On the avoidance of singularities in C-field cosmology

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It is well known that a spherically symmetric imploding cold body collapses into a space-time singularity in general relativity. The singularity does not arise, however, in the modification of the theory proposed in C-field cosmology. Although the C-field has been used to represent creation of matter, the prevention of singularities does not depend on the creation property of the field, but on its negative energy density. It does not seem that singularities can be prevented except by a negative energy field. Internal pressures of the ordinary kind fail to provide support against gravitation provided the mass of the body is sufficiently large.

1. COLLAPSE INTO A SINGULARITY

As was pointed out by Tolman (1934), intrinsic spherical polars give the most convenient choice of co-ordinates for studying the implosion of a cold, spherically symmetric object. This simply means that the r, θ, ϕ co-ordinates are chosen such that

$$r = r_0, \quad \theta = \theta_0, \quad \phi = \phi_0; \quad r_0, \theta_0, \phi_0 \quad \text{constants} \tag{1}$$

is a geodesic. The space co-ordinates of any element of the body stay fixed, and the energy momentum tensor T^i_k is diag $(0, 0, 0, \rho)$, where ρ is the proper density. The set of geodesics obtained from $0 \leqslant r_0 < \infty$, $0 \leqslant \theta_0 \leqslant \pi$, $0 \leqslant \phi_0 < 2\pi$ form a congruence, and a time co-ordinate t can be taken as proper time along the congruence, as is usual in the choice of geodesic co-ordinates in Riemannian geometry. The line element then takes the form

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{e}^{\lambda}\,\mathrm{d}r^2 - \mathrm{e}^{\mu}\,\mathrm{d}\Omega^2, \quad \mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2, \tag{2}$$

where λ, μ are functions of r, t.

Co-ordinates can similarly be chosen when a pressure p is present, but elements of the body do not follow geodesics unless p is uniform, so that in general T^i_k does *not* take the simple form diag$(-p, -p, -p, \epsilon)$. However, the r co-ordinate can be defined within the body such that it remains constant for each material element. This destroys the geodesic property inside the body, and the line element takes the more complex form

$$\mathrm{d}s^2 = \mathrm{e}^{\nu}\,\mathrm{d}t^2 - \mathrm{e}^{\lambda}\,\mathrm{d}r^2 - \mathrm{e}^{\mu}\,\mathrm{d}\Omega^2, \tag{3}$$

with ν, λ, μ all functions of r, t, this being the most general form of the spherically symmetric line element. The energy momentum tensor is now diag $(-p, -p, -p, \epsilon)$, ϵ being the energy density.

The above statements can be proved by starting with (3) and by considering $T^i_{k;i} = 0$. The $k = 1$ and $k = 4$ components give

$$\nu' = -\frac{2p'}{p+\epsilon}, \quad \dot{\lambda} + 2\dot{\mu} = -\frac{2\dot{\epsilon}}{p+\epsilon}, \tag{4}$$

where primes indicate partial differentiation with respect to r, and dots differentiation with respect to t. When $p' = 0$, ν is independent of r, so that a new time coordinate τ, determined by

$$d\tau/dt = e^{\frac{1}{2}\nu},$$

reduces the g_{44} function to unity, and the form (2) is reached (but with τ as the proper time).

In the case $p = 0$, the Einstein field equations give

$$\ddot{\mu}+\tfrac{3}{4}\dot{\mu}^2+e^{-\mu}-\tfrac{1}{4}e^{-\lambda}\mu'^2 = 0, \tag{5}$$

$$2\dot{\mu}'+\dot{\mu}\mu'-\dot{\lambda}\mu' = 0, \tag{6}$$

$$\tfrac{1}{2}(\dot{\mu}\dot{\lambda}+\tfrac{1}{2}\dot{\mu}^2)+e^{-\mu}-e^{-\lambda}(\mu''+\tfrac{3}{4}\mu'^2-\tfrac{1}{2}\mu'\lambda') = 8\pi G\rho. \tag{7}$$

Equation (6) integrates immediately to give

$$e^{\lambda} = e^{\mu}\mu'^2/4(1+f), \quad f \text{ a function of } r. \tag{8}$$

To make further progress with the integration of (5), (7), (8) it is necessary to specify initial conditions. The solution is well known for the case of a body imploding from rest, and with initially uniform density ρ_0, having been considered by Datt (1938) and by Oppenheimer & Snyder (1939). In this case, e^{μ} is separable inside the body, and we can write

$$e^{\frac{1}{2}\mu} = rS(t), \quad S(0) = 1. \tag{9}$$

Also $f = -\alpha r^2$ inside the body, where $\alpha = \frac{8}{3}\pi G\rho_0$. Thus

$$e^{\lambda} = S^2/(1-\alpha r^2). \tag{10}$$

Equation (5) gives a second-order differential equation for $S(t)$, which is easily integrated to give

$$\dot{S}^2 = \alpha(1-S)/S, \tag{11}$$

where the constant of integration has been adjusted to give $\dot{S} = 0$ at $S = 1$, the implosion being taken to start at $t = 0$. The density remains uniform and is given by

$$\rho = \rho_0/S^3. \tag{12}$$

This solution for the interior is similar to the simple elliptic model of cosmology. Writing $k = +1$ in the Robertson–Walker line element for a homogeneous isotropic universe we have

$$ds^2 = dt^2-S^2(t)\left[\frac{dr^2}{1-r^2}+r^2\,d\Omega^2\right]. \tag{13}$$

The field equations give

$$\dot{S}^2 = -1+\tfrac{8}{3}\pi G\rho S^2, \quad \rho S^3 = \text{constant}. \tag{14}$$

Writing ρ_0 for the density at the maximum, and altering S by a scale factor to give $S = 1$ at the maximum, immediately converts (14) to the form of (11). Indeed, the interior solution for the case of a finite body behaves exactly as the matter in the vicinity of the observer in the cosmological case.

