Braneworld models of dark energy

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Abstract

We explore a new class of braneworld models in which the scalar curvature of the (induced) brane metric contributes to the brane action. The scalar curvature term arises generically on account of one-loop effects induced by matter fields residing on the brane. Spatially flat braneworld models can enter into a regime of accelerated expansion at late times. This is true even if the brane tension and the bulk cosmological constant are tuned to satisfy the Randall–Sundrum constraint on the brane. Braneworld models admit a wider range of possibilities for dark energy than standard LCDM. In these models the luminosity distance can be both smaller and larger than the luminosity distance in LCDM. Whereas models with $d_L \leq d_L(\mathrm{LCDM})$ imply $w = p/\rho \geq -1$ and have frequently been discussed in the literature, models with $d_L > d_L(\mathrm{LCDM})$ have traditionally been ignored, perhaps because within the general-relativistic framework, the luminosity distance has this property *only if* the equation of state of matter is strongly negative ($w < -1$). Within the conventional framework, 'phantom energy' with $w < -1$ is beset with a host of undesirable properties, which makes this model of dark energy unattractive. Braneworld models, on the other hand, have the capacity to endow dark energy with exciting new possibilities (including $w < -1$) without suffering from the problems faced by phantom energy. For a subclass of parameter values, braneworld dark energy and the acceleration of the universe are *transient* phenomena. In these models, the universe, after the current period of acceleration, re-enters the matter-dominated regime so that the deceleration parameter $q(t) \to 0.5$ when $t \gg t_0$, where t_0 is the present epoch. Such models could help reconcile an accelerating universe with the requirements of string/M-theory.

I. INTRODUCTION

One of the most sensational discoveries of the past decade is that our universe is currently accelerating. This observation finds support in the luminosity measurements of high-redshift supernovae [1], measurements of degree-scale anisotropies in the cosmic microwave background [2] and, indirectly, in the observations of gravitational clustering [3]. Although a cosmological constant appears to satisfy all current observations, the formidable fine-tuning difficulties which accompany a non-evolving Λ-term have prompted theorists to investigate alternative, evolving forms of dark energy. Theoretical models of dark energy usually involve matter fields possessing unusual properties such as a negative equation of state. Popular dark-energy models include scalar-field-based 'quintessence' models, models based on quantum particle production, Chaplygin gas etc. (see [4–6] for recent reviews). In this paper, we examine a radically different mechanism for the late time acceleration of the universe in which dark energy effectively emerges from the gravity sector and not the matter sector of the theory.

We study a higher-dimensional cosmology in which the observable universe is a four-dimensional 'brane' embedded in a five-dimensional bulk. Models of this kind appeared after the seminal paper by Randall & Sundrum (RS) [7]. Subsequent intensive investigation showed that cosmology based on the RS model exhibits departure from standard general-relativistic behaviour only at very early cosmological times [8–10]. The model which we study in this paper generalizes the RS model in that, besides the brane and bulk cosmological constants, it also includes the scalar curvature term in the action for the brane. In an interesting recent development, Deffayet, Dvali, and Gabadadze (DDG) [11] demonstrated that the presence of the scalar curvature term in the action for the brane can lead to a late-time acceleration of the universe even in the absence of any material form of dark energy [11–13]. The main difference between our model and the DDG model is that, in addition to the scalar curvature term in the action for the brane, we also include the brane and bulk cosmological constants. (The bulk and brane cosmological constants were set equal to zero in the DDG model.)

The presence of bulk and brane cosmological constants in our braneworld (in addition to the brane curvature term) leads to several qualitatively new features that distinguish our model from the DDG model as well as from the scalar-field-based 'quintessence' models. An important property of our braneworld universe is that the (effective) equation of state of dark energy can be $w < -1$. (The DDG braneworld model and quintessence models have $w > -1$.) In addition, the acceleration of the universe in our braneworld scenario can be a *transient* phenomenon. We find that a class of braneworld models accelerate during the present epoch but revert back to matter-dominated expansion at late times. A *transiently accelerating* braneworld does not possess an event horizon and could reconcile a currently accelerating universe with the demands of string/M-theory.

Our paper is organised as follows. After describing the braneworld model in the next section, in Sec. III we proceed to the case of vacuum braneworlds, that is, solutions with zero stress-energy tensor of matter. We derive a general expression for the effective cosmological constant in such models and describe some classes of its symmetric solutions, including *empty* static homogeneous and isotropic universes. We also present simple expressions for a static universe filled with matter. In Sec. IV we study the cosmological evolution of the braneworld. We shall show that braneworld models have properties similar to the observed (accelerating) universe for a large region of parameter space. For instance, the model under consideration can mimic the evolution of a FRW universe in which dark energy has pressure–to–energy-density ratio p/ρ both larger *as well as smaller* than the critical value of -1. For a subclass of parameter values, braneworld dark energy and the acceleration of the universe are *transient* phenomena so that the universe, after the current period of acceleration, re-enters the matter-dominated regime. In Sec. V we formulate our conclusions.

II. BASIC EQUATIONS

In this paper, we consider the case where a braneworld is the timelike boundary of a five-dimensional purely gravitational Lorentzian space (bulk), which is equivalent to the case of a brane embedded in the bulk with Z_2 symmetry of reflection with respect to the brane. The theory is described by the action [14,15]

$$S = \epsilon M^3 \left[\int_{\rm bulk} \left(\mathcal{R} - 2 \Lambda_{\rm b} \right) - 2 \int_{\rm brane} K \right] + \int_{\rm brane} \left(m^2 R - 2 \sigma \right) + \int_{\rm brane} L \left(h_{ab}, \phi \right) . \qquad (1)$$

Here, $\mathcal{R}$ is the scalar curvature of the metric g_{ab} in the five-dimensional bulk, and R is the scalar curvature of the induced metric $h_{ab} = g_{ab} - n_a n_b$ on the brane, where n^a is the vector field of the inner unit normal to the brane, and the notation and conventions of [16] are used. The quantity $K = K_{ab} h^{ab}$ is the trace of the symmetric tensor of extrinsic curvature $K_{ab} = h^c{}_a \nabla_c n_b$ of the brane. The symbol $L(h_{ab}, \phi)$ denotes the Lagrangian density of the four-dimensional matter fields ϕ whose dynamics is restricted to the brane so that they interact only with the induced metric h_{ab}. All integrations over the bulk and brane are taken with the natural volume elements $\sqrt{-g}\, d^5x$ and $\sqrt{-h}\, d^4x$, respectively, where g and h are the determinants of the matrices of components of the corresponding metrics in a coordinate basis. The symbols M and m denote, respectively, the five-dimensional and four-dimensional Planck masses, $\Lambda_{\rm b}$ is the bulk cosmological constant, and σ is the brane cosmological constant, also called the brane tension.

The parameter $\epsilon = \pm 1$ in the first term of action (1) reflects the possibility of different relative signs between the bulk and brane terms in the action. In the context of homogeneous Friedmann cosmology that will be subject of this paper, the differential equations on the brane will not depend on the sign of ϵ. However, different signs of ϵ result in different ways of embedding of a brane with one and the same induced metric in the bulk space, which may be important for studying perturbations of the theory.

The term containing the scalar curvature of the induced metric on the brane with the coupling m^2 in action (1) is often neglected in the literature. However, this term is qualitatively essential for describing the braneworld dynamics since it is inevitably generated as a quantum correction to the matter action in (1) — in the spirit of an idea that goes back to Sakharov [17] (see also [18]). Note that the effective action for the brane typically involves an infinite number of terms of higher order in curvature (this was pointed out in [14] for the case of braneworld theory; a similar situation in the context of the AdS/CFT correspondence is described in [19]). In this paper, we retain only the terms linear in curvature. The linear effects of the curvature term on the brane were studied in [12,21], where it was shown that it leads to four-dimensional law of gravity on sufficiently small scales. For some recent reviews of the results connected with the induced-gravity term in the brane action, one may look into [22].

