THE ASTROPHYSICAL JOURNAL, 512:1–8, 1999 February 10

A FORMAL ANALYSIS OF (2 + 1)–DIMENSIONAL GRAVITY

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Received 1998 May 14; accepted 1998 September 17

ABSTRACT

Several investigations in the study of cosmological structure formation use numerical simulations in both two and three dimensions. In this paper we address the subtle question of ambiguities in the nature of two-dimensional gravity in an expanding background. We take a detailed and formal approach by deriving the equations describing gravity in $(D + 1)$ dimensions using the action principle of Einstein. We then consider the Newtonian limit of these equations and finally obtain the necessary fluid equations required to describe structure formation. These equations are solved for the density perturbation in both the linearized form and in the spherical top-hat model of nonlinear growth. We find that, when the special case of $D = 2$ is considered, no structures can grow. We therefore conclude that, within the frame work of Einstein's theory of gravity in (2 + 1) dimensions, formation of structures cannot take place. Finally, we indicate the different possible ways of getting around this difficulty, so that growing structures can be obtained in two-dimensional cosmological gravitational simulations, and discuss their implications.

Subject headings: cosmology: theory — gravitation — relativity

1. INTRODUCTION

The dominant paradigm for the generation of the observed large-scale structure in the universe is based on the idea that the gravitational instability amplifies the initially small density perturbations. The equations describing the growth of density perturbations in the highly nonlinear stage are analytically intractable, and hence large-scale numerical simulations are resorted to for exploration of this regime.

These N-body simulations require large amount of computing resources (CPU, memory, and storage space) if one is to get the requisite amount of dynamical range, i.e., good resolution in force and mass, large range in values of density, and so on. Time and resource constraints usually limit our ability to probe structure-formation issues more deeply using computers once the required resources are at the limits of technological feasibility. The key parameter that decides the feasibility level of numerical simulations is the size of a simulation, which in turn is characterized by (1) the number of particles in the simulation volume, which is generally specified as N^D, where D is the dimensionality of the simulation (usually 2 or 3), and (2) the number of mesh points (M) along any axis, which determines the minimum length scale at which the results can be treated as reliable indicators of physical phenomena. In order to create a simulation volume that is a fair sample of the universe, one needs about 10^7 particles, and in order to have a high enough force resolution one needs to increase the number of grid points adequately (for a review, see, e.g., Bagla & Padmanabhan 1997). Let us suppose we have 160^3 particles in three dimensions, and our grid is 160 units on a side. Then for the same amount of computational resources one can simulate a two-dimensional situation with 2048^2 particles on a 2048^2 grid ($160^3 \approx 2048^2$). So, if we can extract useful (i.e., generalizable to the three-dimensional case) physical insights from results in two dimensions, then simulations of two-dimensional gravity will be helpful. This hope has led to a large number of two-dimensional simulations in the field of gravitational clustering (Bagla, Engineer, & Padmanabhan 1998; Valinia et al. 1997; Sathyaprakash et al. 1995; Alimi et al. 1990; Shandarin & Zeldovich 1989; Melott 1983).

There are three ways in which two-dimensional gravity can be operationally defined and corresponding numerical simulations undertaken: (1) Consider a system of point particles in a three-dimensional (expanding) background with the force of interaction being given by Newton's law of gravitation (i.e., $F \propto 1/r^2$). The initial positions and velocities of the particles are such that they all lie in the same plane and all the velocities are in the plane; i.e., there are no velocity components orthogonal to the plane. This system will evolve, with the particles being confined to the plane with clustering occurring in the plane. Thus we have a two-dimensional clustering scenario. (2) Another system we can consider consists of infinite, thin "needles" located parallel to each other. The mass elements in the needles still interact through the $1/r^2$ force, but the interaction between needles (obtained by summing over the mass elements) is given by a $1/r$ force. In this case as well, the background space expands uniformly in three dimensions. The two-dimensional clustering that we study is the clustering of these needles, examined by taking a slice orthogonal to the needles. (3) The third possibility involves writing down the Einstein equations in two dimensions, finding the homogeneous and isotropic cosmological solution, taking the Newtonian limit (in which the potentials due to density perturbations and background metric can be superposed), finding the corresponding perfect fluid equations, and solving them. In this case, we will have a background spacetime expanding in *two* dimensions, unlike the other two cases. (There is yet another, fourth possibility, which can be defined only in an ad hoc manner. We will present this in the end.)

The first case is not of much interest for cosmological simulations, since the system is anisotropic, confined to a single plane, and the clustering takes place in a specific plane *only because the initial conditions were specifically selected* to give this result. Hence we will not discuss it, and it is mentioned here only for completeness. The way simulations in two dimensions are carried out usually is by simulating the second case and then defining the "particles" as the intersection of the needles with any plane orthogonal to them. In this case, as in the first case, the background spacetime expands in three dimensions (for a flat dust-dominated universe, the scale factor $a(t)$ goes as $t^{2/3}$). The clustering that we observe and quantify in two dimensions is basically the clustering of these needles in three dimensions. But this is also an anisotropic situation, since the background spacetime is expanding in three dimensions. As an alternative we may try to write down the equations derived from Einstein's equations in two dimensions and examine how a system of particles interacting in a two-dimensional expanding-background spacetime is to be simulated.

In the rest of this paper we shall examine the third alternative. We will take a very general approach by developing the formal theory of $(D + 1)$ gravity and considering $D = 2$ as a special case. (For some previous work on $(2 + 1)$–dimensional gravity see, e.g., Deser, Jackiw, & t'Hooft (1984); Gott & Alpert (1984); Gott (1985); and references cited therein. We shall comment on the connection between our results and the previous ones in the last section).