Equation (8) holds in the exterior, but with

$$f = -\alpha r_b^3/r = -2GM/r, \quad M = \tfrac{4}{3}\pi\rho_0 r_b^3, \tag{15}$$

where r_b is the radial co-ordinate at the boundary of the object. In the exterior e^{μ} is not separable, but can be written as

$$e^{\frac{1}{2}\mu} = rS(tr_b^{\frac{3}{2}}/r^{\frac{3}{2}}). \tag{16}$$

The function S is the same as before, in that it is the same function of its argument, i.e.

$$S(t_1) = S\left(t_2 \frac{r_b^{\frac{3}{2}}}{r^{\frac{3}{2}}}\right), \quad \text{if} \quad \left(\frac{r}{r_b}\right)^{\frac{3}{2}} = \frac{t_2}{t_1}.$$

A simple integration of (11) shows that the interior collapses into a singularity $S \to 0$, $\dot{S} \to \infty$ in a proper time $\pi/2\sqrt{\alpha}$. When S decreases to a value given by

$$S = 2GM/r_b = \alpha r_b^2, \tag{17}$$

an event horizon develops. Integration of (11) shows that this occurs after a proper time

$$\alpha^{-\frac{1}{2}}[\tfrac{1}{2}\pi - \sin^{-1}(\alpha r_b^2)^{\frac{1}{2}} + \alpha^{\frac{1}{2}} r_b (1-\alpha r_b^2)^{\frac{1}{2}}].$$

Signals directed outwards cannot cross the event horizon if they are emitted later than this, although some communication within the event horizon is still possible. However, all communication within the object (in the sense of communication from a particle with radial co-ordinate r_1 to a particle with co-ordinate r_2) ceases at $t = \pi/2\sqrt{\alpha}$.

The development of an event horizon emphasizes the advantage of intrinsic co-ordinates. The more usual Schwarzschild co-ordinates, with the line element represented in the form

$$ds^2 - e^{N}\,dT^2 - e^{\Lambda}\,dR^2 - R^2\,d\Omega^2, \tag{18}$$

fails within the event horizon. However, the form (18) has the advantage that the functions N, Λ are known explicitly for large R, viz.

$$e^{N} = e^{-\Lambda} = 1 - \text{constant}/R. \tag{19}$$

The constant in (19) can be fixed from the initial conditions. Thus $R = r_b$ at the surface initially, and

$$e^{N} = e^{-\Lambda} = 1 - 2GM/R, \quad M = \tfrac{4}{3}\pi\rho_0 r_b^3. \tag{20}$$

Provided a point with co-ordinate R lies both outside the body and outside the event horizon the line element is given by (18), (20). This is the case even when $p \neq 0$ within the body provided radiation does not escape outwards from the surface.

The interior solution can also be expressed in Schwarzschild co-ordinates so long as $S > \alpha r_b^2$, i.e. so long as the event horizon has not developed. Using (9), (10) the line element (2) can be written

$$ds^2 = dt^2 - S^2(t)\,[(1-\alpha r^2)^{-1}\,dr^2 + r^2\,d\Omega^2]. \tag{21}$$

The following transformation converts r, t to new co-ordinates R, T in terms of which the line element has the form (18)

$$R = rS(t), \quad T = \Phi\left[\int_a^r \frac{x\,dx}{1-\alpha x^2} + \int_b^t \frac{dx}{S(x)\,dS/dx}\right], \tag{22}$$

in which the function Φ is arbitrary, and a, b are arbitrary constants. The resulting forms are

$$e^{\Lambda} = \frac{1}{1-\alpha r^2 - r^2\dot{S}^2}, \quad e^{N} = \frac{S^2\dot{S}^2(1-\alpha r^2)}{\Phi_1^2(1-\alpha r^2 - r^2\dot{S}^2)}, \tag{23}$$

where Φ_1 denotes the derivative of Φ with respect to its argument.

Inserting for $\dot{S}^2$ in e^{Λ},

$$e^{\Lambda} = 1/(1-\alpha r^2/S), \tag{24}$$

in which the denominator is zero when $S = \alpha r^2$. Evidently this situation occurs first at $r = r_b$, i.e. for $S = \alpha r_b^2$, which is the event horizon. The above transformation can be applied formally even for $S < \alpha r^2$. The effect is to switch the time-like and space-like properties of the T, R co-ordinates.

It remains to match the internal and external forms of (18) at the surface $r = r_b$. Immediately we have

$$e^{\Lambda}_{\text{boundary}} = \frac{1}{1-\alpha r_b^2/S} = \frac{1}{1-2GM/r_b S} = \frac{1}{1-2GM/R_b}, \tag{25}$$

$$R_b = r_b S, \quad GM = \tfrac{4}{3}\pi G r_b^3 \rho_0 = \tfrac{1}{2}\alpha r_b^3.$$

The internal and external expressions for e^{Λ} are therefore matched at $R = R_b$, irrespective of the choice of Φ. This is not the case, however, for e^{N}. To match e^{N} a particular choice must be made for Φ.

