Variational calculus: a not so conventional but possible tool to explore the coronal magnetic field

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Outline

1. Variational calculus: a brief recap.

2. As a tool to explore different physical systems: two examples.

3. Possibility of applying variational calculus to explore coronal magnetic field, a non conventional approach.

4. A prototype example based on the two-fluid magnetohydrodynamics.

Variational Calculus: a brief recap

$$\text{Let } f \rightarrow f(g, g', h) \text{ with } g \rightarrow g(h), \quad g' \rightarrow g'(h), \quad g'(h) = dg/dh$$

⇩

$$\text{Construct the functional } I = \int_{h1}^{h2} dh\, f(g, g', h)$$

The variational calculus essentially deals with finding extremal dependence of the function f on its variables which extremizes the functional I(g,g').

⇩

$$\delta I = 0$$

Given a physical system, how to choose the functional???

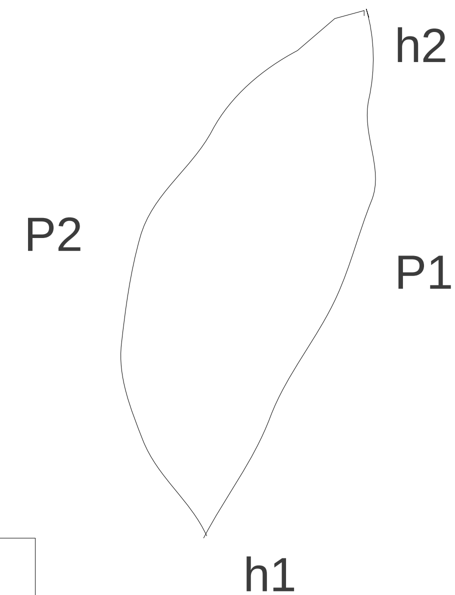
Two examples:

Example1. Application of Hamilton's principle to obtain the equation of motion

$$I=\int_{h1}^{h2} dh\, f(g,g',h)$$

$$g\to g(h,\epsilon),\quad g'\to g'(h,\epsilon)$$

By varying ϵ; $g,g'\to f(g,g',h)$ is varied.
Hence I is different for different paths.



Recipe => Find the path **with fixed end points** for which I is extremum .

Vary I by varying $\epsilon\to$ hence g; $\delta I=0$ yields

$$\int_{h1}^{h2} dh\left[\frac{\partial f}{\partial g}-\frac{d}{dh}\left(\frac{\partial f}{\partial g'}\right)\right]\left(\frac{\partial g}{\partial \epsilon}\right)\delta\epsilon=0$$

$\left(\frac{\partial g}{\partial \epsilon}\right)\delta\epsilon \to \delta g$

For arbitrary variations in g, the integrand must be zero

$$\frac{\partial f}{\partial g}-\frac{d}{dh}\left(\frac{\partial f}{\partial g'}\right)=0$$

$g\rightarrow q_i, \quad g'\rightarrow \dot{q}_i, \quad h\rightarrow t$

$f(g,g',h)\rightarrow L(q_i,\dot{q}_i,t)\equiv T-V$

Euler-Lagrange equation ⇦

$$\frac{\partial L}{\partial q_i}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)=0$$

Example2: Constrained maximization of entropy

Define Gibbs's entropy ⟶ $S=\int \rho \ln \rho \, d^{3N}p \, d^{3N}q$

Define the functional $I=\int \left[\rho \ln \rho - a_1 \rho H - a_2 \rho\right] d^{3N}p \, d^{3N}q$

$Take\, \delta I=0, \quad along\, with\, 1-a_2=-\ln c_1$

$$\int \delta\rho\left[\ln(\rho/c_1)-a_1 H\right] d^{3N}p \, d^{3N}q=0$$

$Choose\, a_1\, such\, that\, \ln(\rho/c_1)-a_1 H=0$

$$\rho=c_1 e^{-a_1 H}$$

⇨ MB distribution

Conventional treatment of the coronal magnetic field:

$$\rho \frac{dv}{dt} = j \times B - \nabla p + \mu \nabla^2 v - \rho g$$

Further approximation: the coronal medium is so tenuous

$$\beta = \frac{\nabla p}{j \times B} = \frac{p}{B^2/2\mu_0} \ll 1$$

⇓

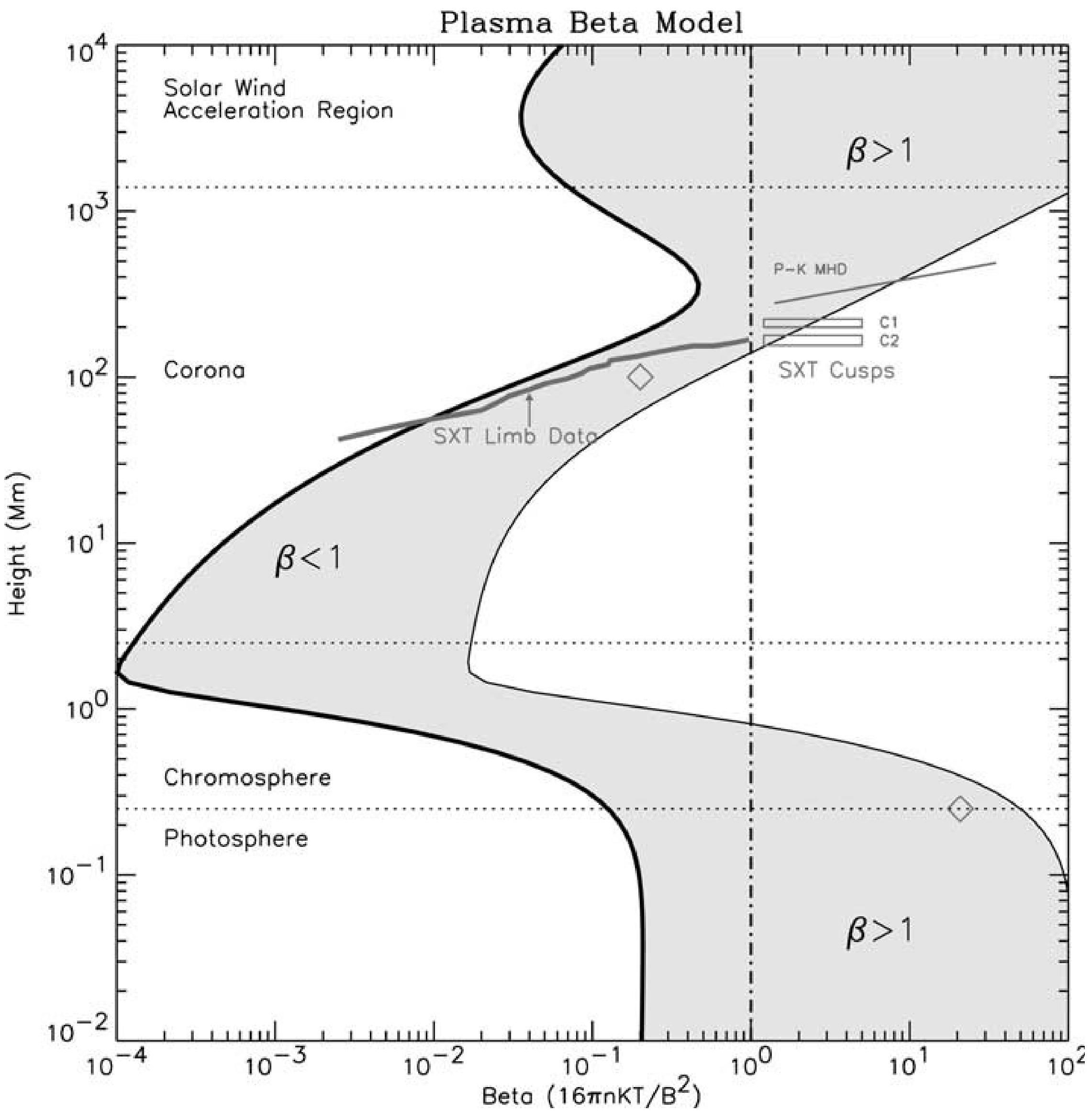
$$j \times B = 0$$ Force-free equation

$$\nabla \times B = 0$$ Potential fields

$$\nabla \times B = \alpha B$$ NLFF

$$\nabla \times B = \alpha(r) B$$ LFF

Is coronal magnetic field really force-free?



• The plasma β in the solar corona is not always less than 1 (G.A. Gary, 2001). It is only the mid-corona where β <<1.

Also

• The extrapolated non force-free magnetic field solution is in good agreement with the observed TRACE image of active region 10987 (G.A. Gary, 2009).

• The non-force-free field fits better to the magnetogram than other three - linear, non linear and the potential magnetic field (Yang Liu, 2002; Pierre & Neukirch's, 2000).

If the coronal field is not globally force-free then how can we model it?

