

On the mass of a uniform density star in higher dimensions

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Abstract

Within the framework of higher dimensions the mass of a uniform density star is evaluated. The four-dimensional upper bound for the mass-to-radius ratio obtained by Schwarzschild is generalized within the framework of higher-dimensional spacetime. It is found that the analogue upper bound for the mass-to-radius ratio in higher dimensions tends to increase at first as the number of dimensions of spacetime increases, it attains a maximum at nine dimensions and thereafter decreases. It is found that $D = 4$ is the lowest number of spacetime dimensions for which the mass-to-radius ratio of a uniform density star can be derived.

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The idea that spacetime dimensions should be extended from four to higher dimensions came from the work of Kaluza and Klein [1] who first tried to unify gravity with electromagnetism. The approach has been revived and considerably generalized after realizing that many interesting theories of particle interactions need spacetime dimensions greater than four for their formulation. For example, superstring theory is considered to be of the promising candidates which may unify gravity with the other fundamental forces in nature which requires 10 dimensions for its consistent formulation. The present ideas in dimensional reduction suggest that our cosmos may be a 3-brane evolving in a D -dimensional spacetime. Cadeau and Woolgar [2] recently addressed this issue in the context of black holes which led to homogeneous but non-FRW-braneworld cosmologies. There has been growing interest in recent years in obtaining a higher-dimensional analogue of a four-dimensional general relativistic result. Several works in the literature have appeared which include the higher-dimensional generalization of the spherically symmetric Schwarzschild and Reisner-Nordström black holes [3, 4], Kerr black

holes [5], Vaidya solution [6], etc. Mandelbrot [7] studied the problem on the variability of dimensions in which he describes how a ball of thin thread is seen as an observer changes scale. An object which seems to be a point object from a far point becomes a three-dimensional ball at a closer distance. Thus as an observer moves down through various scales the ball appears to change shape. While the embedding dimensions for the ball has not changed, the effective dimension of the contents does change. It is possible that there are compact [8] or non-compact [9] dimensions present at a certain point. At this scale, the $(3 + 1)$ metric is simply not true, although one obtains a valid description with general relativity. Liu *et al* [10] reported solar system tests based on a five-dimensional extension of the Schwarzschild metric and Cassisi *et al* [11] have examined the effects of higher dimensions on stellar evolution. Yu and Ford [8] reported that observable effects of higher dimensions may be found from lightcone fluctuations. Günther and Zhuk [12] investigated the observable consequences of spontaneous compactification hypothesis for the extra dimension. At present, dimensional physics has become an active area of investigation with some promise of future experimental insights [13].

In this paper I intend to obtain a higher-dimensional generalization of the Schwarzschild mass formula in four-dimensional spacetime for a uniform density static star. We consider a star in a higher-dimensional universe which is in hydrostatic equilibrium to obtain the mass and radius relation for a stable configuration. The star under study is a D -dimensional ball of uniform density.

Einstein's field equation in higher dimensions is given by

$$R_{ab} - \frac{1}{2}g_{ab}R = 8\pi G_D T_{ab} \quad (1)$$

where D is the total number of dimensions, $G_D = G.V_{D-4}$ is the gravitational constant in D dimensions, G denotes the four-dimensional Newton constant and V_{D-4} is the volume of the extra dimensions, R_{ab} is the Ricci tensor and T_{ab} is the energy-momentum tensor. We consider a higher-dimensional spherically symmetric static spacetime in the form

$$ds^2 = -e^{2\nu(r)} dt^2 + e^{2\mu(r)} dr^2 + f^2(r) d\Omega_n^2 \quad (2)$$

where $n = D - 2$ and $d\Omega_n^2 = d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \sin^2\theta_2(d\theta_3^2 + \dots + \sin^2\theta_{n-1} d\theta_n^2)$ represents the metric on the n -sphere in polar coordinates. The energy-momentum tensor is $T_{ab} = (\rho, -p, -p, -p, \dots, -p)$. The field equation (1) then reduces to the study of the following three equations using $f(r) = r$:

$$\frac{ne^{-2\mu}\mu'}{r} + \frac{n(n-1)}{2} \left(\frac{1 - e^{-2\mu}}{r^2} \right) = 8\pi G_D \rho, \quad (3)$$

$$\frac{ne^{-2\mu}\nu'}{r} - \frac{n(n-1)}{2} \left(\frac{1 - e^{-2\mu}}{r^2} \right) = 8\pi G_D p, \quad (4)$$

$$e^{-2\mu} \left(\nu'' + \nu'^2 - \mu'\nu' - \frac{(n-1)(\mu' - \nu')}{r} \right) - \frac{(n-1)(n-2)}{2} \left(\frac{1 - e^{-2\mu}}{r^2} \right) = 8\pi G_D p, \quad (5)$$

we use $c = 1$ and a prime denotes a derivative with respect to r . The exterior of such a star should match the D -dimensional generalization of the Schwarzschild solution obtained by Myers and Perry [3]. Here we consider an ansatz

$$e^{-2\mu} = 1 - \frac{C}{r^{n-1}} \quad (6)$$

where C is a constant related to the mass of a higher-dimensional star. Using the field equation (4) we obtain

$$v' = \frac{(r^{n+1}/n)8\pi G_D p + [(n - 1)/2]C}{r(r^{n-1} - C)}. \tag{7}$$

For hydrostatic equilibrium, the Tolman–Openheimer–Volkoff (TOV) equation is derived using field equations (3)–(5) which is given by

$$\frac{dp}{dr} = -(\rho + p)v'. \tag{8}$$

We now eliminate v' using equation (7) and obtain

$$\frac{dp}{dr} = -\frac{(\rho + p) [(r^{n+1}/n)8\pi G_D p + [(n - 1)/2]C]}{r(r^{n-1} - C)}. \tag{9}$$

For a star having uniform density we have

$$\begin{aligned} \rho &= \rho_o, & r &\leq R \\ \rho &= 0, & r &> R \end{aligned} \tag{10}$$

where R is the radial size of the star.