First, we note that

$$\int^r \frac{x\,dx}{1-\alpha x^2} + \int^t \frac{dx}{S\,dS/dx} = \int^r \frac{x\,dx}{1-\alpha x^2} + \int^S \frac{dS}{S(dS/dx)^2}$$
$$= -\frac{1}{2\alpha}\ln[(1-\alpha r^2)(1-S)^2] + \text{const.}, \tag{26}$$

where (11) has been used. Hence for a fixed value of r, e.g. $r = r_b$ at the boundary, Φ is a function of S, or of R. Thus any choice of Φ gives a function of S, and hence of R, at the boundary. And Φ_1 is similarly a function of R at the boundary, $G(R)$ say. We therefore write

$$e^{N}_{\text{boundary}} = \left[\frac{\alpha R(r_b - R)(1-\alpha r_b^2)}{r_b^2 G^2(R)(1-\alpha r_b^3/R)}\right]_{R=R_b}. \tag{27}$$

We require (27) to be equal to $1-2GM/R_b$, and this will be the case if

$$G^2(R) = \frac{\alpha R(r_b - R)(1-\alpha r_b^2)}{r_b^2(1-2GM/R)^2}. \tag{28}$$

Equation (28) is the condition on Φ, necessary for the internal solution to fit on to the usual Schwarzschild exterior solution. The usual exterior solution is not of course unique, since a time transformation $\tau = \tau(T)$ can be applied without changing the form of (18).

2. The effect of internal pressure

The proper time $\pi/2\sqrt{\alpha}$ is just the time of collapse of a cold object of uniform density as calculated in Newtonian dynamics for the case $p = 0$. In typical examples $\pi/2\sqrt{\alpha}$ is much less than the cosmological time scale. For the massive objects discussed by Fowler & Hoyle (1963 *a*) the density in the hydrogen-burning phase is

$$\sim 10^2(M_\odot/M)^{\frac{1}{2}}\,\text{g cm}^{-3}. \tag{29}$$

For $M/M_{\odot} = 10^6$, (29) gives $\sim 10^{-1}\,\mathrm{g\,cm^{-3}}$, and the use of this value for ρ_0 in $\alpha = \frac{8}{3}\pi G\rho_0$, gives a collapse time of only 10^4 s. Such explicit cases would seem to force us to ask in a practical way, rather than abstractly, whether space-time singularities really do develop.

At first sight one might think that a sufficiently large positive pressure might halt the implosion. First, we note that the pressure enters the equation for $\ddot{\mu}$, but it enters with the wrong sign—it makes $\ddot{\mu}$ still more negative. Next, we can ask what the structure of the equation for $\dot{S}^2$ would be if pressure were to halt the collapse. A form such as

$$\dot{S}^2 = \alpha(1-S)/S - O(S^{-\sigma}) \quad (\sigma > 1), \tag{30}$$

with the new term becoming dominant as $S \to 0$ seems necessary. This gives a second root for $\dot{S}$ and the object would oscillate between the two roots. But then it should be possible to adjust the pressure to give coincident roots, in which case a static solution would exist. The question would then seem to devolve on whether static solutions with internal pressure can be found. Two possibilities require discussion:

(1) Pressure in cold material due either to a hard-core nucleon potential or to a packing of energy levels according to the exclusion principle.

(2) Pressure in hot material.

For the first case it is sufficient to consider the well-known Schwarzschild interior solution. In this solution the energy density ϵ is taken uniform throughout the body. The radius R_b has a minimum value $(8/9\alpha')^{\frac{1}{2}}$, $\alpha' = \frac{8}{3}\pi G\epsilon$. The pressure at the centre becomes infinite as R_b decreases to this value.

The nucleonic mass M_n of the body is defined as the sum of the rest masses of the constituent particles taken one at a time; i.e.

$$M_n = 4\pi \int_0^{R_b} \rho R^2 \mathrm{e}^{\frac{1}{2}\Lambda}\,\mathrm{d}R, \quad \rho \text{ being the rest mass density.} \tag{31}$$

This is less than

$$4\pi \int_0^{R_b} \epsilon R^2 \mathrm{e}^{\frac{1}{2}\Lambda}\,\mathrm{d}R, \tag{32}$$

since $\rho \leqslant \epsilon$. For ϵ uniform, (32) can be evaluated and is

$$\frac{2\pi\epsilon}{\alpha'}\left[\frac{1}{\sqrt{\alpha'}}\sin^{-1} R_b\sqrt{\alpha'} - R_b\sqrt{(1-\alpha' R_b^2)}\right]. \tag{33}$$

The maximum value of (33) is obtained with R_b at its minimum, $R_b = \sqrt{(8/9\alpha')}$, namely

$$\frac{2\pi\epsilon}{(\alpha')^{\frac{3}{2}}}\left[\sin^{-1}\sqrt{\frac{8}{9}} - \frac{\sqrt{8}}{9}\right] > M_n. \tag{34}$$

Finally, since (34) is proportional to $\epsilon^{-\frac{1}{2}}$ and $\rho < \epsilon$, we have

$$M_n < \frac{2\pi\rho}{\alpha^{\frac{3}{2}}}\left[\sin^{-1}\sqrt{\frac{8}{9}} - \frac{\sqrt{8}}{9}\right], \quad \alpha = \frac{8\pi G\rho}{3}. \tag{35}$$