⇩

A suitably formulated variational problem may come handy

We consider the corona to be a near-ideal electron-proton magnetofluid characterized by a very high but finite magnetic Reynolds number and a moderately high fluid Reynolds number.

Not a bad approximation since for the corona

$$R_M = vL/\eta \approx 10^9, \quad R_F = \rho v L/\mu \approx 10$$

Resistivity ~ 10^{-7} ohm-m
Viscosity ~ 10^9 $m^2 s^{-1}$

- Magnetic field lines are frozen into the fluid to a good approximation.

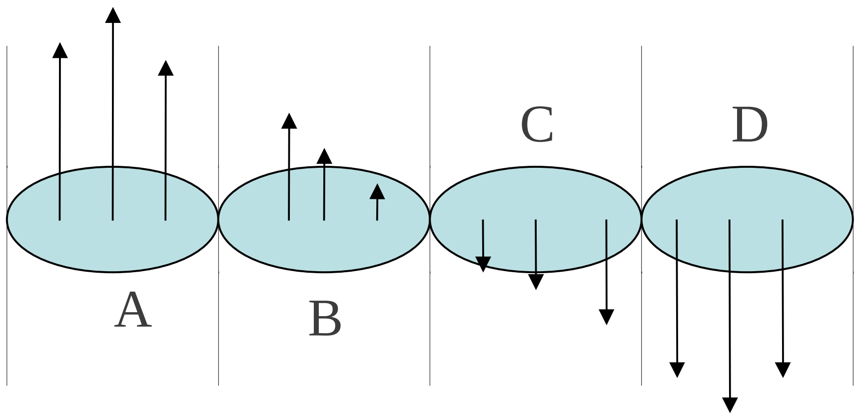
- Viscous effects and hence the velocity-magnetic field coupling cannot be neglected altogether.

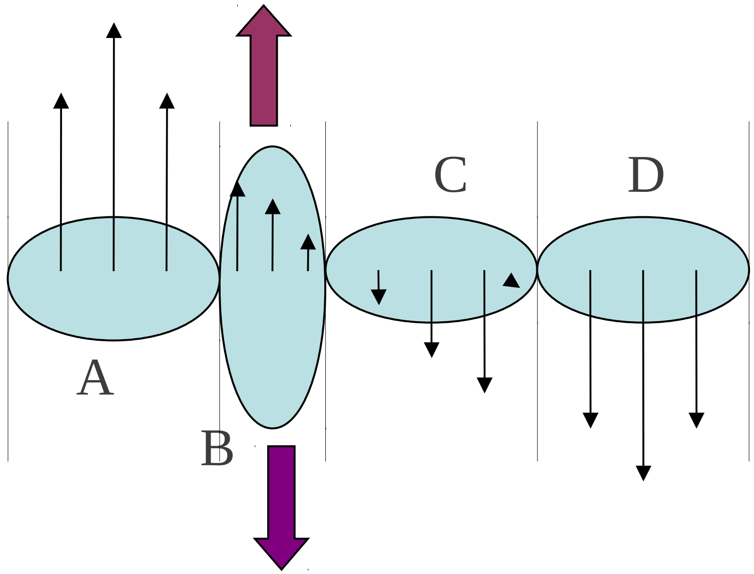
- Consider the magnetofluid to be partitioned into contiguous sub-volumes each entrapping a certain magnetic field.

- The initial field is not in equilibrium and is continuous everywhere.

- Because of the unbalanced Lorentz force, two such sub-volumes may push into each other by ejecting a third interstitial sub-volume. Depending on the orientations of the interacting fields a Current Sheet (CS) may form at the common surface of interaction.

- Small scale effects become important, magnetic fields diffuse from one sub-volume to another and reconnect converting magnetic energy into heat and kinetic energy, the latter being dissipated by viscosity.





Minimizer $\Rightarrow$ $$W=\int\left(\frac{B^2}{2\mu_0}+\frac{\rho\ v^2}{2}\right)d\tau$$

The invariants are ion and electron generalized helicities appropriate for an open system like the solar corona (Dinesh Kumar and R. Bhattacharyya, Phys. Plasmas, in press.)

$$K_i=\int(A+v)\cdot\nabla\times(A+v)d\tau-\int(A'+v')\cdot\nabla\times(A'+v')d\tau$$

$$K_e=\int A\cdot(\nabla\times A)d\tau$$

With the boundary conditions $\Rightarrow$

$$(B-B')\cdot n=0$$
$$(\omega-\omega')\cdot n=0$$
$$\omega=\nabla\times v$$

It can be shown that the total magnetofluid energy decays at a faster rate than both the ion and the electron generalized helicities.

