Higher Dimensional Radiation Collapse and Cosmic Censorship

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We study the occurrence of naked singularities in the spherically symmetric collapse of radiation shells in a higher dimensional spacetime. The necessary conditions for the formation of a naked singularity or a black hole are obtained. The naked singularities are found to be strong in the Tipler's sense and thus violating cosmic censorship conjecture.

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The cosmic censorship conjecture (CCC) put forward some three decades ago by Penrose [1] says that in generic situation all singularities arising from regular initial data are clothed by event horizon. Since CCC remains as an unresolved issue in classical general relativity, examples which appear to violate conjecture remain important and may be valuable when one attempts to formulate notion of CCC in concrete mathematical form. The Vaidya solution [2] is most commonly used as a testing ground for various forms of the CCC. In particular, Papapetrou [3] first showed that this solution can give rise to the formation of naked singularities and thus provided one of the earlier counter examples to CCC. Since then, the solution is extensively studied in gravitational collapse with reference to CCC (see e.g. [4] and references therein). Lately, there has been interest in studying gravitational collapse in higher dimensions [5]. This brief report searches for the occurrence of naked singularities in higher dimensional Vaidya spacetime and if they do, to investigate whether the dimensionality of spacetime has any role in the nature of singularities. We find that higher dimensional Vaidya spacetime admit strong-curvature naked singularities in the Tipler's [6] sense.

The idea that spacetime should be extended from four to higher dimensions was introduced by Kaluza and Klein [7] to unify gravity and electromagnetism. Five dimensional ($5D$) spacetime is particularly more relevant because both $10D$ and $11D$ super-gravity theories yield solutions where a $5D$ spacetime results after dimensional reduction [8]. Hence, we shall confine ourselves to $5D$ case. It should be noted that higher dimensional spacetimes where all dimensions are in the equal foot, like ones to be considered here, are not so realistic, as we are living in effectively 4-dimensional spacetime. So in principle one might expect that by dimensional reduction the higher dimensional spacetime should reduce to our 4-dimensional world. In this sense, the models considered in this paper are ideal.

The metric of collapsing Vaidya models in $5D$ case is [9]

$$ds^2 = -(1 - \frac{m(v)}{r^2})dv^2 + 2dvdr + r^2 d\Omega^2 \qquad (1)$$

where v is null coordinate which represents advanced Eddington time with $-\infty < v < \infty$, r is radial coordinate with $0 \le r < \infty$, $d\Omega^2 = d\theta_1^2 + sin^2\theta_1(d\theta_2^2 + sin^2\theta_2 d\theta_3^2)$ is a metric of 3 sphere and the arbitrary function $m(v)$ (which is restricted only by the energy conditions), represents the mass at advanced time v. The energy momentum tensor can be written in the form

$$T_{ab} = \frac{3}{2r^3}\dot{m}(v)k_a k_b \qquad (2)$$

with the null vector k_a satisfying $k_a = -\delta_a^v$ and $k_a k^a = 0$. We have used the units which fix the speed of light and gravitational constant via $8\pi G = c = 1$. Clearly, for the weak energy condition to be satisfied we require that $\dot{m}(v)$ to be non negative, where an over-dot represents a derivative with respect to v. Thus mass function is a non-negative increasing function of v and that radiation is imploding.

The physical situation here is that of a radial influx of null fluid in an initially empty region of the higher dimensional Minkowskian spacetime. The first shell arrives at $r = 0$ at time $v = 0$ and the final at $v = T$. A central singularity of growing mass is developed at $r = 0$. For $v < 0$ we have $m(v) = 0$, i.e., higher dimensional Minkowskian spacetime, and for $v > T$, $\dot{m}(v) = 0$, $m(v)$ is positive definite. The metric for $v = 0$ to $v = T$ is higher dimensional Vaidya, and for $v > T$ we have the higher dimensional Schwarzschild solution. In order to get an analytical solution for our higher dimensional case, we choose $m(v) \propto v^2$. To be specific we take

$$m(v) = \begin{cases} 0, & v < 0, \\ \lambda v^2 (\lambda > 0) & 0 \le v \le T, \\ m_0 (> 0) & v > T. \end{cases} \qquad (3)$$

and with this choice of $m(v)$ the spacetime is self similar [10], admitting a homothetic Killing vector ξ^a given by

$$\xi^a = r\frac{\partial}{\partial r} + v\frac{\partial}{\partial v} \qquad (4)$$

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which satisfies the condition

$$\text{Ł}_\xi g_{ab} = \xi_{a;b} + \xi_{b;a} = 2g_{ab} \tag{5}$$

, Ł denotes the Lie derivative. It follows that $\xi^a k_a$ is constant along radial null geodesics and hence a constant of motion,

$$\xi^a K_a = rK_r + vK_v = C \tag{6}$$

Let $K^a = \frac{dx^a}{dk}$ be the tangent vector to the null geodesics, where k is an affine parameter. Geodesic equations, on using the null condition $K^a K_a = 0$, take the simple form

$$\frac{dK^v}{dk} + \frac{m(v)}{r^3}(K^v)^2 = 0 \tag{7}$$

$$\frac{dK^r}{dk} + \frac{\dot{m}(v)}{2r^2}(K^v)^2 = 0 \tag{8}$$

Following [11, 12], we introduce

$$K^v = \frac{P}{r} \tag{9}$$

and, from the null condition, we obtain

$$K^r = \left(1 - \frac{m(v)}{r^2}\right)\frac{P}{2r} \tag{10}$$

The function $P(v,r)$ obeys the differential equation

$$\frac{dP}{dk} - \left(1 - \frac{3m(v)}{r^2}\right)\frac{P^2}{2r^2} = 0 \tag{11}$$

