Plane symmetric perfect fluid distributions admitting a one - parameter group of conformal motions

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Abstract. Some exact analytic solutions of Einstein's equations with perfect fluid source have been found, under the assumptions of (i) plane symmetry and (ii) the existence of a one - parameter group of conformal motions, with the generator in the hypersurface. The solutions are algebraically special (Petrov type D) and belong to class I of Wainwright classifications. They are non- static with non - vanishing shear. First of the solutions represent an expanding homogeneous distribution of matter, which evolves from a singular state at t=0. Second one is conformally flat and represents homogeneous density, but inhomogeneous pressure distributions with shear - free motions. The last solution is inhomogeneous in density as well as pressure.

I. Introduction

A space - time that admits the three parameter group of motions of the Euclidean plane is said to passess plane symmetry and is called a plane symmetry space - time. Such space - times have many properties similar to those of spherically symmetric and hyperbolic symmetric ones. For example, they obey the Birkoff theorem [1] , vacuum solutions of the field equations for such space - times are of the same Petrov class as the corresponding spherically symmetric ones [2]. Plane symmetric space - times with perfect fluid as the source, have been studied due to possible applications to Astrophysics, Cosmology and special-relativistic hydodynamics [3]. On account of symmety involved in this case, the complicated non - linear gravitational fluid equations become, with some additional assumptions like shearfree motion, validity of an equation of state or some other type of symmetry etc., mathematically manageable.

Here in this paper we assume that the space - time with perfect fluid as the source of gravitational field admits, besides the plane symmetry, a one - parameter group of conformal motions, i. e.,

$$L_{\xi} g_{ij} = \xi_{i;j} + \xi_{j;i} = \psi g_{ij} \tag{1}$$

where L stands for the Lie derivative and ψ is an arbitrary function of the coordinates. Solutions with spherical symmetry under these assumptions, have been found by Herrera and Leon [4].

Under the above assumptions, we are able to integrate the Einstein's equations and physical properties of the solutions are investigated. It is found that the real null congruences 'l' and 'n' are geodesic and shearfree; the solution are algebraically special (Petrov type D) and belong to class I of Wainwright classifications [5].

The paper is organised as follows: The field equations and conventions are given in section II. The field equations are integrated in section III. Special solutions are derived and their physical and kinematical parameters are given in the section IV. Section V includes conclusion.

II. The field equations and notations

We adopt the notation and convention of Landau and Lifshitz [6]. For field equations, spin coefficients etc., we refer to Kramer et. al. [7]. We assume the coordinates to be comoving.

Let us consider a non - static perfect fluid distribution having plane symmetry with the line - element

$$ds^2 = e^{2\nu}dt^2 - e^{2\lambda}dx^2 - e^{2\mu}(dx^2 + dy^2), \tag{2}$$

with

$$x^i = x^{0,1,2,3} = t, x, y, z$$

and

where ν, λ and μ are functions of 'x' and 't'.

The energy - momentum tensor is given by

$$T^i_j = (\rho + p)U^iU_j - p\delta^i_j, \tag{3}$$

where ρ,p and U^i denote the proper density, pressure and four velocity of the fluid respectively. We have

$$U^i = \delta^i_0 e^{-\nu} \tag{4}$$

The Einstein's field equations

$$G^i_j = 8\pi T^i_j, \tag{5}$$

where we have chosen the units such that c=G=1 read, as

$$-2e^{-2\lambda}(\mu'' + 3/2\mu^2 - \lambda'\mu') + e^{-2\nu}(\dot{\mu}^2 + 2\dot{\mu}\dot{\lambda}) = 8\pi T^0_0 = 8\pi\rho, \tag{6}$$

$$-e^{-2\lambda}(\mu'^2 + 2\mu'\nu') + 2e^{-2\nu}(\ddot{\mu} - \dot{\mu}\dot{\nu} + 3/2\dot{\mu}^2) = 8\pi T^1_1 = -8\pi p, \tag{7}$$

$$-e^{-2\lambda}(\nu''+\nu'^2-\lambda'\nu'+\mu''+\mu'^2+\mu'\nu'-\lambda'\mu')+e^{-2\nu}(\ddot{\lambda}+\dot{\lambda}^2-\dot{\lambda}\dot{\nu}+\ddot{\mu}+\dot{\mu}^2+\dot{\mu}\dot{\lambda}-\dot{\mu}\dot{\nu}) = 8\pi T^2_2$$

$$= 8\pi T^3_3 = -8\pi p, \tag{8}$$

$$2e^{-(\nu+\lambda)}(\dot{\mu}\mu' - \dot{\lambda}\mu' + \dot{\mu}' - \dot{\mu}\nu') = 8\pi T^0_1 = 0. \tag{9}$$

where dots and primes denote differntiation with respect to t and x respectively. Now we assume that the space - time admits a one - parameter group of conformal motion given by (1.1) with the restriction that the vector ξ is orthogonal to fluid velocity vector, i.e.

