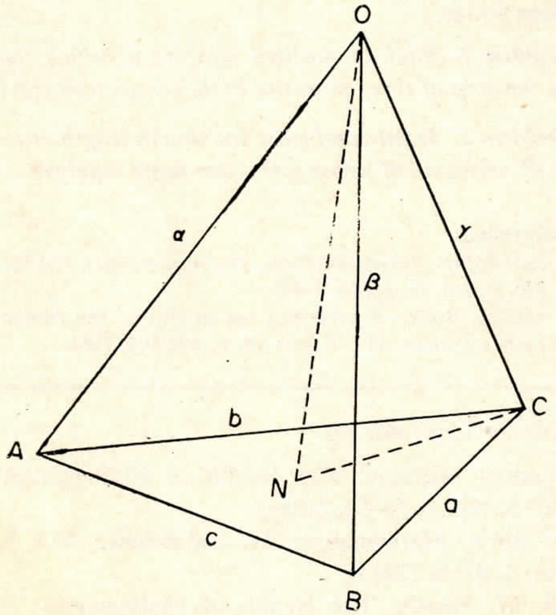
The Volume of Tetrahedron

JAYANT V. NARLIKAR
Tata Institute of Fundamental Research, Bombay

The area of a triangle can be expressed algebraically in terms of its three sides. Similarly the volume of a tetrahedron can be given as an algebraic function of the lengths of its six edges. The proof of this result is usually obtained by using vector methods. The following discussion gives a more elementary derivation of this interesting result.

THE AREA OF a triangle can be expressed algebraically in terms of its three sides. Can the volume of a tetrahedron be given as an algebraic function of its six edges? The answer to this question is 'yes' and the method is outlined below.



Let the edges of the tetrahedron $OABC$ shown in the Figure be $OA = \alpha$, $OB = \beta$, $OC = \gamma$, $BC = a$, $CA = b$, and $AB = c$. Given the six edges we can construct the tetrahedron uniquely, provided certain triangle inequalities guarantee a solution. We will assume that the solution exists.

To compute the volume V draw perpendicular ON from O on the plane ABC. Let $ON = h$. Then

$$V = \tfrac{1}{3} h \left\{ s(s-a)(s-b)(s-c) \right\}^{1/2} \equiv \tfrac{1}{3} h \triangle, \quad (1)$$

say,

where $s = (a + b + c)/2$. Our problem is therefore to compute h in terms of a, b, c, α, b, γ .

It is convenient to use coordinate geometry with N as origin, NC as the x-axis and NO as the z-axis. The y-axis is then specified as the third member of the right handed orthogonal triad. Write the coordinates of O, A, B, C, as (o, o, h), (p, q, o), (l, m, o) and (k, o, o) respectively. We then have the following equations:

$$h^2 + p^2 + q^2 = \alpha^2,\ h^2 + l^2 + m^2 = \beta^2,$$
$$h^2 + k^2 = \gamma^2, (k - l)^2 + m^2 = a^2, (k - p)^2 + q^2$$
$$= b^2, (p - l)^2 + (q - m)^2 = c^2. \quad (2)$$

To solve these equations first eliminate p and l:

$$p = \frac{\alpha^2 - \gamma^2 - b^2}{2k} + k,\ l = \frac{\beta^2 - \gamma^2 - a^2}{2k} + k.$$

Next eliminate m and q:

$$m^2 = a^2 - \left(\frac{\beta^2 - \gamma^2 - a^2}{2k} \right)^2,$$

$$q^2 = b^2 - \left(\frac{\alpha^2 - \gamma^2 - b^2}{2k} \right)^2.$$

The last of the six equations of the set (2) then gives

$$c^2 = \left(\frac{a^2 - \gamma^2 - b^2}{2k} - \frac{\beta^2 - \gamma^2 - a^2}{2k}\right)^2$$

$$+ \left\{\sqrt{a^2 - \left(\frac{\beta^2 - \gamma^2 - a^2}{2k}\right)^2} + \right.$$

$$\left. + \sqrt{b^2 - \left(\frac{a^2 - \gamma^2 - b^2}{2k}\right)^2}\right\}^2, \quad (3)$$

This equation determines k and then h is given by $\sqrt{\gamma^2 - k^2}$. After some tedious algebra we get

$$h = \frac{Q}{4\triangle} \quad (4)$$

where

$$\begin{aligned} Q^2 = {} & a^2\alpha^2(b^2 + \beta^2 + c^2 + \gamma^2 - a^2 - \alpha^2) \\ & + b^2\beta^2(c^2 + \gamma^2 + a^2 + \alpha^2 - b^2 - \beta^2) \\ & + c^2\gamma^2(a^2 + \alpha^2 + b^2 + \beta^2 - c^2 - \gamma^2) \\ & - a^2b^2c^2 - a^2\beta^2\gamma^2 - b^2\gamma^2\alpha^2 - c^2\alpha^2\beta^2. \quad (5) \end{aligned}$$

From equation (1) we therefore find that

$$V = \frac{1}{12}Q. \quad (6)$$

The expression (5) has the relevant combinatorial symmetries of the six edges. Unlike $\triangle$, it does not have any linear or quadratic factors in $a, b, c, \alpha, \beta, \gamma$. However, just as $\triangle$ vanishes when the triangle inequalities become equalities, Q must vanish when the tetrahedron becomes degenerate, i.e. when O, A, B, C become coplaner. I have not been able to demonstrate this result in an elegant manner without recourse to coordinate geometry.

Fiftieth Annual Conference of Indian Mathematical Society, Sardar Patel University, Vallabh Vidyanagar, February 6-8, 1985

The conference was attended by 208 delegates including 35 students and 38 local members. The following addresses were delivered:

Prof. R. P. Aggarwal, Lucknow	Some recent works on Ramanujan—overview
Prof. R. A. Gustafson, USA	Transformation Groups and Representation Theory
Dr. (Miss) S. P. Arya, Delhi	Some variants of continuity and compactness in topology
Dr. O. P. Ahuja, New Guinea	On Bieberbach conjecture and its recent solution by de Branges
Prof. A. R. Rao, Ahmedabad	Geometry and Soap bubbles
Shri S. Ghosh, Delhi	A Mathematical model for Bhopal gas leakage
Prof. B. S. Yadav, Delhi	The lattice $w + i$ of invariant subspaces
Prof. K. S. Padmanabhan, Madras	Univalent functions
Prof. W. H. Abdi, Lucknow	On hypergeometric functions
Dr. S. J. Bhatt, Vallabh Vidyanagar	Unbounded Operator algebras: A survey
Prof. T. Thrivikraman, Cochin	History of Ancient Indian Mathematics

In addition, 76 papers were presented. The paper-reading sessions were chaired by Profs. P. K. Kamthan, J. Gopal Krishna, V. M. Shah and O. P. Ahuja.

A mathematical exhibition and a cultural programme were also organised.

The fifty-first conference of the Society will be held in Cochin in December 1985.