Expressed numerically (35) becomes

$$M_n/M_{\odot} < \sim 10^8 \rho^{-\frac{1}{2}}, \quad \rho \text{ in } \mathrm{g\,cm^{-3}}. \tag{36}$$

The Schwarzschild interior solution has sometimes been criticized on the grounds that uniform ϵ is not consistent with $p \to \infty$ at the centre of the object, i.e. that it implies a physically unrealistic equation of state. However, this is just the situation contemplated for a hard-core nucleon potential. The hard core is supposed to develop at a nucleon separation of $\sim 5 \times 10^{-14}$ cm, implying a nuclear fluid density in the present problem of $\sim 10^{15}\,\mathrm{g\,cm^{-3}}$. Inserting this value for ρ in (36), we see that the nucleonic mass of the object must be less than $\sim 3M_{\odot}$ for a static state to exist. Even a hard-core potential will not prevent collapse to a singularity when the nucleonic mass is large compared with $M_{\odot}$.

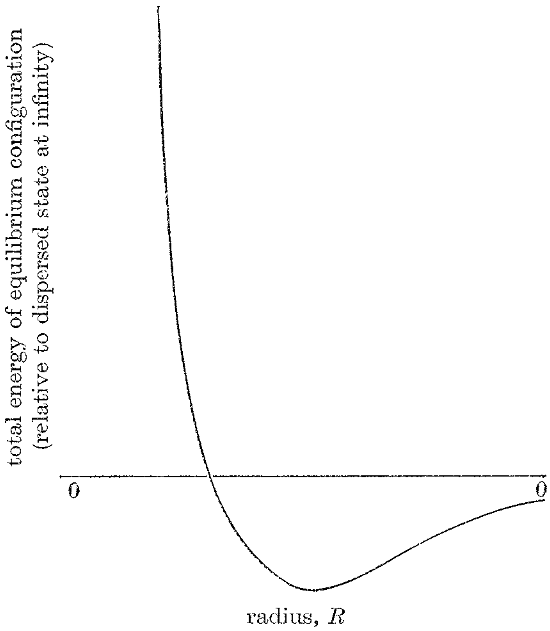

FIGURE 1. The zero level is the energy of the dispersed state at infinity, the nucleonic mass M_n.

Static solutions analogous to the Schwarzschild case can be obtained for any specified equation of state. An equation of state based on the best data then available (including the effect of exclusion packing but not of a hard core potential) was considered by Oppenheimer & Volkoff (1939). In their case the maximum value of M_n for a static model was only $\sim 0{\cdot}7M_{\odot}$. We do not believe that any equation of state can be found for cold material that will change the maximum permitted mass from a value of solar order.

Turning now to a thermal pressure, equilibrium at a sufficiently low density is always possible. However, such an equilibrium state cannot be permanent, since radiation will escape from the system. This leads to shrinkage and to an increase of ρ. As contraction proceeds the effective gravitational mass, or *total* energy as we might call it, defined by

$$M(R_b) = 4\pi \int_0^{R_b} \epsilon R^2 \,\mathrm{d}R \tag{37}$$

behaves in the way shown schematically in figure 1. For large R_b the total energy is less than the nucleonic mass M_n, but as R_b decreases $M(R_b)$ rises above M_n and eventually approaches a vertical asymptote. The fact that $M(R_0)$ rises above M_n was pointed out by Fowler & Hoyle (1963*b*). A more thoroughgoing account of this important property has recently been given by Iben (1963).

Granted that $M(R_b)$ rises above M_n, it is easy to see that there can be no equilibrium models to the left of a certain ordinate. The equation of equilibrium is

$$\frac{\mathrm{d}p}{\mathrm{d}R} = -\frac{4\pi GR(\epsilon+p)}{1-2GM(R)/R}\left[p+\frac{M(R)}{4\pi R^3}\right], \tag{38}$$

where

$$M(R) = 4\pi\int_0^R \epsilon R^2\,\mathrm{d}R.$$

With $M(R_b) > M_n$, a fixed number, it is clear that a singularity must develop in the pressure gradient for sufficiently small R_b. The internal pressure required for equilibrium then diverges, exactly as in the Schwarzschild interior solution. In the present case, however, a divergence of p implies a divergence of ϵ and therefore of $M(R_b)$.†

Now $M(R_b)$ cannot exceed M_n unless nuclear reactions supply the necessary energy, and even nuclear reactions must fail to do this as the vertical asymptote is approached. It follows that thermal energy must eventually fail to support an equilibrium state. We therefore conclude that when the mass appreciably exceeds $M_\odot$ an object eventually implodes to a space-time singularity. Implosion cannot be prevented indefinitely by internal heat, nor can a hard-core nucleon potential prevent implosion. Gravitation is so strong that a sufficiently large aggregate of matter crushes itself to a singularity against all sources of pressure.

One might argue that an imploding object is most unlikely to possess spherical symmetry (rotation, magnetic fields, etc.). While this is true it seems unlikely to us that departures from symmetry can prevent the development of singularities, at any rate for sufficiently large masses, since it should always be possible to choose subregions that possess a greater symmetry than the main object. The above considerations can then be taken as applying to such subregions.