The basic layout of the paper is as follows: In § 2 we first define the analogue of Einstein's gravity in $(D + 1)$ dimensions, discuss the Newtonian limit and its corresponding Poisson's equation, and then go on to analyze the Friedmann metric in $(D + 1)$ dimensions for a flat universe with dust. In § 3 we write down the D-dimensional fluid equations and obtain the equation governing the density perturbations. This equation is then solved in the linear approximation and using the spherical top-hat (STH) model. Then in §§ 4 and 5 we specialize to the cases $D = 3$ and $D = 2$, respectively. Finally, in § 6 we summarize and discuss the implications of the results obtained in the earlier sections.

2. FORMAL $(D + 1)$–DIMENSIONAL GRAVITY

We start our analysis of $(D + 1)$–dimensional (one time dimension and D space dimensions) gravity from the action principle, which we assume has the same form as that used in $(3 + 1)$ dimensions. Using this action we construct the corresponding $(D + 1)$–dimensional Einstein equations, which will be subsequently used to study structure formation and spherical collapse. Thus we begin with the action principle

$$\mathscr{S} = \mathscr{S}_g + \mathscr{S}_m = -\frac{c^4}{2\kappa(D)} \int d^{(D+1)}x R \sqrt{|g|} + \int d^{(D+1)}x \mathscr{L}_m , \quad (1)$$

where $\mathscr{S}_g$ is the action for the gravitational field, $\mathscr{S}_m$ is the action for the matter fields, g is the determinant of the metric tensor g_{ik}, R is the Ricci scalar, $\kappa(D)$ is a suitable constant that can be, in general, a function of D (when $D = 3$, $\kappa = 8\pi G$, G being the usual gravitational constant), and $\mathscr{L}_m$ is the Lagrangian density for the matter fields. The metric signature we adopt is $(+, -, -, -, \ldots, -)$. We adopt the following convention regarding indices: Latin alphabets i, j, k ... are used to represent $(D + 1)$–dimensional indices, which take on the values $(0, 1, 2, \ldots, D)$ while Greek letters are used to denote D-dimensional indices taking on the values $(1, 2, \ldots, D)$. Varying the total action $\mathscr{S}$ with respect to g_{ik}, we obtain Einstein's equations

$$G_{ik} \equiv R_{ik} - \frac{1}{2} g_{ik} R = \frac{\kappa(D)}{c^4} T_{ik} , \quad (2)$$

where T_{ik} is the energy momentum tensor of the matter fields and is defined by

$$\frac{1}{2}\sqrt{|g|}\, T_{ik} = \frac{\partial(\sqrt{|g|}\mathscr{L}_m)}{\partial g^{ik}} - \frac{\partial}{\partial x^l}\left(\frac{\partial(\sqrt{|g|}\mathscr{L}_m)}{\partial(\partial g^{ik}/\partial x^l)}\right) , \quad (3)$$

where G_{ik} is the Einstein tensor and R_{ik} is the usual Ricci tensor. Note that the $\frac{1}{2}$ that appears in Einstein's equations arises because of the square root in the term $(|g|)^{1/2}$ and has nothing to do with the dimension of the spacetime.

We will use the above equations in the subsections to follow. In § 2.1, we will study the Newtonian limit of the metric tensor and then construct the corresponding Poisson equation that relates the Newtonian gravitational field ϕ to the matter density ρ. Then in § 2.2 we analyze the Friedmann metric in $(D + 1)$ dimensions, and the corresponding Newtonian limit of this metric is derived.

2.1. *Poisson Equation in D Dimensions*

In this section, we derive the Poisson equation relating the gravitational potential ϕ to the matter density ρ. We keep all factors of c since the Newtonian limit involves the limit $c \to \infty$. The analysis here follows closely the treatment in Landau & Lifshitz (1975). Consider the metric

$$ds^2 = \left(1 + \frac{2\phi}{c^2}\right)c^2\, dt^2 - dl^2 , \quad (4)$$

where ϕ is a function of space and time with dimensions of velocity square. The term dl^2 is the D-dimensional spatial line element given by the formula

$$dl^2 = \sum_{\alpha=1}^{D} (dx^\alpha)^2 . \quad (5)$$

We will now show that the metric written above is the Newtonian limit of Einstein's gravitational equations. We do this by showing that, in the Newtonian limit, the equation of motion of a particle follows Newton's force law, with the force $-m\nabla\phi$. In relativistic mechanics, the motion of a particle of mass m is determined by the action function S:

$$S = -mc\int ds = -mc\int c\, dt \sqrt{\left(1 + \frac{2\phi}{c^2} - \frac{v^2}{c^2}\right)} , \quad (6)$$

where v^2 is the square of the magnitude of the particle's velocity in D dimensions. In arriving at the second equality, we have used the form of the metric in equation (4). In the limit $c \to \infty$, the action S can be approximated as

$$S \approx -mc^2\int dt\left(1 + \frac{2\phi - v^2}{2c^2}\right) = \int dt\left(-mc^2 + \frac{1}{2}mv^2 - m\phi\right) . \quad (7)$$

The equation of motion for the particle can be immediately written down, and we obtain

$$m\frac{d\boldsymbol{v}}{dt} = -m\nabla\phi , \quad (8)$$

where $\boldsymbol{v}$ is the velocity vector in D space dimensions. Thus, Newton's force law is recovered in the nonrelativistic limit, and from this we conclude that the metric given in equation (4) is the Newtonian limit of Einstein's gravitational equations with ϕ acting as the Newtonian gravitational potential.