$$\xi^i U_i = 0 \tag{10}$$

The equation (10) in view of the plane symmetry implies that

$$\xi^0 = \xi^2 = \xi^3 = 0, \xi^1 = \xi (say). \tag{11}$$

Thus using (1) and (11), we get the following set of equations

$$\xi \nu' = \psi/2, \tag{12}$$

$$\dot{\xi} = 0, \tag{13}$$

$$\xi' + \xi \lambda' = \psi/2, \tag{14}$$

$$\mu' \xi = \psi/2. \tag{15}$$

From (12) and (15) we obtain

$$\nu - \mu = f_1(t), \tag{16}$$

where $f_1(t)$ is an arbitrary function of t.

Now by differentiation of (14) and (15) we get

$$\dot{\lambda}' = \dot{\mu}', \tag{17}$$

from which it follows that

$$\lambda = \mu + f_2(t) + g_1(x). \tag{18}$$

Here $f_2(t)$ and $g_1(x)$ are arbitrary functions of their arguments.

We can perform a coordinate transformation of the form [8].

$$t = t(\bar{t}), r = r(\bar{r}), \tag{19}$$

and may choose $f_1(t) = g_1(x) = 0$

without any loss of generality. We thus obtain

$$\nu = \mu, \lambda = \mu + f(t). \tag{20}$$

Equations (14) and (15) by the use of (20) give $\xi' = 0$, which in view of (13) implies that

$\xi = A = a$ constant, or $\psi = 2A\nu'$ (21)

We now consider the field equation (9) which in view of (20) becomes

$$\dot{\lambda}' - \dot{\lambda}\lambda' = 0. \tag{22}$$

If we define a new variable

$$Z \equiv e^{-\lambda}, \tag{23}$$

then (22) is reduced to

$$\dot{Z}' = 0, \tag{24}$$

whose solution is of the form

$$Z \equiv e^{-\lambda} = h(x) + g(t), \tag{25}$$

where h and g are arbitrary functions of their arguments. Thus we have,

$$e^{-\nu} = e^{-\mu} = e^{f}(h + g). \tag{26}$$

The line - element (2) and the field equations $(6) - (8)$ can be expressed in terms of the functions f, h and g we get

$$ds^2 = \frac{e^{-2f}}{(h+g)^2}[dt^2 - e^{2f}dx^2 - dy^2 - dz^2], \tag{27}$$

$$3(\dot{g}^2 e^{2f} - h'^2) + 2e^{-\lambda}(h'' + 2\dot{f}\dot{g}e^{2f}) + e^{2(f-\lambda)}\dot{f}^2 = 8\pi\rho, \tag{28}$$

$$3(\dot{g}^2 e^{2f} - h'^2) + 2e^{2f-\lambda}(\dot{g}\dot{f} - \ddot{g}) + e^{2(f-\lambda)}(\dot{f}^2 - 2\ddot{f}) = -8\pi p, \tag{29}$$

$$3(\dot{g}^2 e^{2f} - h'^2) + 2e^{-\lambda}(h'' - \ddot{g}e^{2f}) - e^{2(f-\lambda)}\ddot{f} = -8\pi p. \tag{30}$$

III. Integration of the field equations

In this section, we show that the above field equations (28) – (30) can be integrated analytically without any further assumptions of any kind.

The pressure isotropy equations (29) and (30) yield

$$e^{\lambda}(e^{2f}\dot{g}\dot{f} - h'') = \Phi(t), \tag{31}$$

with

$$\Phi(t) = \frac{e^{2f}}{2}(\ddot{f} - \dot{f}^2). \tag{32}$$

Differentiating (31) and (25) with respect to x, we get

$$\lambda'\Phi(t) = h'''e^{\lambda}, \tag{33}$$

and

$$\lambda' = -h'e^{\lambda}, \tag{34}$$

respectively, which give

$$\Phi(t) = -\frac{h'''}{h}. \tag{35}$$

Equation (35) at once implies that

$$\Phi(t) = B = constant. \tag{36}$$

Therefore the first integral of (35) is

$$h'' + Bh = C, \tag{37}$$

where C is a constant. Equation (31), with the help of (25), (36) and (37) is reduced to

$$e^{2f}\dot{g}\dot{f} = Bg + C. \tag{38}$$

Equation (32) has its first integral, and is given as

$$\dot{f}^2 = De^{2f} - Be^{-2f}, \tag{39}$$

where D is a constant.

