) < 1$  or  $> 1$ , i.e., according to  $u(r) < a(r)$  or  $u(r) > a(r)$ . The flow is transonic if at any moment it crosses  $M = 1$ . This happens when supersonic to subsonic or subsonic to supersonic transition takes place either continuously or discontinuously. The points where such crossing continuously takes place are called sonic points and where such crossing takes place discontinuously are called shocks or discontinuities. One crucial difference between the flow characteristics around a black hole and around any other astrophysical object is, as the flow approaches to any type of accretor other than a black hole, it can 'physically' hit the surface of the accretor directly and at the particular moment it collides with the surface, accretion can be either subsonic or supersonic depending on the location of the sonic point (which itself is a function of

various accretion parameters) as well as on the location of the surface of the accretor whereas for accretion onto black holes, flow must fall onto the black hole only supersonically because even for the steepest possible equation of state, the maximum possible sound velocity attained will always be lower than the bulk velocity of the flow at the event horizon. On the otherhand, it is quite possible that the flow at a sufficiently large distance away from the accretor, would be subsonic in general (except for a special case when the supersonic wind from nearby object(s) falls onto the accretor). So black hole accretion is *necessarily transonic* to satisfy the inner boundary condition at the event horizon whereas accretion onto other class of astrophysical objects may not always be transonic.

One standard method to study the classical transonic Bondi flow is to formulate the basic conservation equations for the flow and then to simultaneously solve these conservation equations to get sonic quantities as function of various accretion parameters and also to calculate the values of various dynamical as well as thermodynamic quantities as functions of various accretion parameters or radial distance (measured from the central accretor in the unit of Schwarzschild radius). However, the most effective approach to study the transonicity of spherically symmetric black hole accretion, as we believe, is to investigate the dependence of the location of the flow sonic point on various accretion parameters as well as to study the variation of Mach number of the flow with radial distance; which, perhaps, has not been explicitly discussed in existing literature for full general relativistic description of accretion onto black hole. In this paper, we would like to address this issue by formulating and solving the required equations of motion for a spherically symmetric transonic polytropic fluid accretion in Schwarzschild metric.

## 2 GOVERNING EQUATIONS

We take the Schwarzschild radius  $r_g$  to be:

$$r_g = \frac{2GM_{BH}}{c^2}$$

where  $M_{BH}$  is the mass of the black hole,  $G$  is universal gravitational constant and  $c$  is velocity of light in vacuum. We assume that a Schwarzschild type black hole spherically accretes fluid obeying polytropic equation of state. The density of the fluid is  $\rho(r)$ ,  $r$  being the radial distance measured in the unit of Schwarzschild radius  $r_g$ . We also assume that the accretion rate is not a function of  $r$  and we ignore the self-gravity of the flow. For simplicity of calculation, we choose gravitational unit where unit of length is scaled in units of  $r_g$ , unit of velocity is scaled in units of  $c$  and all other physically relevant quantities can be normalized likewise. We also set  $G = c = 1$  in system of units used here. For a Schwarzschild metric of the form

$$ds^2 = dt^2 \left(1 - \frac{1}{r}\right) - dr^2 \left(1 - \frac{1}{r}\right)^{-1} - r^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

the energy momentum tensor  $T^{\alpha\beta}$  for a perfect fluid can be written as (Shapiro & Teukolsky 1983):

$$T^{\alpha\beta} = \epsilon u^\alpha u^\beta + p (u^\alpha u^\beta - g^{\alpha\beta})$$

where  $\epsilon$  and  $p$  are proper internal energy density and pressure of the fluid (evaluated in the local inertial rest frame of

the fluid) respectively and  $u^\alpha$  is the four velocity commonly known as

$$u^\alpha = \frac{dx^\alpha}{ds}$$

Equations of motion which are to be solved for our purpose are,

1) Conservation of mass flux or baryon number conservation:

$$(\rho u_\alpha)_{;\alpha} = 0 \quad (1a)$$

and

2) Conservation of momentum or energy flux (general relativistic Euler equation obtained by taking the four divergence of  $T^{\alpha\beta}$ ):

$$(\epsilon + p) u_{\alpha;\beta} u^\beta = -p_{,\alpha} - u_\alpha p_{,\beta} u^\beta \quad (1b)$$

where  $\rho$  is the proper matter density and the semicolons denote the covariant derivatives.

