

Energy density in general relativity: a possible role for cosmological constant

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Abstract

We consider a static spherically symmetric charged anisotropic fluid source of radius $\sim 10^{-16}$ cm by introducing a variable Λ dependent on the radial coordinate r under general relativity. From the solution sets a possible role of the cosmological constant is investigated which indicates the dependency of energy density on it.

Key words: Classical electron model, energy density, cosmological parameter.

PACS: 04.20.-q, 04.20.Jb, 98.80.Hw

1 Introduction

The structure of electron was, for a long time, an intrigue problem to the researchers. Many scientists, like Lorentz [1] and even Einstein [2] tried to solve the problem to show that the electron mass is a electromagnetic field dependent quantity (for a detail account see the references [3] and [4]). Lateron, under general relativity some models have been proposed by different authors describing extended electron with its mass entirely of electromagnetic origin [5 - 8]. Recently, based on the experimental upper limits on the size of the electron as $\sim 10^{-16}$ cm [9] it is argued by Cooperstock and Rosen [10], Bonnor and Cooperstock [11] and Herrera and Varela [12] that in the framework of general theory of relativity the electron-like spherically symmetric charged distribution of matter must contain some negative mass density. Motivated by these results with historical and heuristic values we would like to explore a possible role of cosmological constant on the energy density of electron when

it is modeled as a variable dependent on the radial coordinate r of the charged spherical matter distribution.

The basic logic for considering variability of so called cosmological constant, which was introduced by Einstein in 1917 to obtain a static cosmological model, is related to the observational evidence of high redshift Type Ia supernovae [13, 14] for a small decreasing value of cosmological constant ($\Lambda_{present} \leq 10^{-56} cm^{-2}$) at the present epoch. This indicates that instead of a strict constant the Λ could be a function of space and time coordinates. If the role of time-dependent Λ is prominent in the cosmological realm, then space-dependent Λ has an expected effect in the astrophysical context. It is, therefore, argued by Narlikar et al. [15] that the space-dependence of Λ cannot be ignored in relation to the nature of local massive objects like galaxies. Our aim, however, to see if there is any effect of space-dependent Λ on the energy density of the classical electron. This is because cosmological constant is thought to be related to the quantum fluctuations as evident from the theoretical works by Zel'dovich [16]. Moreover, it is believed through indirect evidences that 65 % of the contents of the universe is to be in the form of the energy of vacuum [17]. Thus, the energy density of vacuum due to quantum fluctuation might have, in our opinion, some underlying relation to the energy density of Lorentz's extended electron [1] under general relativistic treatment.

In the present letter we have tried to find out, through some specific case studies, that energy density of classical electron is related to the variable cosmological constant and the gravitational mass of the electron is entirely dependent on the electromagnetic field alone.

2 The field equations

To carry out the investigation we have considered the Einstein-Maxwell field equations for the case of anisotropic charged fluid distribution (in relativistic units $G = c = 1$) which are given by

$$G^i_j = R^i_j - g^i_j R/2 = -8\pi[T^i_j^{(m)} + T^i_j^{(em)} + T^i_j^{(vac)}], \quad (1)$$

$$[(-g)^{1/2} F^{ij}]_{,j} = 4\pi J^i (-g)^{1/2}, \quad (2)$$

$$F_{[ij,k]} = 0 \quad (3)$$

where F^{ij} is the electromagnetic field tensor and J^i , current four vector which is equivalent to $J^i = \sigma u^i$, σ being the charge density and u^i is the four-velocity of the matter satisfying the relation $u_i u^i = 1$.

The matter, electromagnetic and vacuum energy-momentum tensors are, respectively given by

$$T_j^i{}^{(m)} = (\rho + p_\perp)u^i u_j - p_\perp g^i_j + (p_\perp - p_r)\eta^i \eta_j, \quad (4)$$

$$T_j^i{}^{(em)} = -[F_{jk}F^{ik} - g^i_j F_{kl}F^{kl}/4]/4\pi, \quad (5)$$

$$T_j^i{}^{(vac)} = g^i_j \Lambda(r)/8\pi \quad (6)$$

where ρ , p_r and p_\perp are the proper energy density, radial and tangential pressures respectively and also η_i is the unit spacelike vector on which the condition to be imposed is $\eta_i \eta^i = -1$. Here p_r is the pressure in the direction of η_i whereas p_\perp is the pressure on the two-space orthogonal to η_i .