Radial null geodesics of the metric (1), by virtue of Eqs. (9) and (10), satisfy

$$\frac{dr}{dv} = \frac{1}{2}\left[1 - \frac{m(v)}{r^2}\right] \tag{12}$$

Clearly, the above differential equation has a singularity at $r = 0$, $v = 0$. The nature (a naked singularity or a black hole) of the collapsing solutions can be characterized by the existence of radial null geodesics coming out from the singularity. Eq. (12), upon using eq. (3), turns out to be

$$\frac{dr}{dv} = \frac{1}{2}\left[1 - \lambda X^2\right] \tag{13}$$

where $X \equiv v/r$ is the tangent to a possible outgoing geodesic. In order to determine the nature of the limiting value of X at $r = 0$, $v = 0$ on a singular geodesic, we let

$$X_0 = \lim_{r\to 0\ v\to 0} X = \lim_{r\to 0\ v\to 0} \frac{v}{r} \tag{14}$$

Using (13) and L'Hôpital's rule we get

$$X_0 = \lim_{r\to 0\ v\to 0} X = \lim_{r\to 0\ v\to 0} \frac{v}{r} = \lim_{r\to 0\ v\to 0} \frac{dv}{dr} = \frac{2}{1 - \lambda X_0^2} \tag{15}$$

which implies,

$$\lambda X_0^3 - X_0 + 2 = 0 \tag{16}$$

This algebraic equation governs the behavior of the tangent vector near the singular point. Thus by studying the solution of this algebraic equation, the nature of the singularity can be determined. The central shell focusing singularity is at least locally naked (for brevity we have addressed it as naked throughout this paper), if eq. (16) admits one or more positive real roots. When there are no positive real roots to eq. (16), the central singularity is not naked because in that case there are no outgoing future directed null geodesics from the singularity. Hence in the absence of positive real roots, the collapse will always lead to a black hole. The condition under which this locally naked singularity could be globally naked is well discussed [4] and we shall not discuss it here. Thus, the occurrence of positive real roots implies that the strong CCC is violated, though not necessarily the weak CCC. We now examine the condition for the occurrence of a naked singularity. Eq. (16) has two positive roots if $\lambda \leq 1/27$, e.g., the two positive roots of eq. (16) $X_0 = 2.21833$ and 5.69593, correspond to $\lambda = 1/50$. Thus referring to above discussion, gravitational collapse of null fluid in higher dimensions leads to a naked singularity if $\lambda \leq 1/27$ and to formation of a black hole otherwise. The degree of inhomogeneity of collapse is defined as $\mu \equiv 1/\lambda$ (see [13]), we see that for a collapse sufficiently inhomogeneous, naked singularities develop. Comparison with analogous $4D$ case shows that naked singularity occurs for a slightly larger value of inhomogeneity factor in $5D$.

The analysis of geodesics escaping to far away observers emanating from singularity is similar to that discussed in [3, 11]. To see this, we first note that the apparent horizon in our case is given by $X = 1/\sqrt{\lambda}$. We write the equation of geodesics in the form $r = r(X)$. Using eqs. (3) and (13) we obtain,

$$r\frac{dX}{dr} = \frac{\lambda X^3 - X + 2}{1 - \lambda X^2} \tag{17}$$

Integration of the above equation yields the equation of geodesics (integral curves). Here, we briefly describe constraints when geodesics can escape to a far away observer. Basically this follows from analysis of eq. (17) and its solution. We consider separately the following two cases.
Case (a) $\lambda < 1/27$

The solution of eq. (17) is

$$r = \frac{(x - a_1)^{C_1}}{(x - a_2)^{C_2}(x + a_3)^{C_3}} \tag{18}$$

where a_1 and a_2 are two positive roots with $a_1 > a_2$ and $-a_3$ is the negative root of equation eq. (16). C_1, C_2 and C_3 are given by

$$C_1 = \frac{a_2 a_3}{(a_1 - a_2)(a_1 + a_3)}, \quad C_2 = \frac{a_1 a_3}{(a_1 - a_2)(a_2 + a_3)}, \quad C_3 = \frac{a_1 a_2}{(a_1 + a_3)(a_2 + a_3)}$$

For $X \geq 1/\sqrt{\lambda}$, all geodesics are ingoing and no geodesics escapes. For $X < 1/\sqrt{\lambda}$, geodesic families can meet the singularity with tangent $X = a_1$ and $r = \infty$ can be realized in future along the same geodesic at $X = a_2$. Thus the singularity will be globally naked and an infinity of geodesics would escape from singularity to reach far away observer.