The mass of a higher-dimensional star is given by

$$M = \frac{n A_n C}{16\pi G_D} \tag{11}$$

where

$$n = D - 2, \quad A_n = \frac{2\pi^{\frac{1}{2}(n+1)}}{\Gamma((n + 1)/2)}.$$

To determine the constant C we find the volume of an $(n + 1)$ -dimensional ball of radius r which is obtained below

$$V = \int_0^r V_n dr = \frac{2\pi^{\frac{1}{2}(n+1)} r^{n+1}}{(n + 1)\Gamma((n + 1)/2)} \tag{12}$$

where the volume of the n -dimensional surface is $V_n = 2\pi^{\frac{1}{2}(n+1)} r^n / \Gamma(\frac{1}{2}(n + 1))$. Thus using equations (11) and (12) with the energy density condition we obtain

$$C = 16\pi G_D \rho_o r^{n+1} / n(n + 1).$$

The TOV equation can be written as

$$\frac{dp}{dr} = -\frac{(\rho_o + p)[(n + 1)p + (n - 1)\rho_o] r}{2\rho_o(\alpha^2 - r^2)} \tag{13}$$

with $\alpha^2 = n(n + 1)/16\pi G_D \rho_o$. On integrating the differential equation (13) we obtain

$$\frac{(n + 1)p + (n - 1)\rho_o}{p + \rho_o} = K\sqrt{\alpha^2 - r^2} \tag{14}$$

where K is an integration constant. It is now easy to express pressure in terms of ρ_o . Using the boundary condition at $r = R$, i.e. the pressure at the surface of the star should vanish when we

have $p(R) = 0$, one can determine the integration constant, $K = (n - 1)/\sqrt{\alpha^2 - R^2}$. Thus the pressure in terms of the radial coordinate can be expressed as

$$p = \rho_o \frac{(n - 1)(\sqrt{\alpha^2 - r^2} - \sqrt{\alpha^2 - R^2})}{(n + 1)\sqrt{\alpha^2 - R^2} - (n - 1)\sqrt{\alpha^2 - r^2}}. \tag{15}$$

The pressure at the centre $r = 0$ of the star becomes

$$p_c = \rho_o \frac{(n - 1)(\alpha^2 - \sqrt{\alpha^2 - R^2})}{(n + 1)\sqrt{\alpha^2 - R^2} - (n - 1)\alpha}. \tag{16}$$

The important feature of the above solution is that it necessarily imposes a constraint connecting the mass of a given star (M) and radius (R). To derive this constraint, we simply consider the case that at $r = r_o$ the pressure becomes infinite where

$$r_o^2 = \alpha^2 - \left(\frac{n + 1}{n - 1}\right)^2 (\alpha^2 - R^2) \tag{17}$$

which depends on the dimensions of the universe in addition to the density of a star. Since the pressure is a scalar, this infinity will persist all over. To avoid this one obtains a constraint on the mass-to-radius ratio of a star which is

$$\frac{M}{R^{D-3}} < \frac{(D - 2)^2 \pi^{\frac{1}{2}(D-3)}}{2G_D(D - 1)^2 \Gamma((D - 1)/2)}. \tag{18}$$

Thus one obtains a stable configuration for a star satisfying the above limit. It is important to state here that no star of uniform density can have a mass and radius exceeding the above limit. In four dimensions ($D = 4$) the Schwarzschild limit $2M/R < \frac{8}{9}$ is obtained by taking $G_4 = 1$ [14]. The variation of $G_D M/R^{D-3}$ with the dimensions D is given in table 1.

Table 1. Variation of $G_D M/R^{D-3}$ with spacetime dimensions.

$D = n + 2$	n	Upper limit on $G_D M/R^{n-1}$
3	1	0.1250
4	2	0.4444
5	3	0.8836
6	4	1.3404
7	5	1.7135
8	6	1.93364
9	7	1.9783
10	8	1.8666
11	9	1.6438
12	10	1.3630
13	11	1.0714
14	12	0.8027
15	13	0.5757
26	24	1.7560×10^{-4}

We note the following:

- The upper bound on $G_D M/R^{D-3}$ first increases with increasing spacetime dimensions and then attains a maximum value. The maximum value for an integer-dimensional spacetime is found at $D = 9$. Thereafter, it decreases to a very small value as one goes on increasing the dimensions of the universe. Thus compared with the superstring dimensions $D = 10$ the above ratio is found to be least in $D = 4$ dimensions, which is the observable dimension. However, the ratio C/R^{D-3} always remains less than one.

- The mass-to-radius ratio in D dimensions is $M/R < f(D) R^{D-4}$, where $f(D)$ is a function of dimensions of the spacetime in which the star is embedded. Thus the mass-to-radius ratio is a number only for $D = 4$, otherwise it also depends on the radius as R^{D-4} . With $D = 2 + 1$ dimensions, one obtains an upper limit on the mass of the star which is independent of the radius—an unphysical result. However, in $D = 3 + 1$ dimensions, one derives a consistent mass-to-radius ratio for a uniform density star. Thus $D = 4$ is the lowest number of dimensions in this case to formulate the stellar study.
- It may be important to mention here that the four-dimensional analogue of the mass-to-radius ratio in higher-dimensional spacetime gives an interesting result if one varies the number of dimensions along the real axis (i.e. considering fractional spacetime dimensions). The same mass-to-radius ratio obtained in four dimensions is also obtained for fractional higher dimensions which supports the work of Kobelev [15]. This issue will be discussed elsewhere.

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