Action (1) leads to the bulk being described by the usual Einstein equation with cosmological constant:

$$\mathcal{G}_{ab} + \Lambda_{\rm b} g_{ab} = 0 \, , \tag{2}$$

while the field equation on the brane is

$$m^2 G_{ab} + \sigma h_{ab} = \tau_{ab} + \underline{\epsilon M^3 \left(K_{ab} - h_{ab} K \right)} \, . \tag{3}$$

Here, τ_{ab} is the stress-energy tensor on the brane, and it stems from the last term in action (1). In Eq. (3) we have underlined the term whose presence makes braneworld theory different from general relativity. (It should perhaps be mentioned that the variation of action (1) which includes the Gibbons–Hawking surface term $\int_{\rm brane} K$ leads to the Israel junction conditions [20] on the brane, as demonstrated in [14,15].) From (3) one can show that there exists a scale whose value determines the domain in which general relativity is approximately valid [12,21]. Indeed, we can write

$$G_{ab} \sim r_1^{-2} \,, \quad K_{ab} \sim r_2^{-1} \,, \tag{4}$$

where r_1 and r_2 are the characteristic curvature radii of solution of Eq. (3). Then the ratio of the last term on the right-hand side of Eq. (3) to the first term on its left-hand side is of the order $r_1^2/r_2\ell$, where $\ell = 2m^2/M^3$ is a convenient definition to be used later. Thus, for $r_1^2 \ll r_2\ell$, the last term on the left-hand side of Eq. (3) can be neglected, and general relativity is recovered. In typical situations, e.g., in cosmology, we often have $r_1 \sim r_2 \sim r$, and general relativity becomes valid for $r \ll \ell$. This fact was used in [12], where the cosmological model with $\sigma = 0$ and $\Lambda_{\rm b} = 0$ was considered. Therefore, an important distinguishing feature of the braneworld models which we consider, is that they can show departure from standard FRW behaviour at *late times* and on large scales. This property should be contrasted with the braneworld cosmology based on the Randall–Sundrum model [7] (with $m = 0$), in which nonstandard behaviour is encountered at very early times [8–10].

By contracting the Gauss identity

$$R_{abc}{}^{d} = h_a{}^{f} h_b{}^{g} h_c{}^{k} h^{d}{}_{j} \mathcal{R}_{fgk}{}^{j} + K_{ac} K_b{}^{d} - K_{bc} K_a{}^{d} \tag{5}$$

on the brane and using Eq. (2), one obtains the 'constraint' equation

$$R - 2\Lambda_{\rm b} + K_{ab} K^{ab} - K^2 = 0 \,, \tag{6}$$

which, together with (3), implies the following closed scalar equation on the brane:

$$M^6 \left(R - 2\Lambda_{\rm b} \right) + \left(m^2 G_{ab} + \sigma h_{ab} - \tau_{ab} \right) \left(m^2 G^{ab} + \sigma h^{ab} - \tau^{ab} \right) - \frac{1}{3} \left(m^2 R - 4\sigma + \tau \right)^2 = 0 \,, \tag{7}$$

where $\tau = h^{ab} \tau_{ab}$. One method for obtaining solutions of the theory consists in first solving the scalar equation (7) on the brane together with the stress-energy conservation equation, and then integrating the Einstein equations in the bulk with the given data on the brane [9,23].

We note that Eq. (7) describes the evolution on the brane in terms of its intrinsic quantities, and that all homogeneous and isotropic cosmological solutions for the brane can be obtained from this equation [23], as will also be done below. All such solutions are embeddable in the Schwarzschild-AdS five-dimensional bulk. However, one cannot confine oneself to this single equation in the general case, e.g., in studying perturbations of the cosmological solution, where one must also solve equations in the bulk imposing certain boundary and/or regularity conditions. From this general viewpoint, Eq. (7) cannot be regarded as complete: only those of its solutions are admissible which can be developed to regular solutions in the bulk. This illustrates the importance of the precise specification of boundary and/or regularity conditions in the bulk for a complete formulation of the brane-world theory [23].

The gravitational equations in the bulk can be integrated by using, for example, Gaussian normal coordinates, as described, e.g., in [24]. Specifically, in the Gaussian normal coordinates (x, y), where $x = \{x^\alpha\}$ are the coordinates on the brane and y is the fifth coordinate in the bulk, the metric in the bulk is written as

$$ds_5^2 = dy^2 + h_{\alpha\beta}(x, y) dx^\alpha dx^\beta \, . \tag{8}$$

Introducing also the tensor of extrinsic curvature K_{ab} of every hypersurface $y = \text{const}$ in the bulk, one can obtain the following system of differential equations for the components $h_{\alpha\beta}$ and $K^\alpha{}_\beta$:

$$\begin{aligned} \frac{\partial K^\alpha{}_\beta}{\partial y} &= R^\alpha{}_\beta - K K^\alpha{}_\beta - \frac{1}{6} \delta^\alpha{}_\beta \left(R + 2\Lambda_\mathrm{b} + K^\mu{}_\nu K^\nu{}_\mu - K^2 \right) \\ &= R^\alpha{}_\beta - K K^\alpha{}_\beta - \frac{2}{3} \delta^\alpha{}_\beta \Lambda_\mathrm{b} \, , \end{aligned} \tag{9}$$

$$\frac{\partial h_{\alpha\beta}}{\partial y} = 2 h_{\alpha\gamma} K^\gamma{}_\beta \, , \tag{10}$$

where $R^\alpha{}_\beta$ are the components of the Ricci tensor of the metric $h_{\alpha\beta}$ induced on the hypersurface $y = \text{const}$, $R = R^\alpha{}_\alpha$ is its scalar curvature, and $K = K^\alpha{}_\alpha$ is the trace of the tensor of extrinsic curvature. The second equality in (9) is true by virtue of the 'constraint' equation (6). Equations (9) and (10) together with the 'constraint' equation (6) represent the 4+1 splitting of the Einstein equations in the Gaussian normal coordinates. The initial conditions for these equations are defined on the brane through Eq. (3). We emphasise once again that, to obtain a complete braneworld theory in the general case (including a stability analysis), one must also specify additional conditions in the bulk such as the presence of other branes or certain regularity conditions. In our paper, we deal only with the homogeneous and isotropic cosmology on the brane, so this issue does not arise. In this sense, we are studying here the cosmological features common to the whole class of braneworld models described by action (1) with arbitrary boundary conditions in the bulk.

III. VACUUM BRANES AND STATIC BRANES

We proceed to the cosmological implications of the braneworld theory under consideration. In this section, we discuss the situation pertaining to a vacuum brane, i.e., when the matter stress-energy tensor $\tau_{ab} = 0$. It is interesting that the brane approaches this condition during the course of cosmological evolution provided it expands forever and its matter density asymptotically declines to zero. In this case, Eq. (7) takes the form

$$\left(M^6 + \frac{2}{3} \sigma m^2 \right) R + m^4 \left(R_{ab} R^{ab} - \frac{1}{3} R^2 \right) - 4 M^6 \Lambda_{\rm RS} = 0 \, , \tag{11}$$

where

$$\Lambda_{\rm RS} = \frac{\Lambda_{\rm b}}{2} + \frac{\sigma^2}{3 M^6} \, . \tag{12}$$

It is important to note that the second term in Eq. (11) has *precisely* the form of one of the terms in the expression for the conformal anomaly, which describes the vacuum polarization at the one-loop level in curved space-time (see, e.g., [18]).[1] It therefore immediately follows that all *symmetric spaces* are solutions of Eq. (11) with appropriate $\Lambda_{\rm RS}$, just as they are solutions of the Einstein equations with one-loop quantum-gravitational corrections [25].

[1]It is interesting that, while the conformal anomaly term $R_{ab} R^{ab} - \frac{1}{3} R^2$ cannot be obtained by the variation of a local four-dimensional Lagrangian, the very same term is obtained via the variation of a local Lagrangian in the five-dimensional braneworld theory under investigation.

Symmetric spaces satisfy the condition $\nabla_a R_{bcde} = 0$, which implies that geometrical invariants such as $R_{abcd}R^{abcd}$, $R_{ab}R^{ab}$, and R are constants so that Eq. (11) becomes an algebraic equation. Prominent members of this family include:

(i) The homogeneous and isotropic de Sitter space-time

$$ds^2 = -dt^2 + \frac{1}{H^2}\cosh^2 Ht\left[d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] , \tag{13}$$

where $-\infty < t < \infty$, $0 \le \chi, \theta \le \pi$, $0 \le \phi \le 2\pi$. The four dimensional metric (13) has the property $R^a{}_b = 3H^2 h^a{}_b$, and formed the basis for Starobinsky's first inflationary model sustained by the quantum conformal anomaly [26].

(ii) The homogeneous and anisotropic Nariai metric [27,28]

$$ds^2 = k^2\left(-dt^2 + \cosh^2 t dr^2 + d\theta^2 + \sin^2\theta d\phi^2\right) , \tag{14}$$

where $k =$ constant, $-\infty < t < \infty$, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$ and for which $R^a{}_b = h^a{}_b/k^2$. In fact, it is easy to show that any metric for which R and $R_{ab}R^{ab}$ are constants will automatically be a solution to Eq. (11) with an appropriate choice of $\Lambda_{\rm RS}$.

Both de Sitter space and the Nariai metric belong to the class of space-times which satisfy the vacuum Einstein equations with a cosmological constant

$$R_{ab} = \Lambda h_{ab} . \tag{15}$$

Such space-times also satisfy Eq. (11) if

$$\Lambda = \frac{1}{m^2}\left[\left(\frac{3M^6}{2m^2} + \sigma\right) \pm \sqrt{\left(\frac{3M^6}{2m^2} + \sigma\right)^2 - 3M^6\Lambda_{\rm RS}}\right] . \tag{16}$$

Equation (16) expresses the resulting cosmological constant on the brane in terms of the coupling constants of the theory. For the frequently discussed special case $m = 0$, one obtains $\Lambda = \Lambda_{\rm RS}$. The two signs in (16) correspond to the two different ways in which the lower-dimensional brane can form the boundary of the higher-dimensional bulk [14,13]. In the case of a spherically symmetrical bulk, the '$-$' sign corresponds to a brane as a boundary for which the *inner* normal to the brane points in the direction of decreasing bulk radial coordinate.