Following Michel (1972), one can rewrite eq. 1(a) and eq. 1(b) for spherical accretion as

$$4\pi\rho ur^2 = \dot{M}_{in} \quad (2a)$$

and

$$\left(\frac{p+\epsilon}{\rho}\right)^2 \left(1 - \frac{1}{r} + u^2\right) = \mathbf{C} \quad (2b)$$

as two fundamental conservation equations for time independent hydrodynamical flow of matter on to a Schwarzschild black hole without back-reaction of the flow on to the metric itself.  $\dot{M}_{in}$  being the mass accretion rate and  $\mathbf{C}$  is some constant (related to the total enthalpy influx) to be evaluated for a specific equation of state.

It is well known in general theory of relativity that a stationary and axisymmetric space-time is endowed with one space-like  $\left[\left(\frac{\partial}{\partial\phi}\right)^\alpha\right]$  and one time-like  $\left[\left(\frac{\partial}{\partial t}\right)^\alpha\right]$  Killing field where  $\phi$  is the standard azimuthal co-ordinate. For a simpler space-time which is spherically symmetric, only the time like Killing field is of particular interest and corresponding to this field, one can obtain the integral of motion along a streamline as:

$$\mathcal{E} = h\epsilon_b = \frac{p+\epsilon}{\rho}\epsilon_b \quad (3)$$

where  $\mathcal{E}$ ,  $h (= \frac{p+\epsilon}{\rho})$  and  $\epsilon_b$  are the conserved specific energy of the flow including its rest mass, specific enthalpy and specific binding energy respectively. For polytropic equation of state  $p = K\rho^\gamma$  (where  $\gamma$  is the polytropic index and  $K$  is a constant which can be considered as the measure of the entropy of the flow), the generalized expression for the sound velocity  $a = \left(\frac{\partial p}{\partial\epsilon}\right)_S^{\frac{1}{2}}$  gives (Frank et. al. 1992, Weinberg 1972):

$$a = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\frac{\gamma\kappa T}{\mu m_H}} \quad (4)$$

where  $T$  is the flow temperature,  $\mu$  is the mean molecular weight and  $m_H \sim m_p$  is the mass of the Hydrogen atom. The subscript  $S$  indicates that differentiation is performed at constant specific entropy. Also one can easily show that the specific enthalpy  $h$  of the flow is related to the sound speed through the following equation

$$h = \left(1 - \frac{a^2}{\gamma - 1}\right)^{-1} \quad (5)$$

Using the above expression for enthalpy, one can easily rewrite the conservation equation (2b) as the specific energy conservation as

$$\mathcal{E} = \left[ \frac{\gamma - 1}{\gamma - (1 + a^2)} \right] \left( \frac{1 - \frac{1}{r}}{1 - u^2} \right)^{\frac{1}{2}} \quad (6a)$$

Defining  $\dot{M} = \dot{M}_{in} \gamma^{\frac{1}{\gamma-1}} K^{\frac{1}{\gamma-1}}$  to be another constant of motion for a shock free polytropic flow, one can rewrite eqn. (2a) as (see Chakrabarti (1996) and references therein):

$$\dot{M} = 4\pi \left[ \frac{(\gamma - 1) a^2}{\gamma - (a^2 + 1)} \right]^{\left(\frac{1}{\gamma-1}\right)} u \epsilon_b r^2 \quad (6b)$$

The above equation may be considered as the outcome of the conservation of mass and entropy along the flow line.

