

STATIONARY STATES IN A QUANTUM GRAVITY MODEL

T. PADMANABHAN and J.V. NARLIKAR

Tata Institute of Fundamental Research, Bombay 400 005, India

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A model theory for quantized gravity is discussed where only selected degrees of freedom are quantized. The concept of stationary states is introduced. It is shown that the Planck length arises as a lower bound to the space-time length scale in a natural way.

1. Various approaches are being tried to quantize gravity, none with much success (see ref. [1] for a review). Considering the complexities of the problem certain simplified models may be of value. One such approach is to quantize a few selected degrees of freedom (freezing the rest) using Feynman path integrals. This method was used successfully in the past with the conformal degree of freedom (see refs. [2–7]) and leads to the result that (at least for a wide class of metrics [7]) near a classical singularity, transitions to nonsingular space-times can occur.

In order to study the features of the model more quantitatively, the concept of an average metric can be introduced [5,6]. In a way this incorporates the effects of conformal fluctuations. It turns out that, if one starts with the Friedmann metric of the form

$$ds^2 = -Q^2(t) dt^2 + Q^2(t) [dr^2/(1 - kr^2) + r^2(d\theta^2 + \sin^2\theta d\phi^2)], \quad (1)$$

then the effect of conformal fluctuations is to modify $Q^2(t)$ to the form

$$Q_{\text{eff}}^2(t) = Q^2(t) \times \left\{ 1 + \left(\frac{t_1}{t}\right)^4 \left[\frac{A}{\Delta^2} \left(1 - \frac{t}{t_1}\right)^2 + \Delta^2 \left(1 - \frac{2t}{t_1}\right)^2 \right] \right\}, \quad (2)$$

where t_1 , A and Δ are constants and

$$Q(t) \sim t^2, \quad \text{as } t \rightarrow 0. \quad (3)$$

Since $Q(t)$ goes as t^2 near the singularity, this gives a finite value for $Q_{\text{eff}}^2(0)$. The geometry is nonsingular.

The evolution predicts that the collapse proceeds only up to the Planck radius [6].

The purpose of the present note is to introduce the concept of stationary states in this model so as to understand this process better. An analogy with the hydrogen atom may be helpful. Classically, if a proton and electron interact via the Coulomb potential, they collapse towards each other. In quantum mechanics, however, the uncertainty principle prevents this collapse. What is more, there arises a stable stationary state as the ground state of the system [8]. In the following, we shall show that similar results hold in our model.

2. In order to do this, consider the Friedmann metric, written in the form

$$ds^2 = -dt^2 + a^2(t) [d\chi^2 + \sin^2\chi (d\theta^2 + \sin^2\theta d\phi^2)], \quad (4)$$

and treat $a(t)$ as a quantum variable. In the standard path-integral formalism we ask the question “what is the probability amplitude for the variable a to have a value a_1 at time t_1 and a_2 at time t_2 ?” The answer is given by the path integral

$$k(a_2, t_2; a_1, t_1) = \int \mathcal{D}a(t) \exp \frac{i}{\hbar} S[a(t)], \quad (5)$$

where it can be shown that (using the methods of refs. [2] or [4] to omit second derivatives)

$$S = \frac{3}{4G} \int_{t_1}^{t_2} dt (1 - \dot{a}^2)a . \tag{6}$$

The evaluation of (5) will lead to the information about the evolution of the system as reported in refs. [5,6]. Here, since we want the stationary state, we have to write down the ‘‘Schrödinger equation’’ corresponding to the kernel in eq. (5). This can be done by treating $a(t)$ as the quantum variable in the action given by eq. (6). But this action contains a $a\dot{a}^2$ term which will create ambiguity of factor ordering. (As is well known there is no unique quantum theory for such a classical system.) We resolve this ambiguity by a change of variables. Introduce a variable q by the equation (λ is a constant introduced for dimensional reasons, and will be set equal to unity)

$$a = \lambda q^{2/3} . \tag{7}$$

The action reduces to the form

$$S = - \int_{t_1}^{t_2} [\frac{1}{2}m\dot{q}^2 - v(q)] dt , \tag{8}$$

where

$$m = 2\lambda^3/3G , \quad v(q) = (3\lambda/4G)q^{2/3} . \tag{9}$$

Now the ‘‘Schrödinger equation’’ can be written as

$$-(\hbar^2/2m) d^2\psi/dq^2 + V(q)\psi = E\psi . \tag{10}$$

One can write this as

$$d^2\psi/dq^2 + (\epsilon - \alpha^2 q^{2/3})\psi = 0 , \tag{11}$$

$$\epsilon = (2m/\hbar^2)E , \quad \alpha = \lambda^2/G\hbar .$$

One can derive the same equation from the standard effective action method. See ref. [9] where a very similar case is discussed. The boundary conditions are $\psi(q = +\infty) = 0$. We are interested in the ground-state properties of this system, which can be found by the following analysis.