The condition $\Lambda_{\rm RS} = 0$ is the well-known fine-tuning condition of Randall and Sundrum [7] and leads to the vanishing of the cosmological constant on an empty brane if we set $m = 0$ in (1). Note that, under the Randall–Sundrum condition, expression (16) with the sign opposite to the sign of the quantity $3M^6/2m^2 + \sigma$ also gives a zero value for the resulting cosmological constant on the brane, but the other sign usually leads to $\Lambda \neq 0$.

We would like to draw the reader's attention to the fact that Eq. (16) is meaningful only when the expression under the square root is nonnegative. When it is negative, solutions describing the corresponding empty universe simply do not exist. This leads to the following important conclusion: a universe which contains matter and satisfies

$$\frac{3M^6\Lambda_{\rm RS}}{\left(3M^6/2m^2 + \sigma\right)^2} > 1\,, \tag{17}$$

cannot expand forever.

For the special case $3M^6/2m^2 + \sigma = 0$, the expression for Λ is given by

$$\Lambda = \pm \frac{M^3}{m^2} \sqrt{-3\Lambda_{\rm RS}}\,. \tag{18}$$

In this case, both σ and $\Lambda_{\rm RS}$ must be negative in order that the corresponding empty universe exist, but the resulting cosmological constant on the brane can be of any sign.

Another interesting example is that of a static empty universe. The radius (scale factor) a of such a universe is easily determined from (11) to be

$$a^2 = \frac{\kappa}{\Lambda_{\rm RS}} \left(\frac{3}{2} + \frac{\sigma m^2}{M^6} \right)\,, \tag{19}$$

where $\kappa = \pm 1$ is the sign of the spatial curvature. One can see that the radius of the universe can be arbitrarily large. In the general case, the development of this solution to the five-dimensional bulk leads to a Schwarzschild–anti-de Sitter metric. It was shown in [15] that, for $\kappa = 1$, this metric is purely anti-de Sitter (with zero Schwarzschild mass) if the constants of the theory satisfy the condition

$$\frac{\sigma}{m^2} - \frac{\Lambda_{\rm b}}{2} + \frac{3M^6}{4m^4} = 0\,, \tag{20}$$

which implies negative brane tension σ. It should be pointed out that the static and empty braneworld solution described by (19) does not possess a general-relativistic analog, since, in general relativity, a static cosmological model (the 'static Einstein universe') *cannot* be empty (see, for instance, [4]). Furthermore, from (19) we find that the static empty universe can be spatially open ($\kappa = -1$) — for example, in the case $\Lambda_{\rm RS} < 0$ and $\sigma > -3M^6/2m^2$, — again a situation without an analog in general relativity.

For static homogeneous and isotropic braneworlds filled with matter, Eq. (7) gives the following relation:

$$a^2 \left[\rho_{\rm tot}(\rho_{\rm tot} + 3p_{\rm tot}) - 3\Lambda_{\rm b} M^6\right] = 3\kappa \left[m^2(\rho_{\rm tot} + 3p_{\rm tot}) - 3M^6\right]\,, \tag{21}$$

where the total energy density $\rho_{\rm tot}$ and pressure $p_{\rm tot}$ include the contribution from the brane tension, i.e.,

$$\rho_{\rm tot} = \rho + \sigma\,, \quad p_{\rm tot} = p - \sigma\,, \tag{22}$$

and $\kappa = 0, \pm 1$ corresponds to the sign of the spatial curvature. This relation reduces to (19) for $\rho = p = 0$.

Having obtained all these solutions on the brane, one can find the corresponding solutions in the bulk by integrating Eqs. (9) and (10) with the initial conditions on the brane given by Eq. (3), as described in Sec. II. In doing this, one can consider various additional conditions in the bulk, for example, existence of other branes, or one can impose certain regularity conditions. It is worth noting that one and the same cosmological solution on the given brane can correspond to different global solutions in the bulk, for example, other branes may be present or absent, be static or evolving, close or far away from our brane, etc. In the most general case (for instance in the absence of special symmetries on the brane) integration on the brane needs to be performed in conjunction with dynamical integration in the bulk. All such situations must be separately studied and issues such as their stability to linearized perturbations must be examined on a case-by-case basis.

IV. COSMOLOGICAL CONSEQUENCES OF BRANEWORLD DARK ENERGY

A. Cosmological evolution

In the homogeneous and isotropic cosmological setting, Eq. (7) can be integrated with respect to time with the result [23]

$$m^4 \left(H^2 + \frac{\kappa}{a^2} - \frac{\rho + \sigma}{3m^2} \right)^2 = M^6 \left(H^2 + \frac{\kappa}{a^2} - \frac{\Lambda_{\rm b}}{6} - \frac{C}{a^4} \right) , \qquad (23)$$

where C is the integration constant that corresponds to the black-hole mass of the Schwarzschild–(anti)-de Sitter solution in the bulk, $H \equiv \dot{a}/a$ is the Hubble parameter on the brane, and $\kappa = 0, \pm 1$ corresponds to the sign of the spatial curvature on the brane. A generalization of this equation to the case of absence of Z_2 symmetry was obtained in [29,23]. For the frequently considered special case $m = 0$, this equation reduces to the familiar one [8]

$$H^2 + \frac{\kappa}{a^2} = \frac{\Lambda_{\rm b}}{6} + \frac{C}{a^4} + \frac{(\rho + \sigma)^2}{9M^6} \, . \qquad (24)$$

Equation (23) can be solved with respect to the Hubble parameter with the result [14,13]:

$$H^2 + \frac{\kappa}{a^2} = \frac{\rho + \sigma}{3m^2} + \frac{2}{\ell^2} \left[1 \pm \sqrt{1 + \ell^2 \left(\frac{\rho + \sigma}{3m^2} - \frac{\Lambda_{\rm b}}{6} - \frac{C}{a^4} \right)} \right] , \qquad (25)$$

or, equivalently,

$$H^2 + \frac{\kappa}{a^2} = \frac{\Lambda_{\rm b}}{6} + \frac{C}{a^4} + \frac{1}{\ell^2} \left[\sqrt{1 + \ell^2 \left(\frac{\rho + \sigma}{3m^2} - \frac{\Lambda_{\rm b}}{6} - \frac{C}{a^4} \right)} \pm 1 \right]^2 , \qquad (26)$$

where the length scale

$$\ell = \frac{2m^2}{M^3} \qquad (27)$$

was introduced earlier in Sec. II. Again, the '$\pm$'signs in (25) and (26) correspond to two different ways of bounding the Schwarzschild–(anti)-de Sitter bulk space by the brane [14,13]: the lower sign corresponds to the case where the inner normal to the brane (which bounds the bulk) points in the direction of decreasing bulk radial coordinate. Alternatively, the two different signs in (25) and (26) could correspond to the two possible signs of the five-dimensional Planck mass M. Henceforth we shall refer to models with the lower '$-$' sign as BRANE1 and models with the upper '+' sign as BRANE2.

In the ensuing discussion, we shall neglect the decaying term C/a^4 in Eq. (25) (called dark radiation), assuming it to be negligibly small at present. Similarly to [11], we also introduce the dimensionless cosmological parameters

$$\Omega_{\rm m} = \frac{\rho_0}{3m^2 H_0^2}\,, \quad \Omega_\kappa = -\frac{\kappa}{a_0^2 H_0^2}\,, \quad \Omega_\sigma = \frac{\sigma}{3m^2 H_0^2}\,, \quad \Omega_\ell = \frac{1}{\ell^2 H_0^2}\,, \quad \Omega_{\Lambda_{\rm b}} = -\frac{\Lambda_{\rm b}}{6H_0^2}\,, \tag{28}$$

where the subscript '0' refers to current values of cosmological quantities. The system of cosmological equations with the energy density ρ dominated by dust-like matter can now be written in the transparent form using Eq. (25):

- BRANE1

$$\frac{H^2(z)}{H_0^2} = \Omega_{\rm m}(1{+}z)^3 + \Omega_\kappa(1{+}z)^2 + \Omega_\sigma + \underline{2\Omega_\ell - 2\sqrt{\Omega_\ell}\,\sqrt{\Omega_{\rm m}(1{+}z)^3 + \Omega_\sigma + \Omega_\ell + \Omega_{\Lambda_{\rm b}}}}\,, \tag{29}$$

- BRANE2

$$\frac{H^2(z)}{H_0^2} = \Omega_{\rm m}(1{+}z)^3 + \Omega_\kappa(1{+}z)^2 + \Omega_\sigma + \underline{2\Omega_\ell + 2\sqrt{\Omega_\ell}\,\sqrt{\Omega_{\rm m}(1{+}z)^3 + \Omega_\sigma + \Omega_\ell + \Omega_{\Lambda_{\rm b}}}}\,, \tag{30}$$

where z is the cosmological redshift. We have underlined the terms which cause our equations to differ from their general-relativistic counterparts. The constraint equations for the cosmological parameters are most conveniently derived using Eq. (26). We obtain

$$1 - \Omega_\kappa + \Omega_{\Lambda_{\rm b}} = \left(\sqrt{\Omega_\ell + \Omega_{\rm m} + \Omega_\sigma + \Omega_{\Lambda_{\rm b}}} \pm \sqrt{\Omega_\ell}\right)^2 . \tag{31}$$

This equation can be transformed to a more convenient form. In doing this, we note that the theory makes sense only in the case $\Omega_\ell + \Omega_{\rm m} + \Omega_\sigma + \Omega_{\Lambda_{\rm b}} > 0$. Since $\Omega_\ell > 0$, when taking the square root of both sides of this equation, we must consider two possibilities for the sign of the quantity $\Omega_{\mathrm{m}} + \Omega_\sigma + \Omega_{\Lambda_{\mathrm{b}}}$ under the square root in (31).