The relation between ϕ and the mass density ρ is found by taking the $c \to \infty$ limit of Einstein's equations. This procedure, in $(3 + 1)$ gravity, determines the constant $\kappa(D)$, since the Poisson equation is explicitly known. In other dimensions, however, a definite criterion like Gauss's law, for example, must be imposed in order to determine $\kappa(D)$. We now consider the limit $c \to \infty$ of Einstein's equations in the following manner: First, we use the line element given in equation (4) to calculate the Ricci tensor component R_{00}:

$$R_{00} = \frac{1}{c^2} \frac{1}{c^2 + 2\phi} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{c^2} \partial_\mu \partial^\mu \phi , \quad (9)$$

where the summation convention has been invoked in the above equation and the sum over μ is only over the *spatial* dimensions. Then, using equation (2), we obtain

$$R = -\frac{2\kappa(D)}{c^4(D-1)} T , \quad (10)$$

where we have used the fact that $g_{ik} g^{ik} = D + 1$ and assumed $D \neq 1$. Thus Einstein's equations can be written in the equivalent form,

$$R_{ik} = \frac{\kappa(D)}{c^4} \left(T_{ik} - \frac{1}{D-1} g_{ik} T \right) , \quad (11)$$

where T is the trace of T_{ik}. The energy momentum tensor of point particles is $T_{ik} = \rho c^2 u_i u_k$, where ρ is the mass density and u_i is the four velocity. Since in the nonrelativistic limit the macroscopic motion is slow, the space components of u_i can be neglected, and only the time component should be retained. Therefore, $u_0 = (g_{00})^{1/2}$ and $u_\mu \approx 0$ for all μ. Consequently, only $T_{00} = g_{00} \rho c^2$ is nonzero. Substituting for T_{ik} into equation (11) and using the expression in equation (9) for R_{00}, we get

$$\frac{1}{c^2} \partial_\mu \partial^\mu \phi = -\left(\frac{D-2}{D-1}\right)\left(1 + \frac{2\phi}{c^2}\right) \frac{\kappa(D)}{c^4} \rho c^2$$
$$\approx -\left(\frac{D-2}{D-1}\right) \frac{\kappa(D)}{c^2} \rho . \quad (12)$$

That is,

$$\nabla^2 \phi = \left(\frac{D-2}{D-1}\right) \kappa(D) \rho , \quad (13)$$

where ∇^2 is the usual Laplacian operator in D dimensions. This equation is the Poisson equation in D dimensions. Note that, when substituting for the value of R_{00} from equation (9), we neglected the first term in comparison with the second, since the former is of order c^{-4} and the latter is only of order c^{-2}.

2.2. *Friedmann Universe in* $(1 + D)$ *Dimensions*

Let us next consider the maximally symmetric Robertson-Walker metric in $(D + 1)$ dimensions, specializing to flat space with $k = 0$ (we set $c = 1$ in this and in subsequent sections),

$$ds^2 = dt^2 - a^2(t) dl^2 , \quad (14)$$

where $a(t)$ is the scale factor and dl^2 is the D-dimensional line element given in equation (5). Calculating the components of the Einstein tensor, we obtain

$$G_{00} = \frac{D(D-1)\dot{a}^2}{2a^2} ,$$
$$G_{11} = G_{22} = \cdots = G_{DD} = (1 - D) a \ddot{a} + \left(1 - \frac{D}{2}\right)(D-1)\dot{a}^2 , \quad (15)$$

where $\dot{a}$ stands for $da(t)/dt$ and, similarly, $\ddot{a}$ is the second derivative of $a(t)$ with respect to time. All the other components are zero. For consistency, the energy momentum tensor must have the form $T^i_k = \mathrm{diag}\,(\rho, -p, -p, -p, \ldots)$, where ρ is the matter density and p is the pressure.

Substituting in Einstein's equations, we obtain

$$\frac{D(D-1)\dot{a}^2}{2a^2} = \kappa(D)\rho , \quad (16)$$

$$\frac{\ddot{a}}{a} + \frac{D-2}{2} \frac{\dot{a}^2}{a^2} = -\frac{\kappa(D) p}{D-1} . \quad (17)$$

The above two equations, together with the equation of state in the form $p = p(\rho)$, completely specify the system. Solving these three equations, we can determine $a(t)$, $\rho(t)$, and subsequently $p(t)$. Combining equations (16) and (17), we get the single equation

$$\frac{\ddot{a}}{a} = -\frac{\kappa(D)}{D(D-1)} [(D-2)\rho + Dp] . \quad (18)$$

We now specialize to the case of pressureless dust with the equation of state $p = 0$. Using the principle of conservation of energy and momentum expressed by the relation

$$T^k_{i;k} = 0 , \quad (19)$$

we derive the following relation:

$$T^k_{i,k} = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^k} (\sqrt{|g|}\, T^k_i) - \frac{1}{2} \frac{\partial g_{kl}}{\partial x^i} T^{kl}$$
$$= \frac{1}{a^D} \frac{\partial}{\partial x^k} (a^D T^k_i) - \frac{1}{2} \frac{\partial g_{kl}}{\partial x^i} T^{kl} = 0 . \quad (20)$$

Noting that the only nonzero component of T^i_k is $T^0_0 = \rho$, we finally get

$$\rho a^D = \text{constant} = C_1 . \quad (21)$$

Substituting the above relation into equation (16) and solving for $a(t)$ and, subsequently, $\rho(t)$, we obtain the solutions

$$a(t) = \left[\frac{D\kappa(D) C_1}{2(D-1)}\right]^{1/D} t^{2/D} , \quad \rho(t) = \left[\frac{2(D-1)}{D\kappa(D)}\right] t^{-2} . \quad (22)$$