One can now easily derive the expression for velocity gradient  $\left(\frac{du}{dr}\right)$  (by differentiating eq. 6(a) and 6(b)) as

$$\frac{du}{dr} = \frac{u(1 - u^2) [a^2(4r - 3) - 1]}{2r(r - 1)(u^2 - a^2)} \quad (7a)$$

Since the flow is assumed to be smooth everywhere, if the denominator of eq. 7(a) vanishes at any radial distance  $r$ , the numerator must also vanish there to maintain the continuity of the flow. One therefore arrives at the so called ‘sonic point conditions’ by simultaneously making numerator and denominator of eq. 7(a) equal to zero and the sonic point conditions can be expressed as follows

$$u_c = a_c = \sqrt{\frac{1}{4r_c - 3}} \quad (7b)$$

here suffix  $c$  indicates that the values of the respective quantities are measured at the sonic point of the flow. For a specific value of  $\mathcal{E}$  and  $\gamma$ , location of sonic point  $r_c$  can be obtained by solving the following equation algebraically for  $r_c$

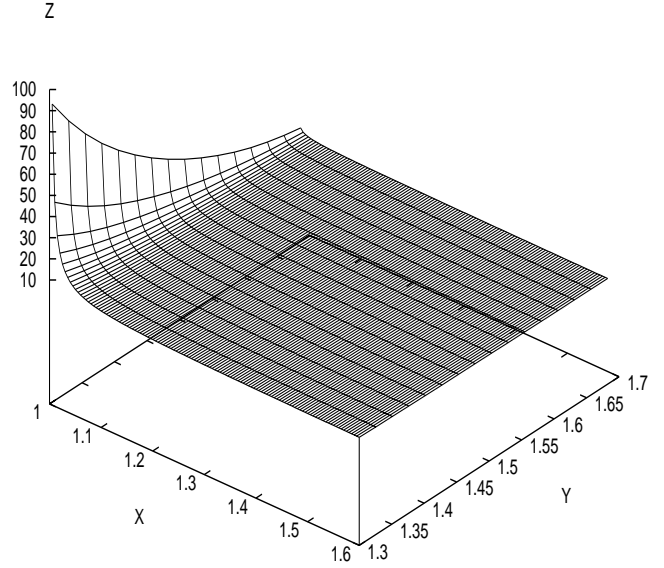
$$64r_c^3 (\mathcal{E}^2 - 1) - 16r_c^2 (2\mathcal{E}^2\psi - 9) + 4r_c (\mathcal{E}^2\psi^2 - 27) + 27 = 0 \quad (7c)$$

where  $\psi = \left(\frac{3\gamma+2}{\gamma-1}\right)$ . It is important to note that though eqn. 7(c) is a third order polynomial in  $r_c$ , for all values of  $\mathcal{E}$  and  $\gamma$ , it gives only one real physical root for  $r_c$  in general, we will return to this issue in next section.

The spherical surface of radius  $r = r_c$  can be defined as ‘acoustic horizon’ because for  $r < r_c$ ,  $u > a$ ; and any acoustic disturbances created in this region are advected towards the black hole. Thus no acoustic disturbances created within this radius can cross the acoustic horizon and escape to the region  $r > r_c$ .

To determine the behaviour of the solution near the sonic point, one needs to evaluate the value of  $\left(\frac{du}{dr}\right)$  at that point (we denote it by  $\left(\frac{du}{dr}\right)_c$ ) by applying L ‘Hospitals’ rule on eq. 7(a). It is easy to show that  $\left(\frac{du}{dr}\right)_c$  can be obtained by solving the following quadratic equation algebraically:

$$\begin{aligned} & \left(\frac{du}{dr}\right)_c^2 + \frac{(\gamma - 1)(16r_c^2 - 16r_c - 8\gamma r_c + 6\gamma + 3)}{3r_c(4r_c - 3)^{\frac{3}{2}}} \left(\frac{du}{dr}\right)_c \\ & + \frac{(\gamma - 1)(2r_c - 1)(24r_c^2 - 28r_c - 8r_c^2\gamma + 4r_c\gamma + 3\gamma + 6)}{2r_c(4r_c - 3)^{\frac{3}{2}}(4r_c - 3)(r_c - 1)} = 0 \end{aligned} \quad (7d)$$