In the case

$$\Omega_{\mathrm{m}} + \Omega_\sigma + \Omega_{\Lambda_{\mathrm{b}}} \geq 0\,, \tag{32}$$

taking the square root of both sides of (31) and rearranging terms, we get

$$\sqrt{1 - \Omega_\kappa + \Omega_{\Lambda_{\mathrm{b}}}} \mp \sqrt{\Omega_\ell} = \sqrt{\Omega_\ell + \Omega_{\mathrm{m}} + \Omega_\sigma + \Omega_{\Lambda_{\mathrm{b}}}}\,. \tag{33}$$

Now taking the square of both sides of the last equation, we finally obtain the constraint:

$$\Omega_{\mathrm{m}} + \Omega_\kappa + \Omega_\sigma \pm 2\sqrt{\Omega_\ell}\,\sqrt{1 - \Omega_\kappa + \Omega_{\Lambda_{\mathrm{b}}}} = 1\,. \tag{34}$$

In the opposite case

$$-\Omega_\ell \;\leq\; \Omega_{\mathrm{m}} + \Omega_\sigma + \Omega_{\Lambda_{\mathrm{b}}} < 0\,, \tag{35}$$

taking the square root of both sides of (31) and rearranging terms, we get

$$\sqrt{1 - \Omega_\kappa + \Omega_{\Lambda_{\mathrm{b}}}} - \sqrt{\Omega_\ell} = \pm\sqrt{\Omega_\ell + \Omega_{\mathrm{m}} + \Omega_\sigma + \Omega_{\Lambda_{\mathrm{b}}}}\,. \tag{36}$$

Taking the square of both sides of the last equation, we finally obtain the constraint:

$$\Omega_{\mathrm{m}} + \Omega_\kappa + \Omega_\sigma + 2\sqrt{\Omega_\ell}\,\sqrt{1 - \Omega_\kappa + \Omega_{\Lambda_{\mathrm{b}}}} = 1\,. \tag{37}$$

To summarise, in the case $\Omega_{\mathrm{m}} + \Omega_\sigma + \Omega_{\Lambda_{\mathrm{b}}} \geq 0$, the constraint equations in the BRANE1 and BRANE2 models read:

$$\Omega_{\mathrm{m}} + \Omega_\kappa + \Omega_\sigma - \underline{2\sqrt{\Omega_\ell}\,\sqrt{1 - \Omega_\kappa + \Omega_{\Lambda_{\mathrm{b}}}}} = 1 \quad \text{(BRANE1)}, \tag{38}$$

$$\Omega_{\mathrm{m}} + \Omega_\kappa + \Omega_\sigma + \underline{2\sqrt{\Omega_\ell}\,\sqrt{1 - \Omega_\kappa + \Omega_{\Lambda_{\mathrm{b}}}}} = 1\,. \quad \text{(BRANE2)} \tag{39}$$

In the opposite case $\Omega_{\mathrm{m}} + \Omega_\sigma + \Omega_{\Lambda_{\mathrm{b}}} < 0$, both constraint equations for BRANE1 and BRANE2 are given by Eq. (39). Again, we have underlined the terms which cause our equations to differ from their general-relativistic counterparts. It is easy to see that the general-relativistic limit of this theory is reached as $\Omega_\ell \to 0$ (i.e., $M \to 0$). In this case, setting $\Omega_\kappa = 0$, $\Omega_\sigma \equiv \Omega_\Lambda$, we find that (29) & (30) give the expression for the Hubble parameter in the LCDM model:

$$H^2_{\rm LCDM}(z) = H_0^2 \left[\Omega_{\rm m}(1+z)^3 + \Omega_\Lambda\right]. \tag{40}$$

An example of the behaviour of the Hubble parameter in the braneworld models is illustrated in Fig. 1.

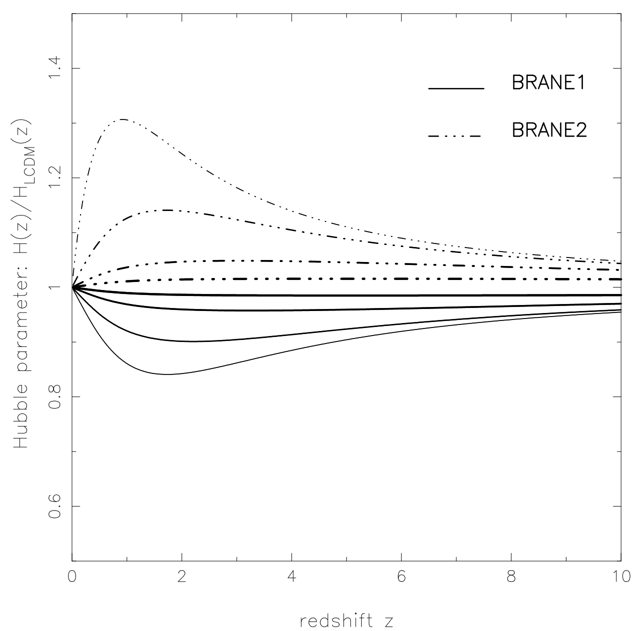

FIG. 1. The Hubble parameter in units of $H_{\rm LCDM}(z)$ is plotted as a function of redshift for the two braneworld models BRANE1 and BRANE2. Whereas $H(z)$ in BRANE2 is *larger* than $H(z)$ in LCDM, in the case of BRANE1 the value of $H(z)$ is *smaller* than its value in LCDM. Parameter values are: $\Omega_\kappa = 0$, $\Omega_\ell = 1.0$, $\Omega_{\rm m} = 0.3$, and $\Omega_{\Lambda_{\rm b}} = 1, 10, 10^2, 10^3$ (top to bottom for BRANE2 and bottom to top for BRANE1). The value of Ω_σ is determined from (38) & (39). Parameter values for the LCDM model are $\Omega_{\rm m} = 0.3$, $\Omega_\Lambda = 0.7$. For large values of the bulk cosmological constant $\Omega_{\Lambda_{\rm b}}$, BRANE1 and BRANE2 become indistinguishable from LCDM.

It is convenient to rewrite Eq. (12) in terms of the dimensionless cosmological parameters

$$\Omega_{\rm RS} = \frac{\Lambda_{\rm RS}}{3H_0^2} = \frac{\Omega_\sigma^2}{4\Omega_\ell} - \Omega_{\Lambda_{\rm b}}, \tag{41}$$

from where it follows that the Randall–Sundrum constraint $\Lambda_{\mathrm{RS}} = 0$ has the simple form

$$\Omega_\sigma^2 - 4\Omega_\ell \Omega_{\Lambda_{\mathrm{b}}} = 0\,. \tag{42}$$

This constraint implies $\Omega_{\Lambda_{\mathrm{b}}} \geq 0$ ($\Lambda_{\mathrm{b}} \leq 0$) and ensures the vanishing of the effective cosmological constant in the future for the braneworld embedding described by the BRANE1 model for $\Omega_\sigma \geq 0$ or for $\Omega_\sigma < 0$ and $\Omega_\ell > \Omega_{\Lambda_{\mathrm{b}}}$, and by the BRANE2 model for $\Omega_\sigma < 0$ and $\Omega_\ell < \Omega_{\Lambda_{\mathrm{b}}}$. The case where the Randall–Sundrum constraint is imposed will be discussed in detail in Sec. IV C below. It will be shown that, in the case of BRANE1 model, for the case $\Omega_\kappa = 0$, Eq. (38) would imply that $\Omega_{\mathrm{m}} > 1$, which apparently contradicts current observations of the large-scale distribution of matter. This is an important result since it suggests that for reasonable values of the matter density ($\Omega_{\mathrm{m}} \ll 1$) and $\Omega_\kappa = 0$, the BRANE1 model demands the presence of a nonvanishing effective cosmological constant. The value of the cosmological constant is given by Eq. (16) and can be determined by using observational data to fix the constants of the theory. On the other hand, the BRANE2 model is in principle observationally compatible with the Randall–Sundrum constraint, which can result in vanishing effective cosmological constant in the future thus making the current acceleration of the universe a transient phenomenon.