3. The avoidance of singularities in C-field cosmology

The above considerations all apply to the usual form of the relativity theory. The C-field cosmology differs from normal relativity in that a new field contribution is added to the energy momentum tensor,

$$R^{ik}-\tfrac{1}{2}g^{ik}R = -8\pi G[T^{ik}_n - f(C^iC^k - \tfrac{1}{2}g^{ik}C^lC_l)], \quad C_i = \partial C/\partial x^i, \tag{39}$$

where T^{ik}_n is the normal tensor, and f is a new coupling constant. The C-field satisfies the source equation

$$fC^i{}_{;i} = j^i{}_{;i}, \quad j^i = \rho(\mathrm{d}x^i/\mathrm{d}s), \tag{40}$$

when matter is represented by a smooth fluid of proper density ρ. The properties of these equations in the case of a homogeneous isotropic line element have been studied

† In order to obtain models in which $M(R_b)$ actually diverges to infinity it is necessary to admit non-zero pressure at $R = R_b$. If zero pressure at $R = R_b$ is demanded the curve of figure 1 simply stops. No models of smaller radius can be found.

by Hoyle & Narlikar (1963). It was shown that if $C^i_{;i} \neq 0$ the only solution of the above equations is:

$$\left.\begin{aligned} &S^3 = A(1+\cosh t), \quad A \text{ a constant,}\\ &t \text{ proper time in units of } (12\pi G f)^{-\frac{1}{2}},\\ &C = t,\\ &\rho = f\cosh t/(1+\cosh t), \end{aligned}\right\} \tag{41}$$

where S is the expansion factor. The universe approaches a steady state for either $t \gg 1$ or $-t \gg 1$. The physical requirement that the density ρ be positive demands f positive.

Equations (39), (40) also have solutions in the case $C^i_{;i} = 0$. Evidently $C = 0$, the normal theory, is such a solution. In the cosmological case such solutions are not of much interest, since the C-field has no source.

The question evidently arises as to whether the arguments of preceding sections require modification if a non-zero C-field exists, as it certainly does if we accept the cosmology set out in (41).

We can hardly suppose that the source term in the wave equation for C is of importance in a local problem. That is to say, creation or annihilation of matter is not of importance at the large densities that arise in local implosions. Locally, it is sufficient to write

$$C^i_{;i} = 0, \tag{42}$$

and to take $C = t$ as the behaviour of C at infinity. The problem is analogous to the cosmologically uninteresting case mentioned above; the field satisfies the homogeneous wave equation and is non-zero at infinity. Since the cosmological case can be solved without difficulty, this case gives a valuable hint as to what may happen in the local problem.

Again consider the line element (13), obtained by writing $k = +1$ in the Robertson–Walker form. The equation for $\dot{S}^2$ is modified to

$$\dot{S}^2 = -1 + \tfrac{8}{3}\pi G S^2(\rho - \tfrac{1}{2}f\dot{C}^2) \tag{43}$$

owing to the presence of the C-field in the energy momentum tensor. With C a function of t only (the homogeneous isotropic case) (42) becomes

$$\ddot{C} + 3(\dot{S}/S)\,\dot{C} = 0 \tag{44}$$

which integrates to give

$$\dot{C} = A/S^3, \quad A \text{ being a constant.} \tag{45}$$

Substitution in (43) then gives

$$\dot{S}^2 = -1 + \tfrac{8}{3}\pi G S^2(\rho - \tfrac{1}{2}fA^2/S^6). \tag{46}$$

Remembering that ρ is proportional to S^{-3}, we see that the $\dot{C}$ term dominates as $S \to 0$. Moreover, the new term enters with the correct sign to provide a second root for $\dot{S}$. In fact, if there is any range of S over which $\dot{S}^2$ is positive (as there must be for a physically meaningful situation) then (46) must have two roots for S. *The introduction of a non-zero C into a closed elliptic universe causes the universe to oscillate periodically.*

This result suggests that the introduction of a non-zero C into the implosion problem might prevent collapse into a singularity, giving an oscillation between two roots of $\dot{S}$. Indeed, if we accept (45) as the solution of $C^i{}_{;i} = 0$ within an object, equations (6) and (8) remain valid. Since $T^{ik}_{n}{}_{;k} = 0$, elements of the body follow geodesics as before, and the line element can still be represented by (2). The equation for $\ddot{\mu}$ is modified, however, to the form

$$\ddot{\mu} + \tfrac{3}{4}\dot{\mu}^2 + e^{-\mu} - \tfrac{1}{4}e^{-\lambda}\mu'^2 = 4\pi G f \dot{C}^2. \tag{47}$$

Again writing $\quad e^{\frac{1}{2}\mu} = rS, \quad e^{\lambda} = e^{\mu}\mu'^2/4(1-\alpha r^2), \quad \alpha = \tfrac{8}{3}\pi G\rho_0,$

we have

$$2\frac{\ddot{S}}{S} + \frac{\dot{S}^2}{S^2} + \frac{\alpha}{S^2} = 4\pi G f \dot{C}^2. \tag{48}$$

With $\dot{C}$ given by (45), (48) can be integrated to give

$$S\dot{S}^2 = \frac{\alpha}{S} - \frac{4\pi}{3}\frac{GfA^2}{S^3} + \text{constant}. \tag{49}$$

The constant of integration can evidently be adjusted so that $\dot{S} = 0$ at $S = 1$, the beginning of the implosion.