Let us next consider the Newtonian limit of the Friedmann metric. This limit is important, because the length scales of interest in structure formation are small compared to the Hubble radius, and the velocities in the system are also much smaller than c. This permits us to study the formation of large-scale structures in the universe in a Newtonian framework, where the effective potential due to the expanding background universe, Φ_{FRW}, and the potential due to the density perturbations, φ, can be simply superposed. In order to obtain Φ_{FRW}, we first recast the Friedmann metric in equation (14) into the more convenient form

$$ds^2 = dt^2 - a^2(t)(dX^2 + X^2\,d\Omega^2)\,, \quad (23)$$

where X is the radial distance in D dimensions and Ω is the corresponding solid angle. We then apply the transformations

$$r = Xa(t)\,, \quad T = t - t_0 + \tfrac{1}{2}a\dot{a}X^2 + \mathcal{O}(X^4)\;; \quad (24)$$

(see Padmanabhan 1996), where only terms up to quadratic in X are retained. Direct calculations, correct up to this order, transforms the Friedmann line element to the form

$$ds^2 \approx \left(1 - \frac{\ddot{a}}{a}r^2\right)dT^2 - dr^2 - r^2\,d\Omega^2\,, \quad (25)$$

which, upon comparison with the metric in equation (4), gives the equivalent Newtonian potential $\Phi_{\rm FRW}$ in D dimensions as

$$\Phi_{\rm FRW} = -\frac{1}{2}\frac{\ddot{a}}{a}r^2\,. \quad (26)$$

We will now use the results developed in the last two subsections to study structure formation and spherical collapse using the STH model.

3. STRUCTURE FORMATION IN D DIMENSIONS

Having determined the form of the Poisson equation in the Newtonian limit and analyzed the Friedmann equations in $(D + 1)$ dimensions, we proceed to derive the equation for the growth of inhomogeneities in the expanding universe. After this we consider a specific model, the STH model, to study spherical collapse of matter.

3.1. *Equation for Density Perturbations in D Dimensions*

Let us assume that matter in the universe is a perfect, pressureless fluid with density ρ_m and flow velocity $\boldsymbol{U}$. We can formally write down the D-dimensional fluid equations describing a perfect fluid in an external potential field $\Phi_{\rm tot}$ in a proper coordinate system labeled by the D-dimensional vector $\boldsymbol{r}$. Therefore, we have

$$\left(\frac{\partial\rho_m}{\partial t}\right)_{\boldsymbol{r}} + \nabla_{\boldsymbol{r}}\cdot(\rho_m\boldsymbol{U}) = 0\,, \quad (27)$$

$$\left(\frac{\partial\boldsymbol{U}}{\partial t}\right)_{\boldsymbol{r}} + (\boldsymbol{U}\cdot\nabla_{\boldsymbol{r}})\boldsymbol{U} = -\nabla_{\boldsymbol{r}}\Phi_{\rm tot}\,, \quad (28)$$

where equation (27) is the usual continuity equation, whereas equation (28) is the Euler equation for the fluid. The potential in equation (28), $\Phi_{\rm tot}$, is the total external Newtonian potential

$$\Phi_{\rm tot} = \Phi_{\rm FRW} + \varphi\,, \quad (29)$$

where $\Phi_{\rm FRW}$ is the background potential associated with the smooth background matter density $\rho_{\rm bm}$ and is given in equation (26), whereas φ is the potential caused by density perturbations $(\rho_m - \rho_{\rm bm})$. The potential φ satisfies the Poisson equation given in equation (13). Thus

$$\nabla_{\boldsymbol{r}}^2\varphi = \left(\frac{D-2}{D-1}\right)\kappa(D)(\rho_m - \rho_{\rm bm}) = \left(\frac{D-2}{D-1}\right)\kappa(D)\rho_{\rm bm}\,\delta\,, \quad (30)$$

where δ is the density contrast defined by

$$\delta = \frac{\rho_m - \rho_{\rm bm}}{\rho_{\rm bm}}\,. \quad (31)$$

We now transform to comoving coordinates defined by $\boldsymbol{x} = \boldsymbol{r}/a(t)$ and define the peculiar velocity $\boldsymbol{v}$ by the relation

$$\boldsymbol{U} = H(t)\boldsymbol{r} + \boldsymbol{v} = \dot{a}\boldsymbol{x} + \boldsymbol{v}\,, \quad (32)$$

where $\boldsymbol{v} = a\dot{\boldsymbol{x}}$ and $H(t) = (\dot{a}/a)$. Then, equation (27) and equation (28) become

$$\left(\frac{\partial\rho_m}{\partial t}\right)_{\boldsymbol{x}} + DH\rho_m + \frac{1}{a}\nabla_x\cdot(\rho_m\boldsymbol{v}) = 0\,, \quad (33)$$

$$\left(\frac{\partial\boldsymbol{v}}{\partial t}\right)_{\boldsymbol{x}} + H\boldsymbol{v} + \frac{1}{a}(\boldsymbol{v}\cdot\nabla_x)\boldsymbol{v} = -\frac{1}{a}\nabla_x\varphi\,, \quad (34)$$

where we have used equation (26) to substitute for $\Phi_{\rm FRW}$. Similarly, in comoving coordinates equation (30) reduces to

$$\nabla_{\boldsymbol{x}}^2\varphi = \left(\frac{D-2}{D-1}\right)\kappa(D)a^2\rho_{\rm bm}\,\delta\,. \quad (35)$$

Using $\rho_m = \rho_{\rm bm}(1 + \delta)$, transforming the time variable from t to $a(t)$, and defining a new velocity variable $\boldsymbol{u}$ by

$$\boldsymbol{u} = \frac{d\boldsymbol{x}}{da} = \frac{\boldsymbol{v}}{a\dot{a}}\,, \quad (36)$$

we can obtain equations for $\delta(a)$ and $\boldsymbol{u}(a)$. Therefore, using equation (21) and performing the transformations, equation (33) and equation (34) further reduce to

$$\frac{\partial\delta}{\partial a} + \nabla_x\cdot[\boldsymbol{u}(1+\delta)] = 0\,, \quad (37)$$

$$\dot{a}^2\frac{\partial\boldsymbol{u}}{\partial a} + \left(\ddot{a} + 2\frac{\dot{a}^2}{a}\right)\boldsymbol{u} + \dot{a}^2(\boldsymbol{u}\cdot\nabla_x)\boldsymbol{u} = -\frac{1}{a^2}\nabla_x\varphi\,. \quad (38)$$