Typical values of the Ω parameters (28) that we consider in this paper are of order $\Omega \sim 1$. For such values, the fundamental constants of the theory have the following orders of magnitude:

$$m^2 \simeq M_{\mathrm{P}}^2 \sim 10^{19}\ \mathrm{GeV}\,, \quad M \sim 100\ \mathrm{MeV}\,, \quad \Lambda_{\mathrm{b}} \sim \frac{\sigma}{m^2} \sim H_0^2 \sim 10^{-56}\ \mathrm{cm}^{-2}\,. \tag{43}$$

The smallness of the bare cosmological constants in the bulk and on the brane represents a fine-tuning similar to what is the case for the cosmological constant in the standard LCDM model. However, even with such small values of the bare cosmological constants in action (1), the braneworld evolution exhibits some qualitatively new properties (discussed in the following subsections) when compared to the case where these constants are set to zero. This is the reason why we find it important to keep open the possibility of nonzero, albeit small, values of Λ_{b} and σ.

A possible procedure for testing the braneworld model against observations is presented in the appendix.

B. Cosmological tests of braneworld models

Observations of high-redshift type Ia supernovae indicate that these objects are fainter than they would be in a standard cold dark matter cosmology (SCDM) with $\Omega_{\rm m} = 1$ [1]. This observation is taken as support for a universe which is accelerating, fuelled by a form of energy with negative pressure (dark energy). In standard FRW cosmology the acceleration of the universe is described by the equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\sum_i (\rho_i + 3p_i), \tag{44}$$

where the summation is over all matter fields contributing to the dynamics of the universe. It is easy to show that a necessary (but not sufficient) condition for acceleration ($\ddot{a} > 0$) is that the strong energy condition is violated for *at least one* of the matter fields in (44), so that $\rho + 3p < 0$. In the case of the popular ΛCDM model this requirement is clearly satisfied since $p_m = 0$ in pressureless (cold) matter, while $p_\Lambda = -\rho_\Lambda \equiv -\Lambda/8\pi G$ in the cosmological constant. The situation with respect to braneworld models is different since, as we have demonstrated in the previous section, braneworld evolution is distinct from FRW evolution at late times. However it is easy to show that braneworld models can accelerate. We demonstrate this by noting that a completely general expression for the deceleration parameter $q = -\ddot{a}/aH^2$ is provided by

$$q(z) = \frac{H'(z)}{H(z)}(1+z) - 1\,, \tag{45}$$

where $H(z)$ is given by (29)–(39) and the derivative is with respect to z. The current value of the deceleration parameter is easily calculated to be

$$q_0 = \frac{3}{2}\Omega_{\rm m}\left(1 \pm \sqrt{\frac{\Omega_\ell}{\Omega_{\rm m} + \Omega_\sigma + \Omega_\ell + \Omega_{\Lambda_{\rm b}}}}\right) - 1\,, \tag{46}$$

where the lower and upper signs correspond to BRANE1 and BRANE2 models, respectively. Thus the present universe will accelerate ($q_0 < 0$) for brane parameter values which satisfy

$$\frac{3}{2}\Omega_{\rm m}\left(1 \pm \sqrt{\frac{\Omega_\ell}{\Omega_{\rm m} + \Omega_\sigma + \Omega_\ell + \Omega_{\Lambda_{\rm b}}}}\right) < 1. \tag{47}$$

It should be pointed out that the proposition to use the induced curvature term on a braneworld to account for the accelerated phase of our universe was first made in [13] in the context of the braneworld model of [12], which is the special case of our BRANE2 model with $\Omega_\sigma = \Omega_{\Lambda_{\rm b}} = 0$. It was subsequently discussed and tested in [11,30]. In the present paper, we allow both $\Omega_\sigma \neq 0$ and $\Omega_{\Lambda_{\rm b}} \neq 0$, and the presence of these two free parameters makes the braneworld model more flexible and allows for qualitatively new behaviour which shall be discussed in detail in this section. The connection between large extra dimensions and dark energy has also been investigated in [31].

Observationally, a pivotal role in the case for an accelerating universe is played by the *luminosity distance* $d_L(z)$, since the flux of light received from a distant source varies inversely to the square of the luminosity distance, $\mathcal{F} \propto d_L^{-2}$. This effect is quantitatively described by the magnitude–luminosity relation: $m_B = M_0 + 25 + 5\log_{10} d_L$, where m_B is the corrected apparent peak B magnitude and M_0 is the absolute peak luminosity of the supernova. A supernova will therefore appear fainter in a universe which possesses a larger value of the luminosity distance to a given redshift.

In a FRW universe, the luminosity distance is determined by the Hubble parameter and three-dimensional spatial curvature [4]:

$$d_L(z) = \frac{1+z}{H_0\sqrt{|\Omega_{\rm total} - 1|}}\, S\left(\eta_0 - \eta\right)\,, \tag{48}$$

where

$$\eta_0 - \eta = H_0\sqrt{|\Omega_{\rm total} - 1|}\int_0^z \frac{dz'}{H(z')}\,, \tag{49}$$

and $S(x)$ is defined as follows: $S(x) = \sin x$ if $\kappa = 1$ ($\Omega_{\rm total} > 1$), $S(x) = \sinh x$ if $\kappa = -1$ ($\Omega_{\rm total} < 1$), and $S(x) = x$ if $\kappa = 0$ ($\Omega_{\rm total} = 1$).

Inflationary models suggest that the universe is nearly flat with $\Omega_\kappa \simeq 0$ and we shall work under this assumption in the rest of this paper (see also [32]). In this case, Eq. (48) simplifies to

$$\frac{d_L(z)}{1+z} = \int_0^z \frac{dz'}{H(z')}\,, \tag{50}$$

where $H(z)$ is determined by (29) for BRANE1, and by (30) for BRANE2. In Fig. 2 we show the luminosity distances for the BRANE1 & BRANE2 models. Also shown for comparison is the value of $d_L(z)$ in a spatially-flat two-component FRW universe with the Hubble parameter

$$H(z) = H_0\left[\Omega_{\rm m}(1+z)^3 + \Omega_X(1+z)^{3(1+w)}\right]^{1/2}, \tag{51}$$

where Ω_X describes dark energy with equation of state $w = p_X/\rho_X$. Three cosmological models will be of interest to us in connection with (51):

(i) SCDM: The standard cold dark matter universe with $\Omega_{\rm m} = 1$ and $\Omega_X = 0$.

(ii) LCDM: Cold dark matter + a cosmological constant with $w = -1$.

(iii) Phantom models: Cold dark matter + 'phantom energy' satisfying $w < -1$ [33].

We find from Fig. 2 that the luminosity distance in both braneworld models exceeds that in SCDM. In fact, BRANE1 models have the unusual feature that their luminosity distance can even exceed that in LCDM (for a fixed value of $\Omega_{\rm m}$)! In fact it can easily be shown that

$$d_L^{\rm dS}(z) \geq d_L^{\rm BRANE1}(z) \geq d_L^{\rm LCDM}(z)\,, \tag{52}$$

where $d_L^{\rm dS}(z)$ refers to the luminosity distance in the spatially flat coordinatization of de Sitter space (equivalently, the steady state universe). The second inequality presumes a fixed value of $\Omega_{\rm m}$. BRANE2 models show complementary behaviour

$$d_L^{\rm LCDM}(z) \geq d_L^{\rm BRANE2}(z) \geq d_L^{\rm SCDM}(z)\,, \tag{53}$$

where the first inequality is valid for a fixed value of $\Omega_{\rm m}$. In the case $\Omega_\ell = 0$, the equations of the braneworld theory formally reduce to those of general relativity, and we have $d_L^{\rm BRANE1}(z) = d_L^{\rm BRANE2}(z) = d_L^{\rm LCDM}(z)$.