Now, for the spherically symmetric metric

$$ds^2 = e^{\nu(r)} dt^2 - e^{\lambda(r)} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (7)$$

the Einstein-Maxwell field equations (1) - (6) corresponding to anisotropic charged fluid with spatially varying cosmological constant i.e. $\Lambda = \Lambda(r)$, are given by

$$e^{-\lambda}(\lambda'/r - 1/r^2) + 1/r^2 = 8\pi T_0^0 = 8\pi\tilde{\rho} + E^2, \quad (8)$$

$$e^{-\lambda}(\nu'/r + 1/r^2) - 1/r^2 = -8\pi T_1^1 = 8\pi\tilde{p}_r - E^2, \quad (9)$$

$$\begin{aligned} e^{-\lambda}[\nu''/2 + \nu'^2/4 - \nu'\lambda'/4 + (\nu' - \lambda')/2r] \\ = -8\pi T_2^2 = -8\pi T_3^3 = 8\pi\tilde{p}_\perp + E^2, \end{aligned} \quad (10)$$

$$[r^2 E]' = 4\pi r^2 \sigma e^{\lambda/2}, \quad (11)$$

where E , the intensity of electric field, is defined as $E = -e^{-(\nu+\lambda)/2}\phi'$ and can equivalently be expressed, from equation (11), as

$$E = \frac{1}{r^2} \int_0^r 4\pi r^2 \sigma e^{\lambda/2} dr. \quad (12)$$

Here prime denotes derivative with respect to the radial coordinate r only.

In the above equations (8) - (10) we have considered that

$$\tilde{\rho} = \rho + \Lambda(r)/8\pi, \quad (13)$$

$$\tilde{p}_r = p_r - \Lambda(r)/8\pi, \quad (14)$$

$$\tilde{p}_\perp = p_\perp - \Lambda(r)/8\pi, \quad (15)$$

where $\tilde{\rho}$, \tilde{p}_r and \tilde{p}_\perp are the effective energy density, radial and tangential pressures respectively.

The equation of continuity $T^i_j; i = 0$, is given by

$$\frac{dp_r}{dr} - \frac{1}{8\pi} \frac{d\Lambda(r)}{dr} + \frac{1}{2}(\rho + p_r)\nu' = \frac{1}{8\pi r^4} \frac{dq^2}{dr} + \frac{2(p_\perp - p_r)}{r} \quad (16)$$

where q is the charge on the spherical system.

We assume the relation between the radial and tangential pressures [12] as

$$p_\perp - p_r = \alpha q^2 r^2, \quad (17)$$

where α is a constant.

Hence, by use of equations (14) and (17), the equation (16) reduces to

$$\frac{d\tilde{p}_r}{dr} + \frac{1}{2}(\tilde{\rho} + \tilde{p}_r)\nu' = \frac{1}{8\pi r^4} \frac{dq^2}{dr} + 2\alpha q^2 r. \quad (18)$$

Again, equation (8) can be expressed in the following form as

$$e^{-\lambda} = 1 - 2M/r, \quad (19)$$

where the active gravitational mass, M , is given by

$$M = 4\pi \int_0^r \left[\tilde{\rho} + \frac{E^2}{8\pi} \right] r^2 dr. \quad (20)$$

3 The solutions

3.1 Model for $\rho + p_r = 0$

Let us now solve the equation (18) under the assumption between the stress-energy tensors as $T^1_1 = T^0_0$, which implies that

$$\tilde{\rho} + \tilde{p}_r = \rho + p_r = 0. \quad (21)$$

Also to make the equation (12) integrable we assume that

$$\sigma = \sigma_0 e^{-\lambda/2}, \quad (22)$$

where σ_0 is the charge density at $r = 0$ of the spherical distribution, i.e. the central density of charge.