Suppose next that the C-field is of only minor importance during the early stages. Then to sufficient accuracy

$$\dot{S}^2 = \frac{\alpha(1-S)}{S} - \frac{4\pi}{3}\frac{GfA^2}{S^4}. \tag{50}$$

As S decreases the term in S^{-4} eventually becomes important, however, and there is a second root for $\dot{S}$. The object oscillates, exactly as in the cosmological case. Implicit in this statement is the condition that the coupling constant f be positive. This is necessary, however, since the density must be positive in the cosmological solution (41).

The above argument does not constitute proof that the solution must be oscillatory because (45) was assumed. What has been shown is that the oscillatory solution satisfies all the necessary equations—(39) and (40)—for the interior. To complete the proof it is necessary to demonstrate that the oscillatory solution for the interior can be matched to an exterior solution in which the boundary conditions at large r are those given in (41). In particular, it is necessary to show that a solution of $C^i{}_{;i} = 0$ can be found that satisfies

$$C = A/S^3 \quad \text{at} \quad r = r_b; \quad C = t \quad \text{at large } r. \tag{51}$$

In considering such an exterior solution C must be taken as a function of both r, t. Equations (39), (40) are then partial, and consequently difficult to handle. We have not succeeded in solving this problem. Nevertheless, we have succeeded in dealing with the asymptotic situation as $S \to 0$.

The exterior solution is needed in the general case to justify (45), rather than some other solution of $C^i{}_{;i} = 0$. However, in the asymptotic situation we can show that (45) becomes the only solution for the interior as the implosion develops. We start the implosion from a dispersed state, so that $\alpha r^2 \ll 1$, and we suppose that the effect of the C-field in the field equations is negligible during the early stages, so that

$$e^{\frac{1}{2}\mu} = rS, \quad e^{\lambda} = S^2/(1-\alpha r^2) \simeq S^2 \tag{52}$$

as before. In its partial form the wave equation for C is

$$\frac{\partial}{\partial r}\left[e^{\mu-\frac{1}{2}\lambda}C'\right] = \frac{\partial}{\partial t}\left[e^{\mu+\frac{1}{2}\lambda}\dot{C}\right]. \tag{53}$$

Inserting (52) we have

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 C'\right] = \frac{1}{S}\frac{\partial}{\partial t}\left[S^3\dot{C}\right]. \tag{54}$$

The general solution of (54) can be represented as a sum of separable solutions of the form $C(r,t) = X(r)\,Y(t)$, because of the linearity of the equation. In terms of such a separable solution

$$\frac{1}{r^2 X}\frac{\partial}{\partial r}(r^2 X') = \frac{1}{SY}\frac{\partial}{\partial t}(S^3\dot{Y}). \tag{55}$$

Evidently

$$\frac{\partial}{\partial r}(r^2 X') = Br^2 X, \quad \frac{\partial}{\partial t}(S^3\dot{Y}) = BSY, \quad B \text{ being a constant.} \tag{56}$$

The solution for X is

$$X(r) = \sinh(\sqrt{B}\,r)/r, \tag{57}$$

there being no term in $\cosh\sqrt{B}\,r$ because of the singularity that would develop as $r \to 0$.

We now show that $\dot{Y}$ tends to the form A/S^3 as $S \to 0$ even if $B \neq 0$. This can be seen from an expansion in series. Write

$$Y = S^\sigma\left(a_0 + \sum_1^\infty a_n S^n\right). \tag{58}$$

Retaining only the lowest power of S on each side of the equation (56) for Y, we have

$$\sigma[S^{\sigma+2}\ddot{S} + (\sigma+2)\,S^{\sigma+1}\dot{S}^2] \simeq BS^{\sigma+1}. \tag{59}$$

Now either the C-field slows the implosion or it does not. If it does not, then

$$\dot{S}^2 = \alpha(1-S)/S, \quad \ddot{S} = -\tfrac{1}{2}\alpha/S^2.$$

Substituting in (59)

$$\alpha\sigma(\sigma+\tfrac{3}{2})\,S^\sigma \simeq BS^{\sigma+1}.$$

The right-hand side becomes negligible as $S \to 0$ so that either $\sigma = 0$ or $\sigma = -\frac{3}{2}$. The case $\sigma = 0$ corresponds to a time independent C-field. A static C-field is always a solution of the wave equation, but a static solution does not satisfy the time dependent cosmological condition given in (41). The static solution was omitted above through the tacit assumption of a non-zero constant of integration in (45). The case $\sigma = -\frac{3}{2}$ gives $Y \propto S^{-\frac{3}{2}}$. Since all separable solutions have this behaviour as $S \to 0$, C is also proportional to $S^{-\frac{3}{2}}$. Hence $\dot{C}$ is proportional to $S^{-\frac{5}{2}}\dot{S}$. Finally, with $\dot{S}$ proportional to $S^{-\frac{1}{2}}$ we have $\dot{C} \propto S^{-3}$, as in (45). Hence the special solution represented by (45) must develop as implosion proceeds, and hence the implosion is finally halted.