Now we use the Friedmann equations in equations (16) and (17) with ρ replaced by $\rho_{\rm bm}$ and with $p = 0$ to substitute for $\ddot{a}$ in the above equation. Further, we define a new potential Ψ by the relation

$$\Psi = \left[\frac{D(D-1)}{6-D}\right]\frac{1}{\kappa(D)\rho_{\rm bm}a^3}\,\varphi\,, \quad (39)$$

so that, upon using equation (35), one obtains

$$\nabla_{\boldsymbol{x}}^2\Psi = \left[\frac{D(D-2)}{6-D}\right]\frac{\delta}{a}\,, \quad (40)$$

where all reference to $\kappa(D)$ has disappeared. Hence the final system of equations we need to tackle are

$$\frac{\partial\delta}{\partial a} + \nabla_x\cdot[\boldsymbol{u}(1+\delta)] = 0\,, \quad (41)$$

$$\frac{\partial\boldsymbol{u}}{\partial a} + (\boldsymbol{u}\cdot\nabla_x)\boldsymbol{u} = -\frac{6-D}{2a}A[\nabla_x\Psi + \boldsymbol{u}]\,, \quad (42)$$

where A is given by the relation

$$A = \left[\frac{2\kappa(D)}{D(D-1)}\right]\frac{a^2}{\dot{a}^2}\rho_{\rm bm} = \frac{\rho_{\rm bm}(t)}{\rho_c(t)}\,, \quad \rho_c \equiv \left[\frac{D(D-1)}{2\kappa(D)}\right]\frac{\dot{a}^2}{a^2}\,. \quad (43)$$

For the $k = 0$ universe, we will set $A = 1$.

To proceed further and determine the equation satisfied by δ, we decompose the term $\partial_\alpha u_\beta$ (where u_β is the βth covariant component of the vector $\boldsymbol{u}$ and ∂_α is short for $\partial/\partial x^\alpha$) as

$$\partial_\alpha u_\beta = \sigma_{\alpha\beta} + \Omega_{\alpha\beta} + \frac{1}{D}\,\delta_{\alpha\beta}\,\theta\,, \quad \alpha, \beta = (1, 2, \ldots, D)\,, \quad (44)$$

where $\sigma_{\alpha\beta}$ is the traceless, symmetric shear tensor, $\Omega_{\alpha\beta}$ is the antisymmetric rotation tensor, θ is the (trace) expansion, and $\delta_{\alpha\beta}$ is the Kronecker delta symbol. Then equation (41) and equation (42) are combined by taking the divergence of equation (42) and using the above decomposition of $\partial_\alpha u_\beta$ to obtain a single equation for δ. Straightforward algebra gives

$$\frac{d^2\delta}{da^2} + \left(\frac{6-D}{2a}\right)\frac{d\delta}{da} - \left[\frac{D(D-2)}{2a^2}\right]\delta(1+\delta) = \left(\frac{D+1}{D}\right)\frac{1}{(1+\delta)}\left(\frac{d\delta}{da}\right)^2 + (1+\delta)(\sigma^2 - 2\Omega^2)\,, \quad (45)$$

where $\sigma^2 = \sigma_{\alpha\beta}\sigma^{\alpha\beta}$ and $\Omega^2 = (1/2)\Omega_{\alpha\beta}\Omega^{\alpha\beta}$. This equation is the full nonlinear equation for δ. Apart from the obvious nonlinear terms containing δ^2 and $(d\delta/da)^2$, the term $(1+\delta)(\sigma^2 - 2\Omega^2)$, which is the contribution from the shear and rotation, is also nonlinear. The nonlinear terms in δ in the above equation render the equation unsolvable in general. Ignoring these nonlinear terms to a first approximation, we can get a linear equation for $\delta(a)$:

$$\frac{d^2\delta}{da^2} + \left(\frac{6-D}{2a}\right)\frac{d\delta}{da} - \left[\frac{D(D-2)}{2a^2}\right]\delta = 0\,. \quad (46)$$

Assuming a power-law solution for delta in the form $\delta \propto a^p$, we get

$$p = \frac{D-4}{4} \pm \frac{1}{4}\sqrt{9D^2 - 24D + 16} \quad (47)$$

as the required values for p. Notice that δ has a growing mode as well as a decaying mode in general. The above solutions hold for all values of $D > 1$ in the linear regime.

Although the full nonlinear equation is not solvable, by neglecting the contribution from the shear and rotation terms and by using a suitable *Ansatz* for δ, the resulting nonlinear equation *can* be solved. We proceed to do this within the framework of the STH model in the next section.

3.2. *The STH Model*

In the STH (spherical collapse) model, we assume spherical symmetry by neglecting the shear and rotation terms in the equation for δ. With this assumption the δ equation can be exactly solved.