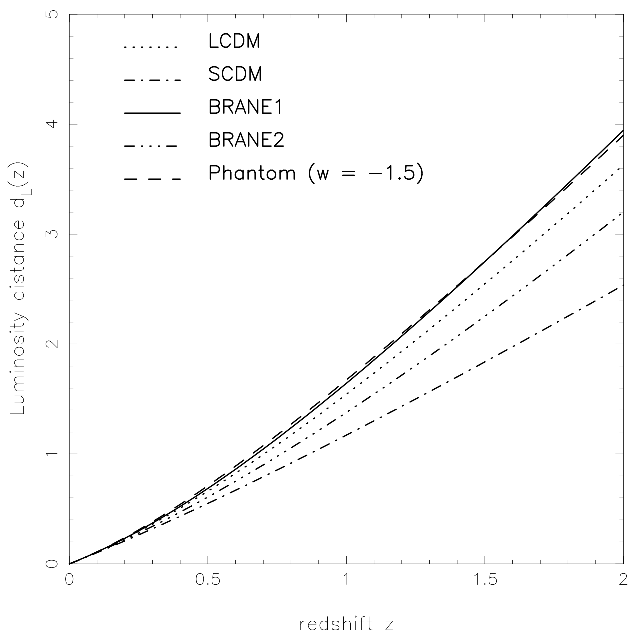

FIG. 2. The luminosity distance is shown as a function of redshift for the two braneworld models BRANE1 & BRANE2, LCDM, SCDM, and 'phantom energy'. All models, with the exception of SCDM, have $\Omega_{\rm m} = 0.3$. SCDM has $\Omega_{\rm m} = 1$. The BRANE1 & BRANE2 models have $\Omega_\ell = 0.3$ and vanishing cosmological constant in the bulk. LCDM and the phantom model have the same dark energy density $\Omega_\Lambda = \Omega_X = 0.7$. The equation of state for dark energy is $w_\Lambda = -1$ for LCDM and $w = p_X/\rho_X = -1.5$ for phantom. The luminosity distance is greatest for BRANE1 & phantom, and least for SCDM. BRANE1 & BRANE2 lie on either side of LCDM.

One might add that the behaviour of BRANE1 is mimicked by FRW models with $w \leq -1$, whereas BRANE2 resembles dark energy with $-1 \leq w \leq 0$ [4]. In fact, from Fig. 2 we see that the luminosity distance in the BRANE1 model is quite close to what one gets from 'phantom energy' described by (51) with $w = -1.5$. (The parameters for this BRANE1 model are $\Omega_{\rm m} = \Omega_\ell = 0.3$, $\Omega_{\Lambda_{\rm b}} = 0$, and $\Omega_\sigma = 1 - \Omega_{\rm m} + 2\sqrt{\Omega_\ell} \approx 1.8$.) It should be pointed out that phantom energy models were introduced by Caldwell [33], who made the important observation that dark-energy with $w < -1$ appears to give a better fit to supernova observations than LCDM (which has $w = -1$). Unfortunately, phantom models have several bizarre properties, which may be the reason why cosmologists have been reluctant to take these models seriously despite their seemingly better agreement with observations (see however [34]). Some strange properties of phantom energy with a very negative equation of state ($w < -1$) are summarised below (see also [33,35]):

(i) A negative equation of state suggests that the effective velocity of sound in the medium $v = \sqrt{|dp/d\rho|}$ can become larger than the velocity of light.

(ii) The expansion factor of a universe dominated by phantom energy grows as

$$a(t) \simeq a\,(t_{\rm eq}) \left[(1+w)\frac{t}{t_{\rm eq}} - w\right]^{2/3(1+w)} \quad , \quad w < -1 \ , \tag{54}$$

where $t_{\rm eq}$ marks the epoch when the densities in matter and phantom energy are equal: $\rho_m(t_{\rm eq}) \simeq \rho_X(t_{\rm eq})$. It immediately follows that the scale factor diverges in a *finite* amount of cosmic time

$$a(t) \to \infty \ \ \text{as} \ \ t \to t_* = \left(\frac{w}{1+w}\right) t_{\rm eq} \, . \tag{55}$$

Substitution of $z \to -1$ and $w < -1$ in (51) shows that the Hubble parameter also diverges as $t \to t_*$, implying that an infinitely rapid expansion rate for the universe has been reached in the *finite* future.

As the universe expands, the density of phantom energy ($w < -1$) *grows* instead of decreasing ($w > -1$) or remaining constant ($w = -1$),

$$\rho(t) \propto \left[(1+w)\frac{t}{t_{\rm eq}} - w\right]^{-2} , \tag{56}$$

reaching a singular value in a finite interval of time $\rho(t) \to \infty$, $t \to t_*$. This behaviour should be contrasted with the density of ordinary matter which drops to zero: $\rho_m \to 0$ as $t \to t_*$. A universe dominated by phantom energy is thus doomed to *expand towards a physical singularity* which is reached in a finite amount of proper time. (An exact expression for the time of occupance of the phantom singularity is given in [36], which also contains an interesting discussion of related issues.)

At this stage one must emphasize that, although the BRANE1 model has several features in common with phantom energy, (which makes us believe that, like the latter, it too is likely to provide a good fit to supernova data), it is not necessarily afflicted with phantom's

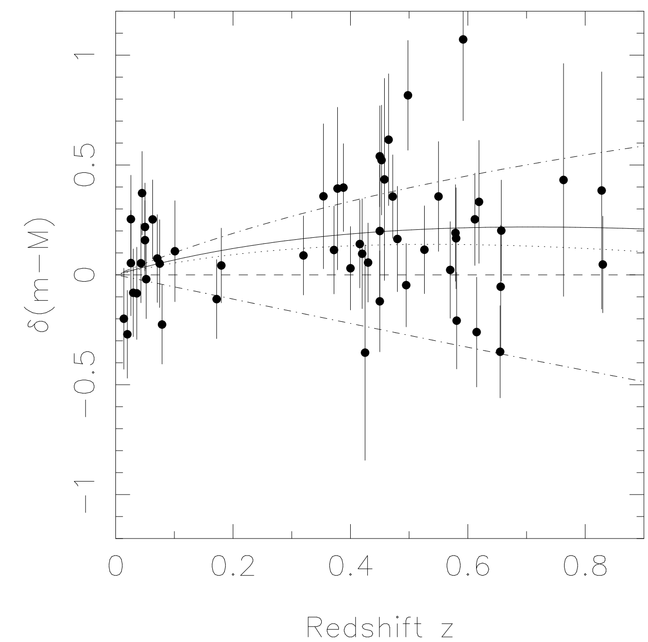

FIG. 3. The distance modulus $(m-M)$ of Type Ia supernovae (the primary fit of the Supernova cosmology project) is shown relative to an empty $\Omega_{\rm m} \to 0$ Milne universe (dashed line). The solid line refers to the distance modulus in BRANE1 with $\Omega_\ell = \Omega_{\rm m} = 0.3$, and vanishing cosmological constant in the bulk. The dotted line (below the solid) is LCDM with $(\Omega_\Lambda, \Omega_{\rm m}) = (0.7, 0.3)$. The uppermost and lowermost (dot-dashed) lines correspond to de Sitter space $(\Omega_\Lambda, \Omega_{\rm m}) = (1, 0)$ and SCDM $(\Omega_\Lambda, \Omega_{\rm m}) = (0, 1)$, respectively.

pathologies. Indeed, in a broad range of parameters, both BRANE1 and BRANE2 are physically well motivated and remain *well behaved during all times*. The present paper therefore adds an important new dimension to the current debate about the acceleration of the universe by showing that cosmological models with $d_L(z) > d_L^{\rm LCDM}(z)$ are possible to construct within the framework of the braneworld scenario and should be taken seriously. Future studies will address the important quantitative issue of whether braneworld models of dark energy provide as good (or better) fit to high-z supernova observations as LCDM (see Fig. 3 as an illustration). (It is appropriate to mention here that the weak energy condition $\rho + p \geq 0$ can also be violated in a class of scalar-tensor theories of gravity, as discussed in [37].)

We should add that the 'angular-size distance' d_A is related to the luminosity distance

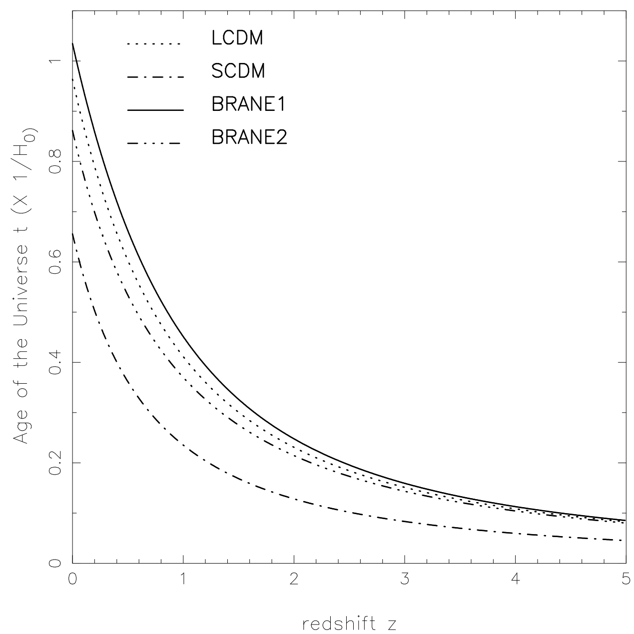

FIG. 4. The age of the universe (in units of the inverse Hubble parameter) is plotted as a function of the cosmological redshift for the models discussed in Fig. 2. (The phantom model is not shown.) BRANE1 models have the oldest age while SCDM is youngest.

d_L through $d_A = d_L(1+z)^{-2}$, therefore much of the above analysis carries over when one discusses properties of the angular-size distance within the framework of braneworld models. Some cosmological features of braneworld models are shown in Figs. 4–6. In Fig. 4, the age of the universe at a given cosmological redshift

$$t(z) = \int_z^\infty \frac{dz'}{(1+z')H(z')}\,, \tag{57}$$

is shown for the two braneworld models and for LCDM & SCDM. We find that the age of the universe in BRANE1 (BRANE2) is larger (smaller) than in LCDM for identical values of the cosmological density parameter $\Omega_{\rm m}$. This is a direct consequence of the fact that the Hubble parameter in BRANE1 (BRANE2) is smaller (larger) than in LCDM. Both braneworld models are significantly older than SCDM.