We conclude that in the presence of a C-field with the cosmological boundary condition (41) there is no collapse into a singularity. Although gravitation is strong enough to crush matter alone, the C-field ultimately dominates. It does so through the effect of the modification of the line element on the wave equation in curved space. The proportionally $\dot{C} \propto S^{-3}$ shows that $\dot{C}$ behaves in the same way as ρ. But the C-field terms affect (39) quadratically, instead of linearly as in the case of matter. For sufficiently small S the C-field terms therefore become overriding. It will be noticed that the contribution of the C-field to the 44-component of (39) is negative—the energy density is negative. Hence the C-field has a repulsive 'gravitational' effect, which explains why implosion can be halted. We believe implosion into singularities to be inevitable if all physical fields have positive energy density. To prevent singularities, a negative energy field is required, one that dominates all positive energy fields as $S \to 0$. In this connexion it is of interest that in the cosmological case electromagnetic fields only add a term in S^{-2} to (50), showing that electromagnetic fields are unable to compete with the C-field as $S \to 0$.

It is worth noting here that the inclusion of the λ-term in Einstein's field equations does give rise to a force of repulsion. This force, however, varies as S and fails to compete with the gravitational force of attraction as $S \to 0$.

4. A static solution for a body of arbitrary mass in the presence of a C-field

In § 2 we argued that if internal pressures could produce an oscillatory solution it should be possible to adjust the pressure to give a static solution. Similarly, it should be possible to obtain a static C-field solution. But it is not intuitively clear that such a field can be matched to the appropriate cosmological boundary condition at large r. Once again the exterior equations are partial in character. The problem is less difficult in this case, however, because the matching of interior and exterior solutions take place at a fixed surface instead of at a moving surface. An explicit solution for this problem will now be given.

As seen in § 3 the line element inside the body can be written in the form

$$ds^2 = dt^2 - S^2[dr^2/(1-\alpha r^2) + r^2 \, d\Omega^2]. \tag{60}$$

The function S satisfies a differential equation

$$\dot{S}^2 = P(S)/S^4, \quad P(S) = \alpha S^3(1-S) - \tfrac{4}{3}\pi G f A^2, \tag{61}$$

with the density ρ and $\dot{C}$ given by

$$\rho = (3\alpha/8\pi G)\, S^{-3}, \quad \dot{C} = AS^{-3}, \tag{62}$$

α, A being constants. The scale of S can always be adjusted to make $S = 1$ at the beginning of the implosion; but for a static solution such a transformation is not important.

For a static solution with $S \equiv S_c$ (say) the quartic $P(S)$ has a double zero at $S = S_c$. Hence $P(S_c) = 0$, $P'(S_c) = 0$. This gives

$$S_c = \tfrac{3}{4}, \tag{63}$$

$$\frac{4\pi G f}{3} A^2 = \frac{27}{256}\alpha, \quad \text{i.e.} \quad A = \left(\frac{3}{4\pi G f}\right)^{\frac{1}{2}} \left(\frac{27\alpha}{256}\right)^{\frac{1}{2}} \equiv A_c \quad \text{(say)}. \tag{64}$$

For $A < A_c$, $P(S)$ has two real positive zeros S_1, S_2 and S oscillates between S_1 and S_2. For $A > A_c$ there is no physical solution at all.

Consider next the external solution. This solution must match the above static internal solution at $r = r_b$ and at infinity it must approach the cosmological solution. As shown in § 3 the cosmological solution is the steady-state solution. In the form originally used by de Sitter the line-element is given by

$$ds^2 = (1-H^2R^2)\,dT^2 - dR^2/(1-H^2R^2) - R^2\,d\Omega^2, \tag{65}$$

where H is the Hubble constant given by

$$H^{-1} = \surd(3/4\pi Gf). \tag{66}$$

The non-zero components of T^{ik} and C_i are

$$\left.\begin{aligned} T^{11} = fR^2H^2, \quad T^{14} = T^{41} = fRH/(1-R^2H^2), \quad T^{44} = f/(1-H^2R^2)^2, \\ C_1 = -RH/(1-R^2H^2), \quad C_4 = 1. \end{aligned}\right\} \tag{67}$$

We now consider the modification produced by the body and take the line-element to be of the form

$$ds^2 = e^N\,dT^2 - e^{-N}\,dR^2 - R^2\,d\Omega^2. \tag{68}$$

We expect the line element (68) to approach the form given by (65) at large R and to behave as the usual exterior Schwarzschild solution near the body where cosmological effects are unimportant. As the line element (65) is stationary, we first try for a solution with N independent of T. This simplifies the problem considerably.

As a generalization of (67) consider the non-zero components of T^{ik} and C_i in the form

$$T^{11} = \rho\xi^2, \quad T^{14} = T^{41} = \rho\xi\eta, \quad T^{44} = \rho\eta^2, \tag{69}$$

$$C_1 = \zeta, \quad C_4 = 1, \tag{70}$$

where ξ, η, ζ, ρ are functions of R only. The field equations (39) reduce to

$$0 = f\zeta + \rho\xi\eta, \tag{71}$$

$$\frac{e^N N'}{R} - \frac{1}{R^2} + \frac{e^N}{R^2} = 8\pi G[\rho\xi^2 e^{-N} - \tfrac{1}{2}f(e^{-N} + \zeta^2 e^N)], \tag{72}$$

$$\frac{e^N N'}{R} - \frac{1}{R^2} + \frac{e^N}{R^2} = -8\pi G[\rho\eta^2 e^N - \tfrac{1}{2}f(e^{-N} + \zeta^2 e^N)], \tag{73}$$

$$\tfrac{1}{2}e^N(N'' + N'^2) + \frac{e^N N'}{R} = -4\pi Gf[e^{-N} - \zeta^2 e^N], \tag{74}$$

where N' stands for dN/dR.