Transforming equation (45) by changing the independent variable back to t, dropping the rotation and shear terms, and using the Friedmann equations given in equations (16) and (17), we get

$$\frac{d^2\delta}{dt^2} + 2\,\frac{\dot a}{a}\frac{d\delta}{dt} - \left(\frac{D+1}{D}\right)\frac{1}{(1+\delta)}\left(\frac{d\delta}{dt}\right)^2 = \left(\frac{D-2}{D-1}\right)\kappa(D)\rho_{\rm bm}\,\delta(1+\delta)\,. \quad (48)$$

We now define a function $R(t)$ by the relation

$$1 + \delta = \frac{\rho}{\rho_{\rm bm}} = \frac{M}{C_D R^D(t)\rho_{\rm bm}}\,, \quad (49)$$

where $C_D = 2\pi^{D/2}/(D\Gamma[D/2])$ is the volume of a unit sphere in D dimensions introduced for later convenience and M is a constant. The expression for δ above can be rewritten using the relation $\rho_{\rm bm} a^D = \rho_0 a_0^D$ from equation (21):

$$1 + \delta = \frac{M}{C_D \rho_0 a_0^D}\left(\frac{a}{R}\right)^D = \lambda\,\frac{a^D}{R^D}\,, \quad (50)$$

where ρ_0 and a_0 are the matter density and scale factor at some (arbitrarily chosen) "present" epoch t_0. Substituting equation (50) in equation (48), we get an equation for the growth of $R(t)$ as

$$\frac{d^2R}{dt^2} = -\frac{D-2}{D(D-1)}\frac{\kappa(D)}{C_D}\frac{M}{R^{D-1}}\,. \quad (51)$$

[As an aside, we may note that if the universe contains matter or fields with equations of state other than $p = 0$, the equation for $R(t)$ becomes

$$\frac{d^2R}{dt^2} = -\frac{D-2}{D(D-1)}\frac{\kappa(D)}{C_D}\frac{M}{R^{D-1}} - \frac{\kappa(D)}{D(D-1)}\left[(D-2)\rho + Dp\right]_{\rm rest} R\,, \quad (52)$$

where the term $[(D-2)\rho + Dp]_{\rm rest}$ comes from the smoothly distributed component with $p \neq 0$.]

From the form of the equation of motion of $R(t)$ we can give the following interpretation: Since the entire system considered above is spherically symmetric, we interpret R as the radius of a D-dimensional spherical region containing a mass M. The equation of motion of R determines the motion of the surface of this region. In general, a spherical overdense region will be expected to initially expand because of the expansion of the background universe, until the excess gravitational force due to the overdensity of enclosed matter stops the expansion and causes the region to collapse back on itself. We will discuss the cases $D = 3$ and $D = 2$ in the subsequent sections and determine the differences in the behavior of the growth of inhomogeneities.

4. SUMMARY OF STANDARD RESULTS IN THREE DIMENSIONS

When $D = 3$, all the standard equations are recovered. First the Poisson equation satisfied by the Newtonian gravitational potential given by equation (13) reduces to the standard form

$$\nabla^2\phi = 4\pi G\rho\,, \quad (53)$$

where we have defined G by relating it to $\kappa(3)$ by $\kappa(3) = 8\pi G$. Similarly, equations (41) and equation (42) reduce to (with $A = 1$) the following:

$$\frac{\partial\delta}{\partial a} + \nabla_x \cdot [\boldsymbol{u}(1+\delta)] = 0\,, \quad (54)$$

$$\frac{\partial\boldsymbol{u}}{\partial a} + (\boldsymbol{u}\cdot\nabla_x)\boldsymbol{u} = -\frac{3}{2a}\left[\nabla_x \cdot \Psi + \boldsymbol{u}\right]\,, \quad (55)$$

while the equation for Ψ becomes

$$\nabla_x^2\Psi = \frac{\delta}{a}\,. \quad (56)$$

In a similar manner, the δ equation reduces to

$$\frac{d^2\delta}{da^2} + \frac{3}{2a}\frac{d\delta}{da} - \frac{3}{2a^2}\,\delta(1+\delta) = \frac{4}{3(1+\delta)}\left(\frac{d\delta}{da}\right)^2 + (1+\delta)(\sigma^2 - 2\Omega^2)\,, \quad (57)$$

and the solutions to the linear perturbation equation, which is obtained by dropping the nonlinear terms and the $(\sigma^2 - 2\Omega^2)$ term, are

$$\delta \propto a^p\,, \quad p = 1\,, \quad -\tfrac{3}{2}\,, \quad (58)$$

which are well known. The STH model for $D = 3$ also reduces to the standard form

$$\frac{d^2R}{dt^2} = -\frac{GM}{R^2} - \frac{4\pi G}{3}(\rho + 3p)_{\rm rest} R\,, \quad (59)$$

which is again well known (Padmanabhan 1996). Therefore, it is seen that the full D-dimensional equations reduce to the correct equations in three dimensions. Now we will go on to discuss the important case of $D = 2$.

5. TWO-DIMENSIONAL GRAVITY

If we naively consider the limit $D \to 2$ in the D-dimensional equations, assuming that $\kappa(D)$ is finite in this limit, we obtain the following results: First, the Poisson equation (13) reduces to

$$\nabla^2\phi = 0\,. \quad (60)$$

The above result shows that in two dimensions, the gravitational potential does *not* couple to the matter density ρ. In structure formation, this means that inhomogeneities cannot grow, since the perturbed potential φ is not related to δ at all. The second interesting result is that the background Newtonian potential $\Phi_{\rm FRW}$ vanishes. This occurs because, referring back to equation (18), $\ddot{a} = 0$ for pressureless dust, and hence the background potential is zero. Further, the δ equation reduces to

$$\frac{d^2\delta}{da^2} + \frac{2}{a}\frac{d\delta}{da} = \frac{3}{2(1+\delta)}\left(\frac{d\delta}{da}\right)^2 + (1+\delta)(\sigma^2 - 2\Omega^2)\,. \quad (61)$$

Linearizing the equation as before by dropping the $(\sigma^2 - 2\Omega^2)$ and $(d\delta/da)^2$ terms, we obtain

$$\frac{d^2\delta}{da^2} + \frac{2}{a}\frac{d\delta}{da} = 0\,. \quad (62)$$

The solutions to the linearized equation are

$$\delta \propto a^p\,, \quad p = 0\,, \quad -1\,. \quad (63)$$

Thus only a constant or the decaying mode is present. This is consistent with the result that the perturbed gravitational potential does not couple to δ. If one considers the STH model, it is easy to see that the growth equation for $R(t)$ reduces to

$$\frac{d^2R}{dt^2} = -\kappa(2)p_{\rm rest} R\,. \quad (64)$$

For $p_{\rm rest} = 0$, the solution to the above equation is just $R(t) = B_1 t + B_2$, where B_1, B_2 are constants. This is to be expected, since there is no gravitational force that can lead to clustering, and as a consequence the radius simply grows with time just like the background universe. Thus if $\kappa(D)$ is *finite* in the limit $D \to 2$, it is not possible to have gravitational clustering that can grow with time.