With $\delta[W+\lambda_e K_e+\lambda_i K_i]=0$

The Euler Lagrange equations are obtained as

$$\nabla \times B+(\lambda_e+\lambda_i)B+\lambda_i \nabla \times v=0$$

$$v+\lambda_i \nabla \times v+\lambda_i B=0$$

$For\ \lambda_i=0,\ \ \nabla \times \boldsymbol{B}=\lambda_e \boldsymbol{B}$ ⇒ Linear force-free equation

The Euler-Lagrange equations are solved in terms of

- the non force-free parameter a
- The twist parameter γ

in Cartesian coordinates where z=0 plane represents the photosphere and z>0 half volume denotes the corona.

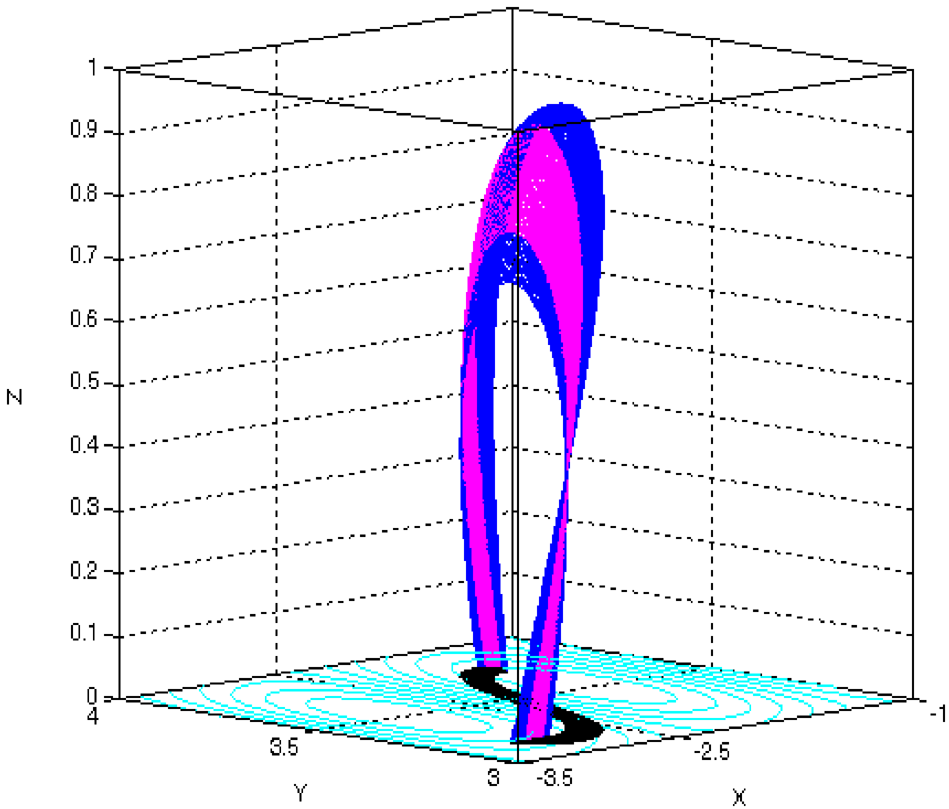


Figure-1

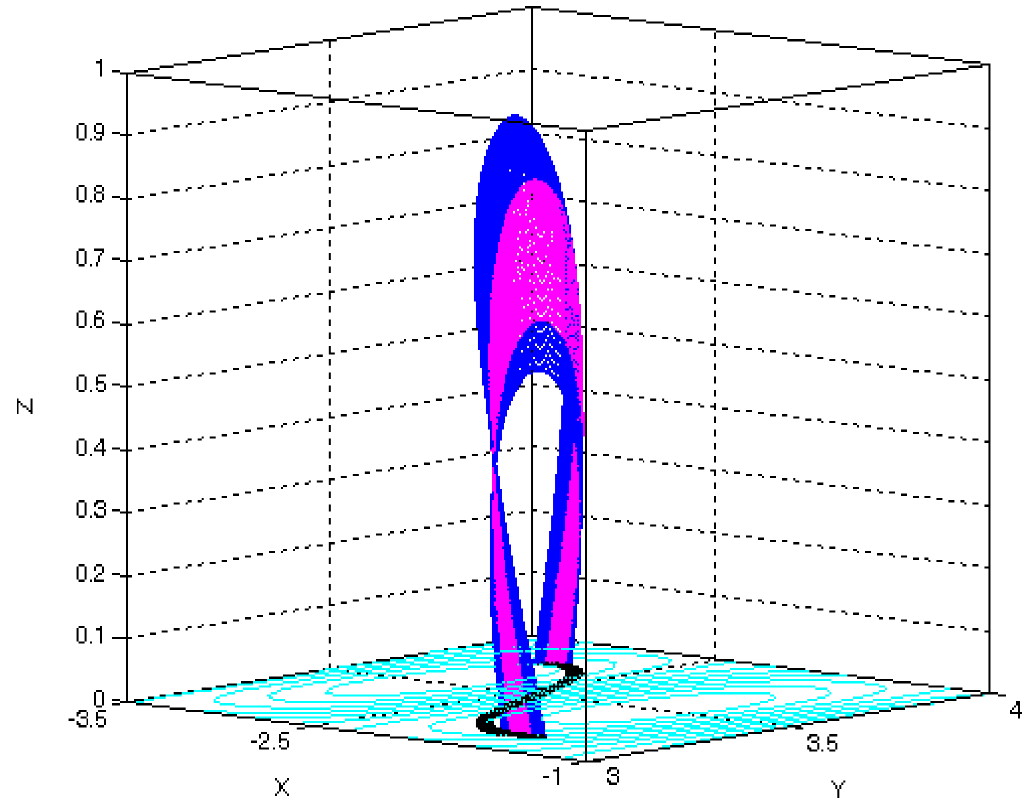


Figure-2

Figure-1 corresponds to the non-force-free twisted magnetic ribbons with a=0.25 and γ=3.7, of which the projection the xy-plane generates the forward sigmoids. Figure-2 characterizes the same non-force-free solution with a=0.25 and γ=-3.7, of which the projection on the xy-plane generates the backward sigmoidal structure.