Considerable insight into the dynamics of the universe is provided by the cosmological deceleration parameter (45). Our results, shown in Fig. 5, indicate that at late times the BRANE1 (BRANE2) universe accelerates at a faster (slower) rate than LCDM (with identical $\Omega_{\rm m}$). Curiously, the BRANE1 universe shows an earlier transition from deceleration to

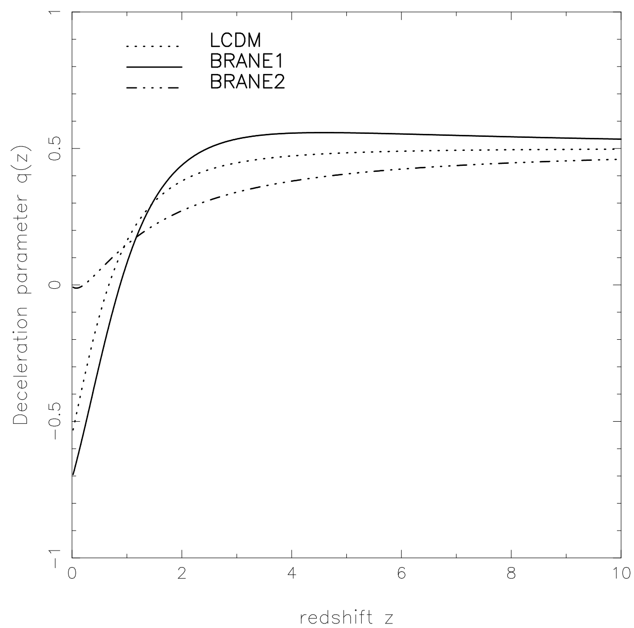

FIG. 5. The deceleration parameter $q(z)$ is shown for BRANE1, BRANE2 and LCDM. The model parameters are as in Fig. 2. For reference it should be noted that $q = 0.5$ for SCDM while de Sitter space has $q = -1$.

acceleration than any of the other models. (For the given choice of parameters this transition takes place at $z \simeq 1$ for BRANE1 and at $z \simeq 0.7$ for LCDM. The BRANE2 model begins accelerating near the present epoch at $z \simeq 0$.) A related point of interest is that at $z \gtrsim 2$ the deceleration parameter in BRANE1 marginally exceeds that in SCDM indicating that the BRANE1 model is decelerating at a faster rate than SCDM ($q = 0.5$). In conventional models of dark matter this behaviour can occur only if the equation of state of the dark component is stiffer than dust, implying $w > 0$ in (51), or if the universe is spatially closed. On the other hand, the current *acceleration* rate of BRANE1 in our example ($q_0 \simeq -0.7$) significantly exceeds that of LCDM ($q_0 \simeq -0.55$) with an identical value of $\Omega_{\rm m} = 0.3$ in both models. Within the framework of four-dimensional Einstein gravity, this situation can only arise if the equation of state of dark energy is strongly negative: $w < -1$ in (51).

The unusual high-z behaviour of the deceleration parameter in BRANE1 can be better understood if we consider the cosmological density parameter

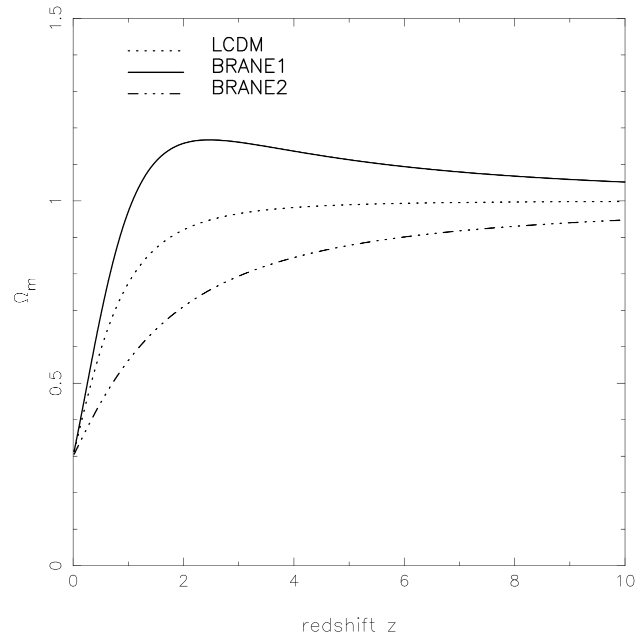

FIG. 6. The dimensionless matter density $\Omega_{\rm m}(z)$ is shown for the two braneworld models and LCDM. ($\Omega_{\rm m} = 1$ in SCDM.) Parameter values are the same as in previous figures. BRANE1 has the interesting feature that $\Omega_{\rm m}(z)$ *exceeds unity* for $z \gtrsim 1$.

$$\Omega_{\rm m}(z) = \left[\frac{H_0}{H(z)}\right]^2 \Omega_{\rm m}(0)(1+z)^3\,, \tag{58}$$

where $H(z)$ is given by (29)–(39) for braneworld models. From Fig. 6 we notice that, for $z \gtrsim 1$, the value of $\Omega_{\rm m}(z)$ in BRANE1 *exceeds* its value in SCDM ($\Omega_{\rm m} = 1$). This is precisely the redshift range during which $q(z)_{\rm BRANE1} > q(z)_{\rm SCDM}$. Thus, the rapid deceleration of BRANE1 at high redshifts can be partly attributed to the larger value of the matter density $\Omega_{\rm m}(z)$ at those redshifts, relative to SCDM.

Having established partial similarity of BRANE1 with phantom models at low redshifts, we can investigate the analogy further and calculate the effective equation of state of dark energy

$$w(z) = \frac{2q(z) - 1}{3\left[1 - \Omega_{\rm m}(z)\right]}\,, \tag{59}$$

where $\Omega_{\rm m}(z)$ is given by (58). One notes that $w(z)$ has a pole-like singularity at $z \simeq 1$ for BRANE1, which arises because $\Omega_{\rm m}(z)$ crosses the value of unity at $z \simeq 1$ (see Fig. 6).

This demonstrates that the notion of 'effective equation of state' is of limited utility for this model. Equations (29), (30), (45), and (59) also illustrate the important fact that dark energy in braneworld models, though similar to phantom energy in some respects, differ from it in others. For instance, in both braneworld models, $w(z) \to -0.5$ at $z \gg 1$ and $w(z) \to -1$ as $z \to -1$, whereas phantom energy has $w(z) < -1$ at *all* times.

A useful quantity is the *current value* of the effective equation of state of dark energy in braneworld theories:

$$w_0 = \frac{2q_0 - 1}{3\,(1 - \Omega_{\rm m})} = -1 \pm \frac{\Omega_{\rm m}}{1 - \Omega_{\rm m}} \sqrt{\frac{\Omega_\ell}{\Omega_{\rm m} + \Omega_\sigma + \Omega_\ell + \Omega_{\Lambda_{\rm b}}}}\,, \tag{60}$$

where the lower and upper signs, as usual, correspond to BRANE1 and BRANE2 models, respectively. We easily see that $w_0 < -1$ for BRANE1, whereas $w_0 > -1$ for BRANE2.

C. Disappearing Dark Energy

The braneworld models under consideration also admit the intriguing possibility that the current acceleration of the universe may not be a lasting feature. It may be recalled that most models of dark energy including the cosmological constant have the property that, once the universe begins to accelerate, it accelerates forever. As shown in a number of recent papers, an eternally accelerating universe is endowed with a cosmological event horizon which prevents the construction of a conventional S-matrix describing particle interactions within the framework of string or M-theory [39]. In this section we show that, provided the Randall–Sundrum constraint relation (42) is satisfied, the acceleration of the universe can be a transient phenomenon in braneworld models. An anisotropic solution of Bianchi V class with the same feature was described in [40].