We can, however, try some alternative approaches to examine whether it is possible to have a consistent physical picture of growing structures for two-dimensional gravity.

One possibility is that instead of assuming $\kappa(D)$ to be finite, let us assume that the expression $[\kappa(D)(D-2)]$ remains finite when $D \to 2$. This finite value can be fixed, for example, by invoking Gauss's theorem in D dimensions. This gives

$$\kappa(D) = \left(\frac{D-1}{D-2}\right)\frac{2\pi^{D/2}G}{\Gamma[D/2]}\,. \quad (65)$$

Thus $\kappa(D) \to \infty$ when $D \to 2$, but the Poisson equation acquires the form

$$\nabla^2\phi = 2\pi G\rho\,. \quad (66)$$

This is, of course, the same form that is obtained by applying Gauss's law in two dimensions. Hence, as in three dimensions, the gravitational potential is determined by the matter density, and thus inhomogeneities can in principle grow. There are, however, difficulties with this approach. To begin with, the constant factor $c^4/[2\kappa(D)]$ in the action $\mathscr{S}$ in equation (1) vanishes for $D = 2$. But this is not too serious a problem. The gravitational part of the action certainly vanishes, but because only the variations about the action are of significance, this difficulty can be ignored. But a more serious problem arises when the solutions to the Friedmann equation are considered. The solutions for $a(t)$ and $\rho(t)$ in D dimensions are given in equation (22). Using equation (65), these reduce to

$$a(t) = \left\{\frac{D\pi^{D/2}GC_1}{(D-2)\Gamma[D/2]}\right\}^{1/D} t^{2/D}\,,$$

$$\rho(t) = C_1 a^{-D} = \frac{(D-2)\Gamma[D/2]}{D\pi^{D/2}G}\,t^{-2}\,. \quad (67)$$

When $D \to 2$ then $a \to \infty$ and $\rho \to 0$, irrespective of the dependence on t. This implies that one cannot solve the equations describing the growth of structure in a consistent and nonsingular way. Hence we conclude that it is not possible to have a theoretical formulation of two-dimensional gravity as the Newtonian limit to Einstein's equations in two dimensions.

An alternative that remains is to use the Newtonian fluid equations in D dimensions directly and rewrite them for an expanding background with an *arbitrary* scale factor $a(t)$. Note that $a(t)$ is not obtained from the Friedmann equations and is completely arbitrary. We can superpose the potentials for the background universe and the perturbations in this case as before. The further assumptions we need to make are (1) the potential of the background universe $\Phi_{\rm bg}$ is of the form

$$\Phi_{\rm bg} = -\frac{1}{2}\frac{\ddot{a}}{a}r^2\,, \quad (68)$$

and (2) the Poisson equation is given by

$$\nabla^2\phi = \kappa(D)\rho_{\rm bm}\delta\,, \quad (69)$$

where $\kappa(D) = 2\pi^{D/2}G/\Gamma[D/2]$. This form of $\kappa(D)$ is obtained from the use of Gauss's law in D dimensions. We also need to specify how the background density $\rho_{\rm bm}$ depends on time. In analogy with the usual Friedmann equations, we will assume $\rho_{\rm bm}\, a^D = C_1$, where C_1 is a constant. This gives an equation for δ with an arbitrary scale factor $a(t)$, as

$$\frac{d^2\delta}{da^2} + \left(\frac{\ddot{a}a + 2\dot{a}^2}{a\dot{a}^2}\right)\frac{d\delta}{da} - \kappa(D)C_1\,\frac{1}{a^D\dot{a}^2}\,\delta(1+\delta) = \left(\frac{D+1}{D}\right)\frac{1}{1+\delta}\left(\frac{d\delta}{da}\right)^2 + (1+\delta)(\sigma^2 - 2\Omega^2)\,. \quad (70)$$

The above equation can be solved in any dimension D if the form of $a(t)$ is given. This gives us a nonsingular way to analyze growth of structures in D dimensions, including the case $D = 2$. But in three dimensions we observe that the above equation does not correctly reduce to equation (57). We may obtain the correct equation in three dimensions by making an additional *Ansatz*, namely, that

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{2\kappa(D)}{D}\,\rho_{\rm bm}\,. \quad (71)$$

With this, equation (70) reduces to

$$\frac{d^2\delta}{da^2} + \left(\frac{6-D}{2a}\right)\frac{d\delta}{da} - \frac{D}{2a^2}\,\delta(1+\delta) = \left(\frac{D+1}{D}\right)\frac{1}{1+\delta}\left(\frac{d\delta}{da}\right)^2 + (1+\delta)(\sigma^2 - 2\Omega^2)\,. \quad (72)$$

Notice that the above equation differs from the earlier equation for δ in D dimensions, equation (45), in that there is a factor of $(D-2)$ missing from the coefficient of the $\delta(1+\delta)$ term. Therefore, the above equation does correctly reduce to equation (57), since $(D-2)$ equals unity when $D = 3$. When $D = 2$, equation (72) gives

$$\frac{d^2\delta}{da^2} + \frac{2}{a}\frac{d\delta}{da} - \frac{1}{a^2}\,\delta(1+\delta) = \frac{3}{2(1+\delta)}\left(\frac{d\delta}{da}\right)^2 + (1+\delta)(\sigma^2 - 2\Omega^2)\,. \quad (73)$$

On linearizing this equation by dropping the $(\sigma^2 - 2\Omega^2)$ term and the nonlinear terms and solving it, we get both a growing mode as well as a decaying mode for δ. The solutions are

$$\delta \propto a^q\,, \quad q = \frac{(-1 \pm \sqrt{5})}{2}\,. \quad (74)$$

(It is interesting to note that one of the power-law exponents is the golden ratio.)