It is also seen that for very small value of λ, say $O(10^{-17})$, $X \geq 1/\sqrt{\lambda}$ and hence singularity is only locally naked.

Case (b) $\lambda = 1/27$

In this case, the eq. (17) takes the form

$$r\frac{dX}{dr} = \frac{(X-3)^2(X+6)}{27 - X^2} \tag{19}$$

which has a trivial solution $X = 3$. In the variables (t, r), it takes the form $t = 2r$ which is a special null geodesic. In analogous $4D$ case one gets $t = 3r$ [3]. For $X \neq 3$, eq. (17) can be integrated to

$$r = \frac{Exp\left(\frac{2}{3-X}\right)}{(3-X)^{8/9}(X+6)^{1/9}} \tag{20}$$

As in $4D$ case [3], we can conclude that there is an infinite number of geodesics escaping to a far away observer.

The strength of singularity is an important issue because there have been attempts to relate it to stability [14]. A singularity is termed gravitationally strong or simply strong, if it destroys by crushing or stretching any object which falls in to it. A sufficient condition [15] for a strong singularity as defined by Tipler [6] is that for at least one non-space like geodesic with affine parameter k, in limiting approach to singularity, we must have

$$\lim_{k\to 0} k^2\psi = \lim_{k\to 0} k^2 R_{ab}K^aK^b > 0 \tag{21}$$

where R_{ab} is the Ricci tensor. Eq. (21), with the help of eqs. (2), (3) and (9), can be expressed as

$$\lim_{k\to 0} k^2\psi = \lim_{k\to 0} 3\lambda X(\frac{kP}{r^2})^2 \tag{22}$$

Our purpose here is to investigate the above condition along future directed null geodesics coming out from the singularity. For this, solution of eq. (11) is required. Eq. (6), because of eqs. (3), (9) and (10), yields

$$P = \frac{2C}{2 - X + \lambda X^3} \tag{23}$$

which is a solution of eq. (11) and geodesics are completely determined. Further, we note that

$$\frac{dX}{dk} = \frac{1}{r}K^v - \frac{X}{r}K^r \tag{24}$$

which, on inserting the expressions for K^v and K^r, become

$$\frac{dX}{dk} = (2 - X - \lambda X^3)\frac{P}{2r^2} = \frac{C}{r^2} \tag{25}$$

Using the fact that as singularity is approached, $k \to 0$, $r \to 0$ and $X \to a_+$ (a root of (16)) and using L'Hôpital's rule, we observe

$$\lim_{k\to 0}\frac{kP}{r^2} = \frac{2}{1+\lambda X_0^2} \tag{26}$$

and hence eq. (22) gives

$$\lim_{k\to 0} k^2\psi = \frac{12\lambda X_0}{(1+\lambda X_0^2)^2} > 0 \tag{27}$$

Thus along radial null geodesics strong curvature condition is satisfied.

Recently, Nolan [16] gave an alternative approach to check the nature of singularities without having to integrate the geodesic equations. It was shown in [16] that a radial null geodesic which runs into $r = 0$ terminates in a gravitationally weak singularity if and only if $\dot{r}$ is finite in the limit as the singularity is approached (this occurs at $k = 0$), the over-dot here indicates differentiation along the geodesics. So assuming a weak singularity, we have

$$\dot{r} \sim d_0 \qquad r \sim d_0 k \tag{28}$$

Using the asymptotic relationship above and the form of $m(v)$, the geodesic equations yield

$$\frac{d^2v}{dk^2} \sim \alpha k^{-1} \tag{29}$$

where $\alpha = -\lambda X_0^4 d_0$ = a non zero const., which is inconsistent with $\dot{v} \sim d_0 X_0$, which is finite. Since, the coefficient α of k^{-1} is non-zero, the singularity is gravitationally strong. Having seen that naked singularity is a strong curvature singularity, we now turn our attention to check it for scalar polynomial singularity. We therefore examine the behavior of Kretschmann scalar ($K = R_{abcd}R^{abcd}$, R_{abcd} is the Riemann tensor). The Kretschmann scalar with the help of eq. (3), takes the form

$$K = \frac{72}{r^4}X^4 \tag{30}$$

which diverges at the naked singularity and hence the singularity is a scalar polynomial singularity. The Weyl scalar ($C = C_{abcd}C^{abcd}$, C_{abcd} is the Weyl tensor) has the same expression as Kretschmann scalar and thus the Weyl scalar also diverges at the naked singularity and so the singularity is physically significant [17].

The Vaidya metric in $4D$ case has been extensively used to study the formation of naked singularity in spherical gravitational collapse [4]. We have extended this study to higher dimensional Vaidya metric, and found that strong curvature naked singularities do arise for slightly higher value of inhomogeneity parameter. We have checked, for naked singularities to be gravitationally strong, by method in [15] and by alternative approach proposed by Nolan [16] as well, and both seem to be in agreement. It is straight forward to extend above analysis for non radial causal curves. Further, the Kretschmann and Weyl scalars blows up as singularity is approached. In conclusion, this offers a counter example to CCC.

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