Hence, using condition (22) in equation (12), we get for the expression of electric charge and intensity of the electric field as

$$q = Er^2 = \frac{4}{3}\pi\sigma_0 r^3. \quad (23)$$

With the help of equations (21) and (23), the equation (18) reduces to

$$\frac{d\tilde{p}_r}{dr} = \frac{4}{3}\pi\sigma_0^2 r + 2\alpha q^2 r, \quad (24)$$

Thus the solution set is given by

$$e^{-\lambda} = e^\nu = 1 - \frac{16}{45}\pi^2\sigma_0^2 r^2(5a^2 - 2r^2) - \frac{8}{15}\pi\alpha q^2 r^2(5a^2 - 3r^2), \quad (25)$$

$$p_r = -(\alpha q^2 + \frac{2}{3}\pi\sigma_0^2)(a^2 - r^2) + \frac{\Lambda(r)}{8\pi}, \quad (26)$$

$$p_\perp = \alpha q^2 r^2 - (\alpha q^2 + \frac{2}{3}\pi\sigma_0^2)(a^2 - r^2) + \frac{\Lambda(r)}{8\pi}, \quad (27)$$

$$\rho(r) = (\alpha q^2 + \frac{2}{3}\pi\sigma_0^2)(a^2 - r^2) - \frac{\Lambda(r)}{8\pi}. \quad (28)$$

The active gravitational mass which is defined in the equation (20), then, by virtue of the equations (23) and (28), takes the form as

$$M(r) = \frac{8}{135}\pi^2\sigma_0^2 r^3[8\pi\alpha a^6(5a^2 - 3r^2) + 3(5a^2 - 2r^2)]. \quad (29)$$

Thus, the metric potentials λ and ν are given by

$$e^{-\lambda} = e^\nu = 1 - \frac{2M(r)}{r}. \quad (30)$$

The total effective gravitational mass can be obtained, after smoothly matching of the interior solution to the exterior Reissner-Nordström solution on the

boundary, as

$$m = M(a) + \frac{q^2(a)}{2a} = \frac{64}{45}\pi^2\sigma_0^2a^5\left(1 + \frac{2}{3}\pi\alpha a^6\right), \quad (31)$$

which corresponds to the second case (B) of Herrera-Varela model [12] and represents “electromagnetic mass” model such that gravitational mass of a charged fluid sphere originates from the electromagnetic field alone [1, 18]. This again corresponds to the Tiwari-Rao-Kanakamedala model [7] with $\alpha = 0$ case and thus the present model reduces to isotropic one.

Now, considering the observed values of mass, charge and radius of the electron (in relativistic units) as $m = 6.76 \times 10^{-56}$ cm, $q = 1.38 \times 10^{-34}$ cm and $a = 10^{-16}$ cm the value of α , from the equation (31), is given by

$$\alpha = -4.77 \times 10^{95} \text{ cm}^{-6}. \quad (32)$$

For the above value of constant α , the energy density in equation (28) becomes

$$\rho(r) = -6.81 \times 10^{27}(a^2 - r^2) - \frac{\Lambda(r)}{8\pi}. \quad (33)$$

The central energy density, ρ_0 , at $r = 0$, then can be calculated as

$$\rho_0 = -6.81 \times 10^{-5} - \frac{\Lambda_0}{8\pi}. \quad (34)$$

Thus, from the equation (34) one can see that for $\Lambda_0 > 0$ the energy density of the electron is a negative quantity. It is to be noted here that in the cosmological context Λ positive is related to the repulsive pressure and hence an acceleration dominated universe as suggested by the SCP and HZT project report [13, 14, 19, 20]. However, equation (34) indicates that this negativity of energy density is also obtainable for $\Lambda_0 < 0$ (which indicates a collapsing situation of the universe [21]) for its very small value. In this context it is also possible to show that at an early epoch of the universe when the numerical value of negative Λ was higher than that of the first term of ρ (i.e. $\sim 10^{-5}$ at $r = 0$) obviously energy density was a positive quantity. Thus, in the case of decreasing negative value of Λ it is clear that there was a smooth crossover from positive energy density to a negative energy density via a phase of null energy density! However, these results confirm the vacuum equation of state $\rho + p_r = 0$ [22 - 25].