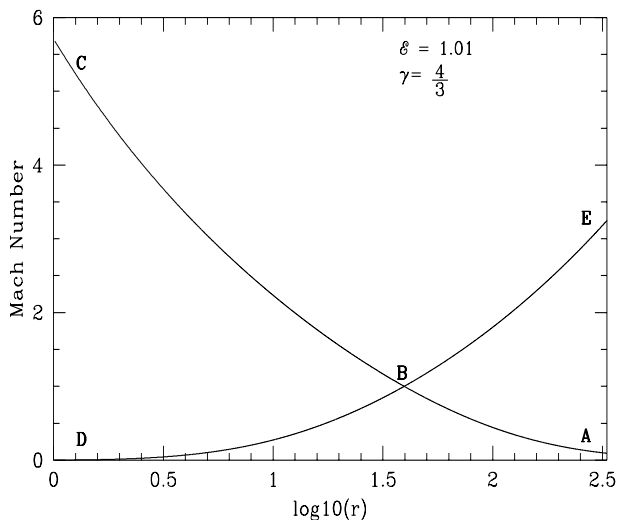
**Fig. 1:** Variation of the location of the accretion sonic point  $r_c$  with conserved specific energy  $\mathcal{E}$  (which includes the rest mass energy of matter) and polytropic index  $\gamma$  of the flow. While  $r_c$  is plotted along  $z$  axis,  $\mathcal{E}$  and  $\gamma$  are plotted along  $x$  and  $y$  axis respectively. The curves shown in the figure are plotted corresponding to the increments of 0.005 and 0.025 in  $\mathcal{E}$  and  $\gamma$  respectively. Note that  $r_c$  is produced even for flow with  $\gamma \leq 4/3$  and  $\gamma \geq 5/3$ , though not all sonic points shown here allow real physical transonic accretion. See text for detail.

It is now quite straightforward to simultaneously solve eq. 6(a) and eq. 6(b) to get the integral curves of the flow (curves showing the variation of Mach number with radial distance) for a fixed value of  $\mathcal{E}$  and  $\gamma$ . Detail methodology for this purpose will be discussed in §3.

### 3 SOLUTION PROCEDURE AND RESULTS

One can obtain the location of the flow sonic point by solving eqn. 7(c) for a fixed value of the conserved specific energy  $\mathcal{E}$  and polytropic index  $\gamma$  of the flow. As already mentioned, only one physical value of  $r_c$  would be obtained which lies outside the event horizon. We solve eqn 7(c) for a range of values of  $\mathcal{E}$  and  $\gamma$  for which a real physical solution is possible. There are cases where only one real solution for  $r_c$  is present but it lies inside the event horizon so we exclude those solutions. In Fig. 1., we show the variation of  $r_c$  (plotted along  $z$  axis) with  $\mathcal{E}$  (plotted along  $x$  axis) and  $\gamma$  (plotted along  $y$  axis). It is observed that the location of the sonic point non-linearly anti-correlates with both  $\mathcal{E}$  and  $\gamma$  which implies that ultra-relativistic flow\* with low total specific energy will produce the sonic point located furthest distance away from the black hole. One thing is very interesting in

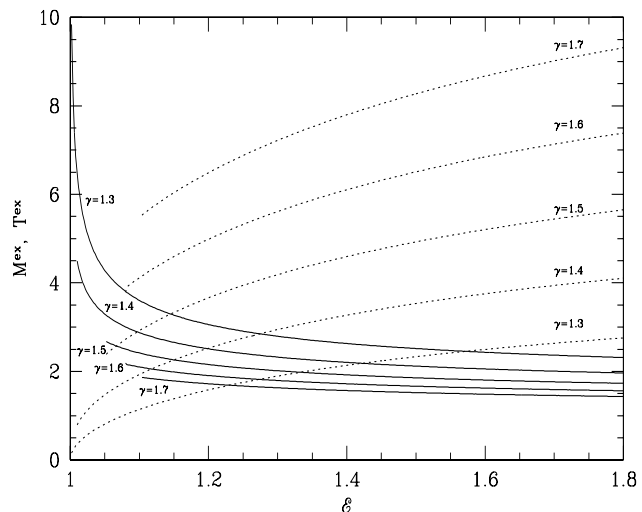
\* Hereafter, we will describe the flow to be ultra-relativistic for  $\gamma = \frac{4}{3}$  and purely non-relativistic for  $\gamma = \frac{5}{3}$  according to standard practice (Frank. et. al 1992).