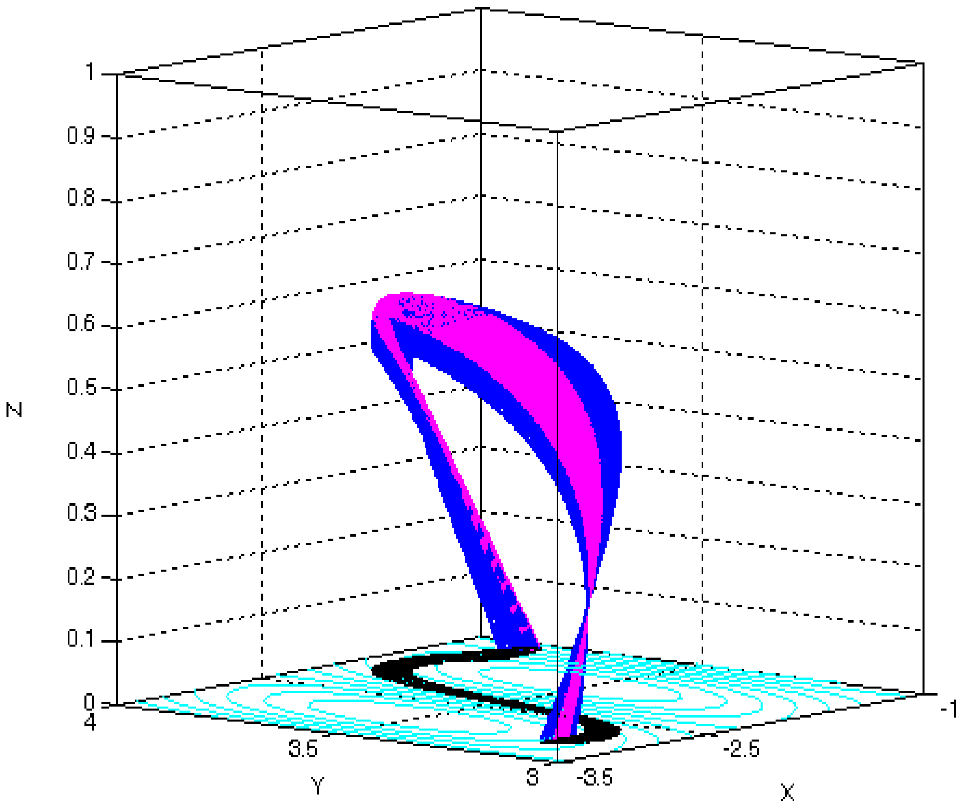


Figure-3

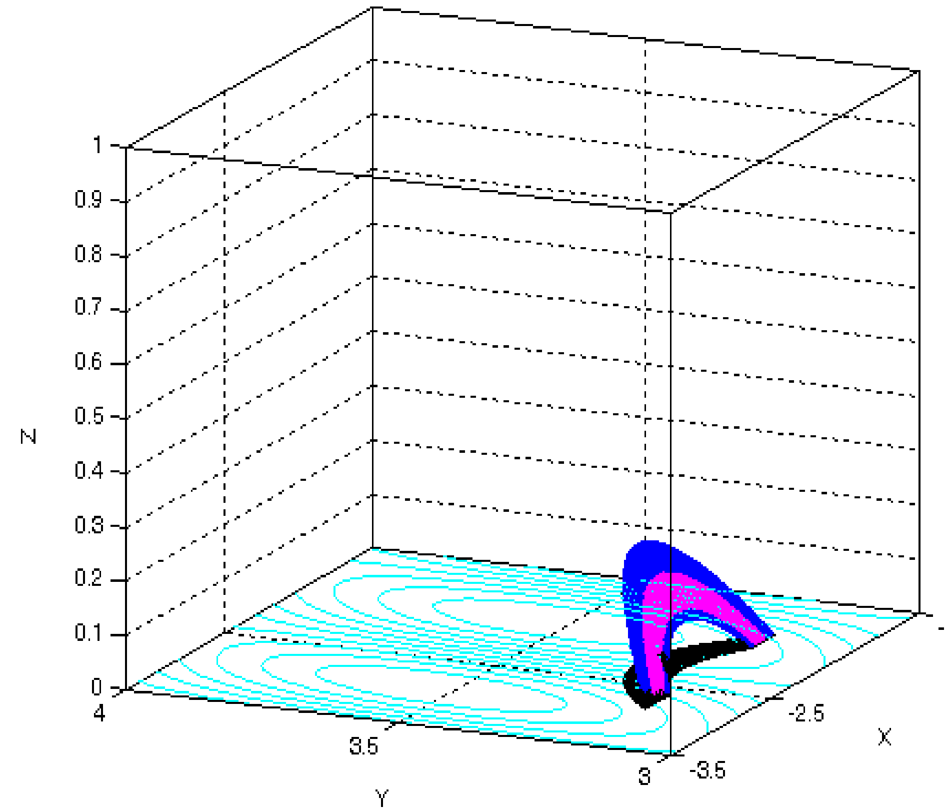


Figure-4

Figure-3 represents the twisted force-free magnetic ribbons with a=0 and γ=3.7, the projection on the xy-plane trace the form of the sigmoids. Figure.-4 corresponds to the potential magnetic field with α=0 and γ=0 which is untwisted.