Thus the symmetry assumptions reduce the nonlinear Einstein equations to ordinary differential equations (37) – (39) for three unknown functions f, g and h. Let us integrate Equation (37). Its first integral is

$$h'^2 = 2ch - Bh^2 + E, E = constant. \tag{40}$$

By defining a new variable

$$R(t,x) = \frac{e^{f(t)}}{h(x) + g(t)}, \tag{41}$$

Eq.(40) may be expressed as

$$h'^2 = 2C(\frac{e^{-f}}{R} - g) - B(\frac{e^{-f}}{R} - g)^2 + E. \tag{42}$$

Now the expressions for density and pressure are given of the equations (28), (29),(37),(38),(41) and (42) by

$$8\pi\rho = \frac{De^{2f}}{R^2} + F(t), \tag{43}$$

$$8\pi p = \frac{De^{2f}}{R^2} - \frac{2De^{f}}{R\dot{f}^2}(C + Bg) - F(t) \tag{44}$$

respectively, and where

$$\frac{F(t)}{3} = 2Cg + Bg^2 - E + (\frac{C + Bg}{\dot{f}e^{f}})^2. \tag{45}$$

For a realistic case $\rho \geq p, F(t)$ must satisfy the following inequality

$$\frac{2De^{f}(C + Bg)}{R\dot{f}^2} + 2F \geq 0. \tag{46}$$

We notice that for the case $\dot{f} \neq 0$, we should have $D \neq 0$, otherwise pressure would be negative.

The gravitational field represented by the metric (2) with the assumed conformal motion, is of Petrov type D. The non - vanishing Weyl's coefficients [7] is Ψ_2 given by

$$\Psi_2 = \frac{1}{3}e^{-2\mu}(\ddot{f} + \dot{f}^2). \tag{47}$$

We have also investigated that the fluid velocity lies in the 2-space Σ spanned by the principal null directions l and n of the gravitational field, and hence the solutions are of the class I of Wainwright's classification [5]. The null congruences of curves determined by these vectors are geodesic and shearfree (spin coefficients $x = \sigma = \nu = \lambda = 0$).

The kinematical parameters of the fluid are given by

$$\theta = -e^{f}[3\dot{g} + 2\dot{f}(h+g)], \tag{48}$$

$$\sigma = \frac{\dot{f}}{\sqrt{3}} e^{f-\lambda}, \tag{49}$$

$$a^{i} \equiv \dot{U}^{i} = -\delta^{i}_{1} h'(h+g). \tag{50}$$

IV. Solutions and their properties

We may obtain different particular solutions by different choices of the constants namely B, C, D and E.

Solution 1: Let us first consider the case

$$B = C = 0. \tag{51}$$

In this case Eqs. (40), (45) and (46) imply that

$$F(t) = E = 0, \tag{52}$$

and so we get from (38) and (40)

$$h(x) = M, g(t) = N, \tag{53}$$

where M and N are integration constants.

We also obtain from (39) and (51) and by redefining the origin of time

$$e^f = \frac{1}{\omega t}, \tag{54}$$

where $$\omega^2 = D. \tag{55}$$

The density and pressure are given by the help of Eqs. (41), (43), (44) and (52) – (54) as

$$\rho = p = \frac{G}{t^4}, \tag{56}$$

where $G = \frac{[\omega(M+N)]^2}{8\pi}$

The line - element has the form

$$ds^2 = (Ht)^2[dt^2 - \frac{dx^2}{(\omega t)^2} - (dy^2 + dz^2)], \tag{57}$$

where $H \equiv \frac{\omega}{M+N}$.

We have

$$\Psi_2 = \frac{2}{3H^2t^4}, \tag{58}$$

$$\lambda' = \nu' = \mu' = \psi = 0. \tag{59}$$

Thus the generator ξ is a space - like killing vector. The kinematical quantities are obtained as

$$\sigma = (\sqrt{3}Ht^2)^{-1}, \tag{60}$$

$$\theta = 2(Ht^2)^{-1}, \tag{61}$$

$$a^i = 0. \tag{62}$$

The solution represents an expanding (ρ is a decreasing function of t) configuration,matter being stiff all the time. It has non - vanishing shear. Shear and expansion both decrease and vanish asymtotically with time.