We can also see that on the boundary, $r = a$, the total energy density becomes

$$\rho_a = -\frac{\Lambda_a}{8\pi}, \quad (35)$$

which shows its clear dependency on the cosmological constant. However, for $\Lambda_a > 0$, ρ_a is negative whereas for $\Lambda_a < 0$, ρ_a is as usual a positive quantity. As a simple and interesting exercise (as all the parameters related to the electron are known) one can find out the numerical value of Λ_a , at the boundary of the spherical system from the equation (35), which equals $\sim 10^{-7} cm^{-2}$. This constant value of Λ_a is too large and might be related to an early epoch of the universe. Here for finding out the total energy density ρ_a it is considered that $\rho_a \leq \rho_{average}$, where $\rho_{average}$ is equal to $m/\frac{4}{3}\pi a^3$ as the energy density of the spherical distribution is decreasing from centre to boundary.

3.2 Model for $\rho + p_r \neq 0$

Now using equation (23) the equation (18) can be written as

$$\frac{d}{dr} \left[\tilde{p}_r - \frac{E^2}{8\pi} \right] + \frac{1}{2}(\tilde{\rho} + \tilde{p}_r)\nu' = \frac{E^2}{2\pi r} + 2\alpha q^2 r. \quad (36)$$

Assuming that the radial stress-energy tensor $T^1_1 = 0$, one gets

$$\nu' = \frac{(e^\lambda - 1)}{r}, \quad (37)$$

$$\tilde{p}_r = \frac{E^2}{8\pi}. \quad (38)$$

Using equations (37) and (38) in equation (36), we then have

$$\tilde{\rho} + \tilde{p}_r = \rho + p_r = \frac{(4\alpha q^2 r^2 + E^2/\pi)}{e^\lambda - 1}. \quad (39)$$

Thus, equation (20) takes the form as

$$M = 4\pi \int_0^r \left[\frac{(4\alpha q^2 r^2 + E^2/\pi)}{e^\lambda - 1} \right] r^2 dr. \quad (40)$$

To make equation (40) integrable we assume that

$$E^2 = \pi k(e^\lambda - 1)(1 - R^2) - 4\pi\alpha q^2 r^2, \quad (41)$$

where k is a constant and $R = r/a$, a being the radius of the sphere.

Thus, the solution set is given by

$$e^{-\lambda} = 1 - AR^2(5 - 3R^2), \quad (42)$$

$$e^\nu = (1 - 2A)^{5/4} e^{\lambda/4} \exp[5B \tan^{-1} B(6R^2 - 5) - \frac{1}{2} \tan^{-1} B], \quad (43)$$

$$p_r = \frac{1}{8}k(e^\lambda - 1)(1 - R^2) - \frac{1}{2}\alpha q^2 r^2 + \frac{\Lambda(r)}{8\pi}, \quad (44)$$

$$p_\perp = \frac{1}{8}k(e^\lambda - 1)(1 - R^2) + \frac{1}{2}\alpha q^2 r^2 + \frac{\Lambda(r)}{8\pi}, \quad (45)$$

$$\rho = k(1 - R^2)[1 - \frac{1}{8}(e^\lambda - 1)] + \frac{1}{2}\alpha q^2 r^2 - \frac{\Lambda(r)}{8\pi}, \quad (46)$$

where the constant $A = 8\pi k a^2 / 15$.

By application of the matching condition at the boundary we again get the total effective gravitational mass, which in the present case takes the form

$$m = \frac{8}{15}\pi k a^3 + \frac{q^2}{2a}. \quad (47)$$

In view of the equation (41), for vanishing charge the constant k vanishes and hence makes the gravitational mass in the equation (47) to vanish. Thus, the present case $\rho + p_r = k(1 - R^2) \neq 0$ also represents electromagnetic mass model.

Now, the constant k can be expressed in terms of the known values of the electric mass, radius and charge as

$$k = \frac{15}{16\pi a^4}(2am - q^2). \quad (48)$$

At $r = 0$, the energy density, from the equation (46), is then given by

$$\rho_0 = -5.68 \times 10^{-5} - \frac{\Lambda_0}{8\pi}. \quad (49)$$

As, in the case of electron, k is a negative quantity so for $\Lambda_0 > 0$ the central energy density ρ_0 is negative only. However, for $\Lambda_0 < 0$ the central energy density may respectively be negative and positive depending on the numerical value of k whether it is higher and lower than that of Λ_0 .