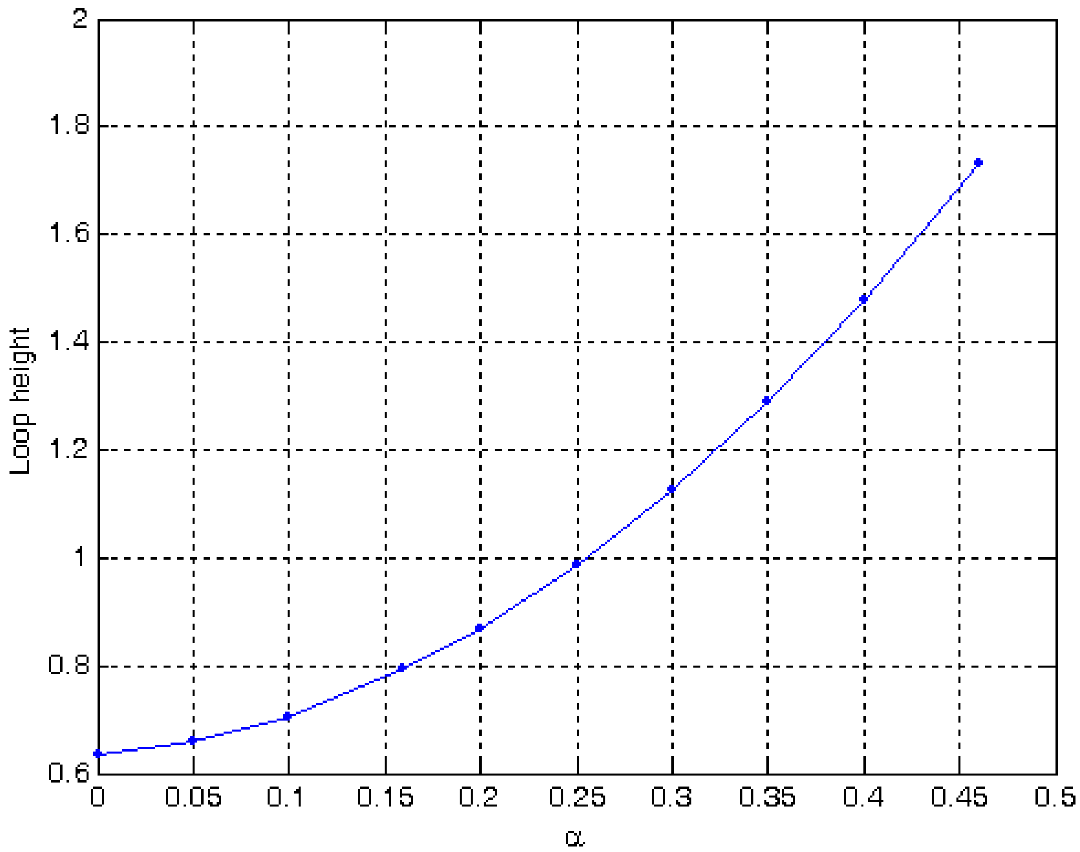


Figure-5

This typical plot is for the variation of the loop height of the non-force-free field lines against the non-force-free parameter a.

Summary

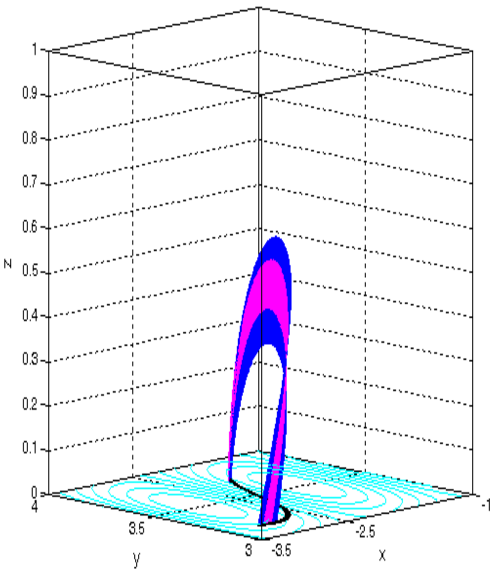
- We started with a brief recap of the variational calculus.

- The conventional treatment of the coronal magnetic field renders it force-free.

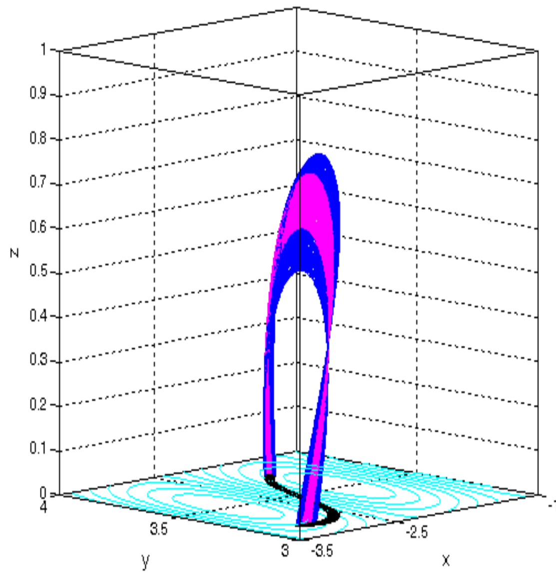
- There are instances/interpretations where the coronal magnetic field is not force-free