A solution of these equations is obtained with

$$\left.\begin{aligned} \rho = f, \quad e^N = 1 - \frac{\gamma}{R} - \beta R^2, \quad \xi = \surd\left(\frac{\gamma}{R} + \beta R^2\right), \\ \eta = e^{-N}, \quad \zeta = -\surd\left(\frac{\gamma}{R} + \beta R^2\right)\Big/\left(1 - \frac{\gamma}{R} - \beta R^2\right), \end{aligned}\right\} \tag{75}$$

provided $3\beta = 4\pi Gf$. But, since $3H^2 = 4\pi Gf$, this requires $\beta = H^2$, so that e^N tends to the required form at large R.

This solution also satisfies the equation (40) which determines the rate of creation.

The constant γ is determined only by fitting the boundary conditions on the surface of the body. We now turn to the question of matching the internal and external solutions.

Continuity of the coefficients of $\mathrm{d}\Omega^2$ in the two solutions requires

$$R = R_b = r_b S_c = \tfrac{3}{4} r_b. \tag{76}$$

Continuity of the coefficient of $\mathrm{d}r^2$ at $r = r_b$ gives

$$1 - \alpha r_b^2 = 1 - \frac{\gamma}{R_b} - \beta R_b^2 = 1 - \frac{4\gamma}{3r_b} - \frac{9H^2 r_b^2}{16}. \tag{77}$$

For continuity of g_{44} we need

$$t = \sqrt{\left(1 - \frac{4\gamma}{3r_b} - \frac{9H^2}{16} r_b^2\right)}\, T = \sqrt{(1 - \alpha r_b^2)}\, T. \tag{78}$$

It is also necessary to make the C-field continuous across the boundary. For the external solution C is given by

$$C = T + \int \frac{\sqrt{\{(\gamma/R) + H^2R^2\}}\,\mathrm{d}R}{1 - (\gamma/R) - H^2R^2} = T + F(R) \quad \text{(say)}. \tag{79}$$

In the internal solution C is given by

$$C = \int \frac{A}{S^3}\,\mathrm{d}t = \frac{A_c}{S_c^3}\, t + B, \quad B = \text{constant}. \tag{80}$$

For continuity at $r = r_b$ at all times we therefore require

$$B = F(R_b); \quad (A_c/S_c^3)\sqrt{(1 - \alpha r_b^2)} = 1. \tag{81}$$

Substituting for A_c, S_c from (64) and using (66) we get

$$\frac{4}{3\sqrt{3}} \frac{\sqrt{\alpha}\sqrt{(1 - \alpha r_b^2)}}{H} = 1. \tag{82}$$

This relation determines the radius of the body in terms of α. Using (62) this can be written as a radius-density relation.

$$\sqrt{(\tfrac{2}{3}\pi G\rho)}\sqrt{(1 - \tfrac{9}{8}\pi G\rho r_b^2)} = H. \tag{83}$$

The constant γ determines the effective gravitational mass of the body as 'seen' by an external observer. From (77) we get

$$\gamma = \tfrac{3}{4}(\alpha - \tfrac{9}{16}H^2)\, r_b^3. \tag{84}$$

Writing

$$M = \tfrac{4}{3}\pi r_b^3 S_c^3 \rho \tag{85}$$

we get, using (62),

$$\gamma = 2G\{\tfrac{3}{4}M - \tfrac{27}{128}(H^2/G)\, r_b^3\}. \tag{86}$$

The gravitational mass is therefore $(\tfrac{3}{4}M - \tfrac{27}{128}(H^2/G)\, r_b^3)$. The nucleonic mass M_n as defined by (31) is given by

$$M_n = \frac{3M}{2}\left(\sqrt{\frac{r_b}{2GM}}\right)^3 \left[\sin^{-1}\sqrt{\frac{2GM}{r_b}} - \sqrt{\frac{2GM}{r_b}}\sqrt{\left(1 - \frac{2GM}{r_b}\right)}\right]. \tag{87}$$

It is interesting to consider the case where $r_b \simeq 2GM$, but $r_b \ll H^{-1}$, the cosmological distance scale. Writing

$$2GM = (1-\eta)\, r_b \quad (\eta \ll 1), \tag{88}$$

we get, from (83) and (85),

$$\eta = \tfrac{27}{16} H^2 r_b^2. \tag{89}$$

The nucleonic mass is given by

$$M_n \simeq \tfrac{3}{4}\pi M. \tag{90}$$

To this approximation

$$M = \tfrac{4}{3}(2G)^{-\frac{3}{2}}\, (\pi\rho)^{-\frac{1}{2}}, \tag{91}$$

and the gravitational mass as seen by an external observer is close to $\frac{3}{4}M$. The effective gravitational mass of the body is reduced owing to the negative energy density of the C-field which has a repulsive gravitational effect mentioned earlier.

5. Conclusion

The static solution in the previous section demonstrates that the repulsive gravitational force of the C-field can be adjusted to balance the usual gravitational force inside a body of arbitrary mass. The matching of interior and exterior solutions in the oscillatory case is much more difficult, though not different in principle. An explicit solution in the oscillatory case will yield information about the maximum density attained in the body during its oscillations and will be of relevance to the massive objects considered by Fowler & Hoyle (1963*a*, *b*).

The above considerations also appear to show that a negative energy field like the C-field is necessary in order to prevent singularities from developing in space-time.

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