Solution 2: For the choice,

$$B = D = 0, \tag{63}$$

We get from the Eq. (39) $f = constant$ which may be taken to be zero without loss of any generality. Then (39) at ones implies that

$$C = 0. \tag{64}$$

Therefore from (40) we have

$$h = kx + b, k^2 \equiv E. \tag{65}$$

where b is a constant of integration.

The line - element for this case is

$$ds^2 = (kx + b + g)^{-2}[dt^2 - dx^2 - dy^2 - dz^2], \tag{66}$$

where g(t) is arbitrary. The space - time is conformally flat or the gravitational field is Petrov type O.

The expressions for density and pressure are obtained with the help of (28) and (29) and they read

$$8\pi\rho = 3(\dot{g}^2 - k^2), \tag{67}$$

$$8\pi p = -3(\dot{g}^2 - k^2) + 2\ddot{g}(kx + b + g)^{-1}, \tag{68}$$

with the equation of state given by

$$8\pi(\rho + p) = 2\ddot{g}(kx + b + g)^{-1}. \tag{69}$$

The dominant energy condition [9] demands that the space - time region is limited by an inequality in g and its derivatives given by

$$k^2 \leq \dot{g}^2 \leq \frac{2/3\ddot{g} + k^2}{kx + b + g} \leq (2\dot{g}^2 - k^2). \tag{70}$$

The density distribution is homogeneous whereas the pressure distribution is inhomogeneous.

The kinematical quantities are

$$\theta = -3\dot{g}, \tag{71}$$

$$\sigma = 0, \tag{72}$$

$$a^x = -k(kx + b + g), a^y = a^z = a^t = 0. \tag{73}$$

If k is taken to be zero, then acceleration vanishes and pressure distribution too is homogeneous.

Solution 3: Finally we consider the case

$$C = g = 0. \tag{74}$$

Eq. (37) gives the following solutions for h, namely,

$$h = \begin{cases} A_1 \sin ax + A_2 \cos ax, & B = a^2, \\ B_1 \sinh ax + B_2 \cosh ax, & B = -a^2, \end{cases} \tag{75}$$

Where $A_1, A_2, B_1 and B_2$ are constants of integration.

Density and pressure are given by

$$8\pi\rho = \frac{De^{2f}}{R^2} - 3E, \tag{76}$$

$$8\pi p = \frac{De^{2f}}{R^2} + 3E, \tag{77}$$

where $D > 0, E < 0$.

Let us define a new function R_0 by

$$R_0 = \sqrt{-D/3E}e^f, \tag{78}$$

then in view of (39), R_0 satifies the differential equation

$$\ddot{R}_0 = -6ER_0^3. \tag{79}$$

A particular solution of (79) is

$$R_0 = \frac{-1}{\sqrt{-3E}t}, \tag{80}$$

which gives

$$e^f = \frac{-1}{\sqrt{D}t}. \tag{81}$$

Now the explicit expressions for density and pressure are, in view of (41),(75) –

(77) and (81)

$$8\pi\rho = \begin{cases} \frac{(A_1 \sin ax + A_2 \cos ax)^2}{Dt^4} - 3E, & B = a^2, \\ \frac{(B_1 \sinh ax + B_2 \cosh ax)^2}{Dt^4} - 3E, & B = -a^2, \end{cases} \tag{82}$$

$$8\pi p = 8\pi\rho + 6E. \tag{83}$$

The Weyl's coefficients is

$$\Psi_2 = \frac{2}{3Dt^4} \begin{cases} (A_1 \sin ax + A_2 \cos ax)^2, & B = a^2 \\ (B_1 \sinh ax + B_2 \cosh ax)^2, & B = -a^2 \end{cases} \tag{84}$$

The kinematical parameters read

$$\theta = \frac{-2h}{\sqrt{D}t^2}, \tag{85}$$

$$\sigma = \frac{h}{\sqrt{3D}t^2}, \tag{86}$$

$$a^x = -hh', a^y = a^z = a^t = 0. \tag{87}$$

We observe that both density and pressure distributions are inhomogeneous .Expansion and shear scalars of the fluid decrease to zero asymtotically with time.

V. Conclusion.

We have obtained exact plane symmetric perfect fluid solutions of Einstein's equations assuming one - parameter group of conformal motions. Particular solutions and their geometrical and physical properties have been discussed at length. Distribution is nonstatic shearing and expanding, some solutions are homogeneous while others are inhomogeneous.

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