The STH equation in this case turns out to be

$$\ddot{R} = -\frac{GM}{R} + \frac{1}{2}\frac{R}{t^2}\,, \quad (75)$$

where M is the constant mass inside a "spherical" shell of radius R. This equation, unfortunately, has no simple analytic solution.

While this procedure leads to nontrivial results, it has many ad hoc assumptions and cannot be obtained by taking appropriate limits of Einstein's theory in a systematic manner. Consequently, it cannot be applied to numerical investigations of two-dimensional gravity with the confidence that the results will have some implications for the three-dimensional case.

6. CONCLUSIONS

In this paper we have analyzed the case of two-dimensional gravitational clustering starting from a formulation of the D-dimensional Einstein's equations and taking the proper limits. The system of equations thus arrived at for a $(D+1)$–dimensional universe has been shown to reduce to the correct equations in three dimensions. But when the $D \to 2$ limit of these equations is taken, we are forced to conclude that, irrespective of the value of $\kappa(D)$, a consistent two-dimensional gravity theory in a cosmological context that supports growth of structures cannot be constructed.

If $\kappa(2)$ is assumed to be finite, we observe that the coefficient in Poisson's equation goes to zero, thus decoupling the potential from the density. This implies that perturbations do not grow but decay in time because of the expansion of the background spacetime. The alternative that is obtained by using the expression for $\kappa(D)$ given by equation (65) gives rise to solutions for the scale factor, which are singular and therefore unacceptable.

We have discussed all the ways in which two-dimensional gravity may be simulated, including an ad hoc procedure without a strong foundation, which can give nonsingular results as far as structure formation scenarios in two dimensions are concerned. The results presented in the paper leads us to conclude that the only way to do a numerical simulation of two-dimensional gravity is to simulate infinite "needles" in a background spacetime expanding in three dimensions and consider the "particles" in the system to be intersections of the needles with any plane orthogonal to them.

Finally, we shall comment on some connection between the current work and the earlier work in this subject, especially that by Gott & Alpert (1984) and Gott (1985). Gott and Alpert obtained the field equations from general considerations rather than from a formal analysis involving a $(D+1)$–dimensional action principle. The fact that they obtained the same equations shows the consistency of the two approaches. Further, they note that point masses do not attract each other in the $(2+1)$–dimensional theory, since the potential remains a constant. This ties in nicely with our result that fluctuations do not grow even when the source is distributed in space. In their analysis the spacetime was treated as empty between the point masses and—since the vanishing of the Ricci tensor implies a vanishing of the Riemann tensor in the same region for $(2+1)$ dimensions—the spacetime turns out to be flat. We note that a distributed source can lead to a nontrivial metric, but gravity is still not effective in forming structures in a $(2+1)$ spacetime! (There exists a closed cosmological model that is static and contains dust, as pointed out by Gott and Alpert. It can also be derived from our approach.) The fact that the $(2+1)$–dimensional simulations are done with needles moving in $(3+1)$–dimensional space connects up with earlier results (Gott 1985, 1991) in which a point mass in $(2+1)$ dimensions is connected with cosmic strings in $(3+1)$ dimensions. There has been an extensive investigation on this theme in the context of static and moving strings that agrees with the results in the present paper. Our analysis puts these earlier results in a physical context and also uses a language and formalism that is more familiar to astronomers and cosmologists from the corresponding (3 + 1) analysis.

The application of the anthropic principle to the growth of fluctuations and the consequent complexity, raised by Gott and Rees (Gott & Rees 1975), is another interesting point, which we do not discuss in this paper for sake of brevity.

K. S. and S. E. thank CSIR, India, for support during the period of work. We would also like to thank an anonymous referee for an extremely useful and detailed set of comments.

REFERENCES

Alimi, J. M., Bouchet, F. R., Pellat, R., Sygnet, J. F., & Moutarde, F. 1990, ApJ, 354, 3
Bagla, J. S., Engineer, S., & Padmanabhan, T. 1998, ApJ, 495, 28
Bagla, J. S., & Padmanabhan, T. 1997, Pramana J. Phys, 42, 161
Deser, S., Jackiw, R., & t'Hooft, G. 1984, Ann. Phys., 152, 220
Gott, J. R. 1985, ApJ, 288, 422
Gott, J. R. 1991, Phys. Rev. Lett., 66, 1126
Gott, J. R., & Alpert, M. 1984, Gen. Relativ. Gravitation, 16, 243
Gott, J. R., & Rees, M. J. 1975, A&A, 45, 365
Landau, L., & Lifshitz, E. 1975, Course of Theoretical Physics, Vol. 2 (4th ed.; Oxford: Pergamon Press)
Melott, A. L. 1983, MNRAS, 202, 595
Padmanabhan, T. 1996, Cosmology and Astrophysics through Problems (Cambridge: Cambridge Univ. Press)
Sathyaprakash, B. S., Sahni, V., Munshi, D., Pogosyan, D., & Melott, A. L. 1995, MNRAS, 275, 463
Shandarin, S. F., & Zeldovich, Ya. B. 1989, Rev. Mod. Phys., 61, 185
Valinia, A., Shapiro, P. R., Martel, H., & Vishniac, E. T. 1997, ApJ, 479, 46