**Fig. 2:** Integral curves of motion for a fixed value of  $\mathcal{E} = 1.01$  and  $\gamma = 4/3$ . While Mach number of the flow  $M$  is plotted along  $y$  axis, distance from the central accretor (in units of  $r_g$ ) is plotted along  $x$  axis in logarithmic scale.

Fig. 1. We observe that sonic point is produced even for flows with  $\gamma \leq 4/3$  and  $\gamma \geq 5/3$  though not all such sonic points allow steady physical transonic flows passing through them, which would be clear from the following discussion and from Fig. 3.

As we have mentioned above,  $r_c$  can be obtained from eqn. 7(c). Now the value of  $(\frac{du}{dr})$  at  $r_c$  can be obtained by solving eqn. 7(d) for a particular value of  $\mathcal{E}$  and  $\gamma$ . Fourth order Runge-Kutta method is then employed to integrate from sonic point to get the dynamical flow velocity and the acoustic velocity (so also the Mach number of the flow) at any point of the flow. It is well known that for spherically symmetric accretion in Newtonian potential, two solutions are obtained while solving the conservation equations, one out of which corresponds to the accretion and the other one to the ‘self-wind’. Same kind of situation is obtained here also for flows in general relativity. In Fig. 2 we plot variation of Mach number (plotted along  $y$  axis) with radial distance from the accretor (along  $x$  axis in unit of  $r_g$ ) in logarithmic scale for accretion (ABC) and wind (DBE) branches for a fixed value of  $\mathcal{E}$  ( $=1.01$ ) and  $\gamma$  ( $=4/3$ ). The location of the sonic point B comes out to be  $39.53 r_g$ . It is important to note that though from eqn 7(c), sonic points are obtained for a wide range of values of  $\mathcal{E}$  and  $\gamma$ , in reality, not all sonic points allow a real physical flow to pass through them. We find that for accretion passing through some of the sonic points (obtained for a specific region of parameter space spanned by  $\mathcal{E}$  and  $\gamma$ ), dynamical velocity or acoustic velocity, or both, becomes superluminal hence causality relation is violated which indicates that only a subset of the sonic points shown in Fig. 1 can be considered to study the transonicity of a real physical accretion flow. Here we would like to emphasize that still some sonic points, which allow real physical flow passing through them, are obtained for accretion with  $\gamma \leq \frac{4}{3}$  or  $\gamma \geq \frac{5}{3}$ , see Fig. 3.



**Fig. 3:** Behaviour of Mach number of the flow  $M$  (solid lines) and the flow temperature  $T$  (dotted lines) for a set of values of  $\mathcal{E}$  and  $\gamma$  marked in the figure.  $M^{ex}$  and  $T^{ex}$  for the values of  $M$  and  $T$  respectively measured at a distance  $0.01 r_g$  away from the event horizon.  $T^{ex}$  is scaled as  $T^{ex} \rightarrow 0.22 T^{ex}/T_{11}$  (where  $T_{11} = 10^{11}$  °K) to fit in the same graph; see text for details.