Since eq. (11) is symmetric under $q \rightarrow -q$, the uniqueness of the eigensolutions tells us that they are either odd or even. The ground state $\psi_0(q)$ will never be zero (not counting $q = \pm\infty$, of course) and will have $d\psi_0/dq = 0$ at $q = 0$ by symmetry. As shown in fig. 1 $\psi_0(q)$ has a point of inflexion at q_0 where

$$\epsilon_0 = \alpha^2 q_0^{2/3} . \tag{12}$$

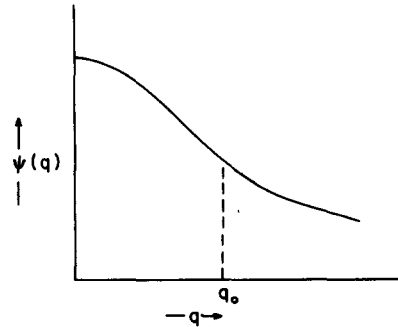


Fig. 1.

For $q < q_0$ the curve has negative curvature while for $q > q_0$ it has positive curvature. We can rewrite the equation:

$$d^2\psi_0/dq^2 + \epsilon_0 [1 - (q/q_0)^{2/3}] \psi_0 = 0 . \tag{13}$$

We now compare this equation with the wave equation of a harmonic oscillator written in the form

$$d^2\phi/dq^2 + \epsilon_0 [1 - (q/q_0)^2] \phi = 0 . \tag{14}$$

In fig. 2 we plot $-d^2\psi_0/dq^2$ and $-d^2\phi/dq^2$ against q . The curve for ψ_0 (dotted) lies below the curve for ϕ , for $q < q_0$. So if both are normalized in fig. 3 to have the same value at $q = 0$, $\phi(q)$ will be below $\psi_0(q)$ as far as $q < q_0$. Beyond this point $\phi(q)$ has a stronger exponential curvature than $\psi_0(q)$ so that it will drop to zero faster than $\psi_0(q)$. However, $\psi_0(q)$ has been adjusted to be zero only at $q = \infty$, so $\phi(q)$ will hit the q -axis at a finite point.

This means that $\phi(q)$ cannot describe the lowest-energy state of a harmonic oscillator. In other words,

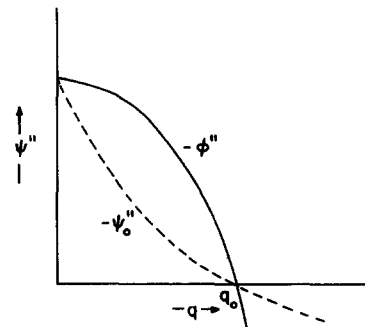


Fig. 2.

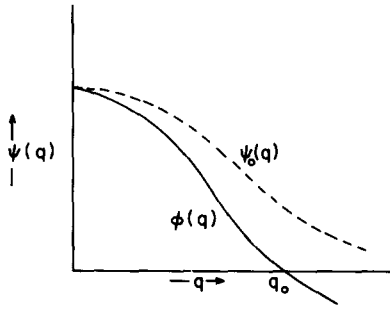


Fig. 3.

ϵ_0 exceeds the lowest-energy eigenvalue of the harmonic oscillator with potential function

$$\tilde{V}(q) = (\epsilon_0/q_0^2)q^2. \tag{15}$$

If the zero-level solution is $\phi_0(q) = e^{-\mu q^2}$, then direct comparison gives

$$\tilde{\epsilon}_0 = 2\mu, \quad \epsilon_0/q_0^2 = 4\mu^2. \tag{16}$$

Hence the condition $\epsilon_0 > \tilde{\epsilon}_0$ gives

$$\epsilon_0 > q_0^{-2}. \tag{17}$$

This leads to the result for the lowest-energy level

$$E_0 > \frac{3}{4}\sqrt{\hbar/G}. \tag{18}$$

Again a comparison of $\psi_0(q)$ with $\phi_0(q)$ tells us that for large q , the latter is more strongly damped than the former. Thus,

$$\overline{q^2} \equiv \langle q^2 \rangle_{\psi_0(q)} > \langle q^2 \rangle_{\phi_0(q)} = \frac{1}{2}\mu^{-1}. \tag{19}$$

So, $\overline{q^2} > (G\hbar)^{3/2}/\lambda^3$. We define the average value of a^2 by the equation

$$\langle a^2 \rangle \equiv \lambda^2(\overline{q^2})^{2/3}.$$

This means $\overline{a^2} > G\hbar/c^3$ (putting in the c factor), thereby establishing a lower bound on the mean square value of the scale factor for the metric. The average value of the scale factor in the lowest stationary state is of the order of the Planck length.

3. Two points are worth noting about this result. First, this result helps us to understand the behaviour of conformal fluctuations in the Friedmann universe from a different point of view. Since the lowest stationary state has the scale factor of the order of the Planck length, it is understandable that the collapsing system settles down to that state (cf. analogy with the hydrogen atom presented before). Secondly, one should notice that distances below the Planck length have no operational significance. Any procedure to probe into such small distances would require very large energy densities; contribution of this energy density to curvature cannot be neglected. In this way also the Planck length comes in as a lower limit to the length scale.

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