and we believe that in those cases an approach based on variational calculus may become helpful.

- Here, we have presented one such prototype example where the Euler-Lagrange equations are obtained by minimizing the total magnetofluid energy (magnetic + kinetic) while keeping the ion and electron generalized helicities invariant.

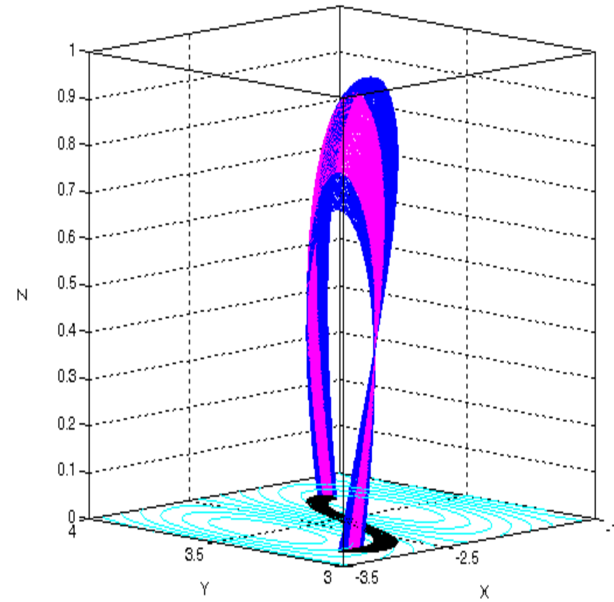
- The resulting magnetic field is non force-free in nature and mimics, at least qualitatively, realistic coronal structures.