At $r = a$, the total energy density is given by

$$\rho_a = -4.54 \times 10^{-5} - \frac{\Lambda_a}{8\pi}. \quad (50)$$

Similarly, for $\Lambda_a > 0$, the energy density is negative whereas for $\Lambda_a < 0$, it may either be negative or positive depending on the numerical value of Λ_a as discussed in the previous case.

3.3 A test model

In the previous two cases we have qualitatively discussed the effect of cosmological parameter $\Lambda(r)$ on the energy density $\rho(r)$ of the electron. Let us now explore some quantitative effect and hence treat the equation (24) in a different way. If we substitute the value of \tilde{p}_r , from equation (14), then integrating equation (24) we get

$$\Lambda_{eff} = \Lambda(a) - \Lambda(r) = 8\pi\rho(r) - 8\pi(\alpha q^2 + \frac{2}{3}\pi\sigma_0^2)(a^2 - r^2), \quad (51)$$

where Λ_{eff} is the effective cosmological parameter.

We study the following cases:

For the central value of the energy density of the spherical distribution, i.e. at $r = 0$, the effective cosmological parameter becomes

$$\Lambda_{eff}^0 = \Lambda(a) - \Lambda(0) = 8\pi\rho_0 - 8\pi(\alpha q^2 + \frac{2}{3}\pi\sigma_0^2)a^2. \quad (52)$$

Considering that $\rho_0 \geq \rho_{average}$ the effective cosmological parameter, at $r = 0$, for the proper numerical values of the charge and radius of the electron can be found out as

$$\Lambda_{eff}^0 = 1.71 \times 10^{-3} cm^{-2}. \quad (53)$$

On the other hand, at the boundary, $r = a$, of the spherical distribution the effective cosmological parameter becomes

$$\Lambda_{eff}^a = \Lambda(a) - \Lambda(a) = 0. \quad (54)$$

Thus, from the equations (53) and (54) it is shown that the effective cosmological parameter has a finite value at the centre of the electron which decreases radially and becomes zero at the boundary.

4 Discussions

We see from the above analysis that the cosmological parameter Λ has a definite role even on the energy density of micro-particle, like electron. We, therefore, feel that it may also be possible to extrapolate the present investigation to the massive astrophysical bodies to see the effect of spatially varying cosmological parameter on their energy densities and vice versa.

The proper pressure p_r , in general, being positive as evident from the equation (26) is in accordance with the condition (21) which may be explained as due to vacuum polarization [26]. In this connection it is mentioned by Bonnor and Cooperstock [11] that the negativity of the active gravitational mass and hence negative energy density for electron of radius $a \sim 10^{-16}$ is consistent with the Reissner-Nordström repulsion. We would also like to mention here that the equation of state in the form $p + \rho = 0$ is discussed by Gliner [27] in his study of the algebraic properties of the energy-momentum tensor of ordinary matter through the metric tensors and called it the ρ -vacuum state of matter. It is also to be noted that the gravitational effect of the zero-point energies of particles and electromagnetic fields are real and measurable, as in the Casimir Effect [28]. According to Peebles and Ratra [29], like all energy, this zero-point energy has to contribute to the source term in Einstein's gravitational field equation. This, therefore, demands inclusion of vacuum energy related term cosmological constant in the field equation. In this regard it is interesting to recall the comment made by Einstein [2] where he stated that "... of the energy constituting matter three-quarters is to be ascribed to the electromagnetic field, and one-quarter to the gravitational field" and did "disregard" the cosmological constant in his field equation is in contradiction to the present result as shown in the equation (31) and (47).

Acknowledgments

One of the authors (SR) is thankful to the authority of IUCAA, Pune, India for providing Associateship programme under which a part of this work was carried out. Authors thanks are also due to Dr. B. Bhattacharya, Dr. K. K. Nandi, Dr. R. G. Vishwakarma and the referee for their valuable comments which made it possible to improve the article. Special thanks are due to the CTS group, IIT Kharagpur for constant inspiration throughout the work.

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