We would now like to investigate the behaviour of flow variables close to the inner boundary of accretion. For this purpose we would like to study the the Mach number of the flow and the flow temperature very close to the event horizon of the black hole. As all equations diverge on the event horizon, we would like to tackle the problem in the following way:

Let  $r_h$  be the actual location of the event horizon (in units used here,  $r_h = r_g$ ) and let  $\Delta r$  be equal to  $\delta r_g$  where  $\delta$  is a small number less than one. We then define  $r^{ex} = r_h + \Delta r$  to be the extreme point for our calculation of any flow variable  $V_{flow}$  so that  $V_{flow}^{ex}$  refers to the value of  $V_{flow}$  measured at a radial distance  $(1 + \delta) r_g$ . Here we would like to calculate  $M^{ex}$  and  $T^{ex}$  for  $\delta = 0.01$  for a fixed values of  $\mathcal{E}$  and  $\gamma$  and to study the variation of  $M^{ex}$  and  $T^{ex}$  with  $\mathcal{E}$  and  $\gamma$ . Lower value of  $\delta$  can also be taken but it would only increase the computational cost, the general solution profile would remain unaffected.

In what follows, for a particular value of  $\mathcal{E}$  and  $\gamma$ , we solve eqn 6(a) and 6(b) simultaneously upto  $r^{ex}$  to get  $M^{ex}$  and in the same way we calculate  $M^{ex}$  for various values of  $\mathcal{E}$  and  $\gamma$  which allows real physical solutions. Similarly, as the sound speed  $a$  can be calculated easily while calculating  $M$ , one can obtain  $T^{ex}$  (using eqn. (4)) for various values of  $\mathcal{E}$  and  $\gamma$ . In Fig. 3 we show the variation of  $M^{ex}$  (solid line) and  $T^{ex}$  (dotted line) with  $\mathcal{E}$  for a set of equi-spaced values of  $\gamma$  shown in the figure.  $T^{ex}$  has been normalized as  $T^{ex} \rightarrow 0.22 T^{ex}/T_{11}$  (where  $T_{11} = 10^{11}$  °K) to fit in the same graph. One can note that the value of minimum energy  $\mathcal{E}_{min}$  for which real physical accretion solution is available, non-linearly increases with increase of  $\gamma$ . We see that  $M^{ex}$  non-linearly anti-correlates with  $\mathcal{E}$  and  $\gamma$ . This is obvious because as we have seen from Fig. 1, flow with low energy and low  $\gamma$  produces the sonic point far away from the black

hole so that the flow becomes more and more supersonic as it approaches the event horizon. However,  $T^{ex}$  follows exactly the opposite profile, it non-linearly correlates with both  $\mathcal{E}$  and  $\gamma$  because for high energy accretion as well as for high enthalpy accretion (which is equivalent to high  $\gamma$  flow), flow temperature becomes higher. It is important to note that for a lower value of  $\delta$ , both  $M^{ex}$  and  $T^{ex}$  keeps increasing keeping the general ( $M^{ex}, T^{ex}$ ) vs ( $\mathcal{E}, \gamma$ ) profile remain unaltered.

#### 4 CONCLUSION

In this paper we have solved the equations governing general relativistic spherically symmetric hydrodynamic accretion of polytropic fluid on to a Schwarzschild black hole to investigate some of the transonic properties of the flow. We could study the variation of flow critical points with the conserved specific energy  $\mathcal{E}$  and the polytropic index  $\gamma$  of the flow and showed that while high energy purely-non-relativistic accretion produces sonic points closer to the event horizon of the black hole, low energy ultra-relativistic flows pushes the sonic point far away from the hole. Also it is shown that not all sonic points allows a real physical flow to pass through them. Perhaps the most important finding in this paper regarding the transonicity of the flow is that it is possible to obtain real physical sonic points (which allows a real physical general relativistic spherically symmetric transonic accretion through them) even for accretion with  $\gamma < \frac{4}{3}$  or  $\gamma > \frac{5}{3}$ . Also we study the behaviour of the flow temperature and Mach number of the flow in close vicinity of the event horizon of the black hole as a function of  $\mathcal{E}$  and  $\gamma$  and show that while low energy ultra-relativistic flow produces high Mach number at event horizon, high energy purely-non-relativistic flow produces high temperature at the horizon.

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