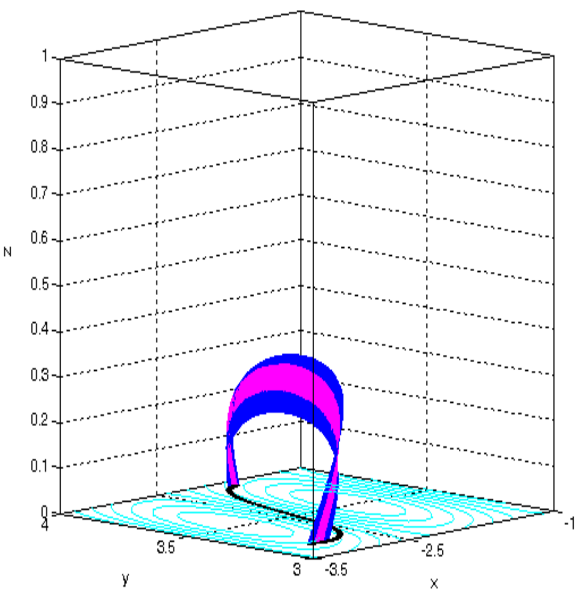
α=0.25, γ=3.6



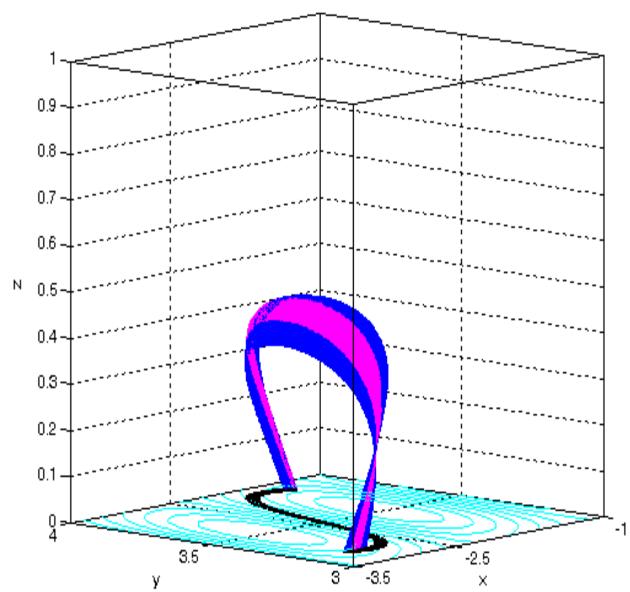
α=0.25, γ=3.65



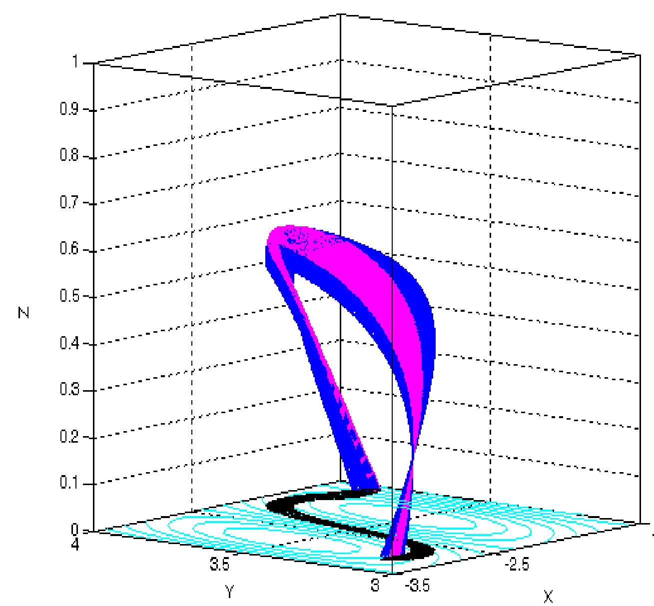
α=0, γ=3.7



α=0, γ=3.6



α=0, γ=3.65



α=0, γ=3.65