From Eqs. (29) and (30) we obtain the following asymptotic expressions for the Hubble parameter H_∞ as $z \to -1$, assuming that the universe expands forever:

$$\left(\frac{H_\infty}{H_0}\right)^2 = \Omega_\sigma + 2\Omega_\ell \pm 2\sqrt{\Omega_\ell}\,\sqrt{\Omega_\sigma + \Omega_\ell + \Omega_{\Lambda_{\rm b}}}\,, \tag{61}$$

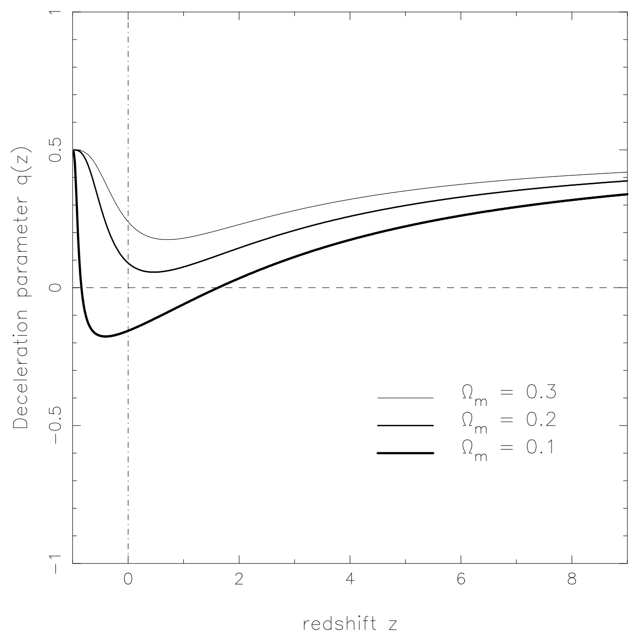

FIG. 7. The deceleration parameter is plotted as a function of redshift for the BRANE2 model with the Randall–Sundrum constraint (42), $\Omega_{\Lambda_b} = 2$, and $(\Omega_m, \Omega_\ell) = (0.3, 1.2)$, $(0.2, 1.6)$, $(0.1, 1.98)$ (top to bottom). The vertical (dot-dashed) line at $z = 0$ marks the present epoch, while the horizontal (dashed) line at $q = 0$ corresponds to a Milne universe $[a(t) \propto t]$ which neither accelerates nor decelerates. Note that the universe *ceases to accelerate* and becomes matter dominated in the future.

where the lower and upper signs correspond to BRANE1 and BRANE2 models, respectively. In applying the Randall–Sundrum constraint (42), we first consider the case where $\Omega_\sigma > 0$. Then

$$\Omega_\sigma = 2\sqrt{\Omega_\ell \Omega_{\Lambda_b}} \tag{62}$$

and

$$\left(\frac{H_\infty}{H_0}\right)^2 = 2\sqrt{\Omega_\ell}\left[\sqrt{\Omega_\ell} + \sqrt{\Omega_{\Lambda_b}} \pm \left(\sqrt{\Omega_\ell} + \sqrt{\Omega_{\Lambda_b}}\right)\right] . \tag{63}$$

One can see that this expression vanishes for the lower sign. Thus, for positive Ω_σ, it is the BRANE1 model that leads to vanishing effective cosmological constant in the future. However, in this case, the constraint equation (38) with $\Omega_\kappa = 0$ becomes

$$\Omega_m - 2\sqrt{\Omega_\ell}\left(\sqrt{1 + \Omega_{\Lambda_b}} - \sqrt{\Omega_{\Lambda_b}}\right) = 1 \tag{64}$$

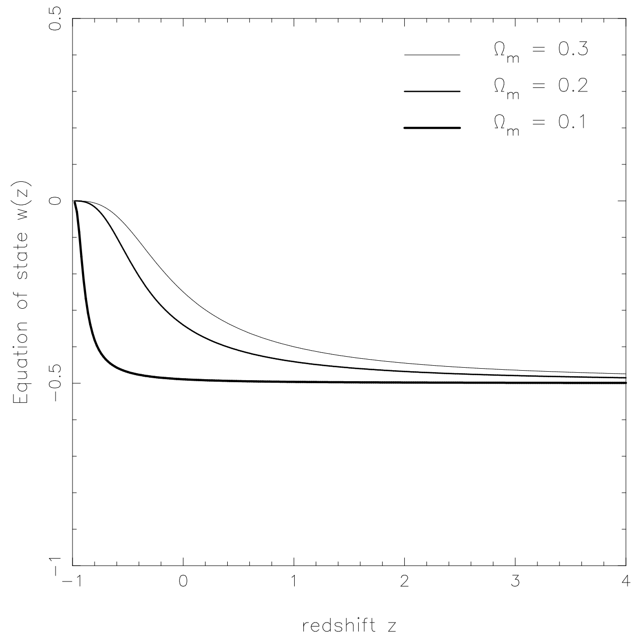

FIG. 8. The effective equation of state for dark energy in the BRANE2 model is shown as a function of redshift. Model parameters are as in the previous figure. Note that the past and future asymptotes of $w(z)$ are quite different: $w(z) \to -1/2$ for $z \gg 0$, while $w(z) \to 0$ for $z \to -1$. Braneworld dark energy therefore effectively disappears in the future, giving rise to a matter-dominated universe.

and implies $\Omega_{\rm m} > 1$, which is hardly compatible with the observations.

In the case of $\Omega_\sigma < 0$, we have

$$\Omega_\sigma = -2\sqrt{\Omega_\ell \, \Omega_{\Lambda_{\rm b}}} \tag{65}$$

and

$$\left(\frac{H_\infty}{H_0}\right)^2 = 2\sqrt{\Omega_\ell}\left(\sqrt{\Omega_\ell} - \sqrt{\Omega_{\Lambda_{\rm b}}} \pm \left|\sqrt{\Omega_\ell} - \sqrt{\Omega_{\Lambda_{\rm b}}}\right|\right) . \tag{66}$$

If $\Omega_\ell > \Omega_{\Lambda_{\rm b}}$, then this expression vanishes for the lower sign, which brings us back to the nonphysical BRANE1 models with $\Omega_{\rm m} > 1$. The case $\Omega_\ell > 1 + \Omega_{\Lambda_{\rm b}}$ is compatible with the condition $\Omega_{\rm m} + \Omega_\sigma + \Omega_{\Lambda_{\rm b}} < 0$, which corresponds to BRANE1 (29) with the constraint equation (39). However, it can be shown that these conditions imply $\Omega_{\rm m} < 0$, which is also physically unacceptable.

There remains the case of $\Omega_\sigma < 0$ and $\Omega_\ell \le \Omega_{\Lambda_b}$. In this case, expression (66) vanishes for the upper sign, which corresponds to BRANE2 models. The constraint equation (39) with $\Omega_\kappa = 0$ now reads

$$\Omega_m + 2\sqrt{\Omega_\ell}\left(\sqrt{1+\Omega_{\Lambda_b}} - \sqrt{\Omega_{\Lambda_b}}\right) = 1 \tag{67}$$

and implies $\Omega_m < 1$.

Therefore, BRANE2 with $\Omega_\sigma < 0$ and $\Omega_\ell \le \Omega_{\Lambda_b}$, provides us with an interesting example of a physically meaningful cosmological model in which the current acceleration of the universe is a *transient phenomenon*. An example of this behaviour as probed by the deceleration parameter is shown in Fig. 7, which demonstrates that the current period of cosmic acceleration takes place between two matter-dominated epochs. We emphasize that these models require negative brane tension σ. Since an observer in this model resides on a negative-tension brane one must ponder over the issue of whether such a braneworld will be perturbatively stable and hence physically viable. We consider this to be an open question for future investigations. Remarks made at the end of Sec. III are relevant, however, since one and the same cosmological solution on the 'visible' (negative tension) brane can correspond to many different global conditions in the bulk, for instance, other ('hidden') branes may be present or absent, static or evolving, close to or far away from our brane, etc. Moreover, here the sign of the parameter $\epsilon = \pm 1$ in action (1) is expected to be of crucial importance, since, in particular, the bulk gravity will be localised around the *negative* tension (visible) brane in the case of $\epsilon = -1$. Issues of perturbative stability involve the examination of all such distinct global solutions on a case-by-case basis. Such a study, though interesting and important, clearly lies beyond the scope of the present paper.

Useful insight into the BRANE2 model is also provided by the effective equation of state of dark energy (59). Our results shown in Fig. 8 indicate that the past and future behaviour of dark energy in the braneworld universe can be very different. The past behaviour $w(z) \to -0.5$ for $z \gg 1$ arises because, in a spatially flat braneworld, the second most important contribution to braneworld expansion at high redshifts is caused by the $(1+z)^{3/2}$ term in

(30); see also [11]. The future behaviour $w(z) \to 0$ as $z \to -1$, on the other hand, reflects the decreasing importance of dark energy as the universe expands. The acceleration of the universe is therefore a transient phenomenon which ends once the universe settles back into the matter-dominated regime.

Finally, we should mention that a transiently accelerating regime also arises in a class of BRANE2 models which do not satisfy the Randall-Sundrum constraint (42). In these models the current epoch of acceleration is succeeded by an epoch during which the deceleration parameter grows without bound. This unusual 'future singularity' is reached in a *finite* interval of expansion time and is characterised by the fact that both the matter density and the Hubble parameter remain finite, while $\ddot{a} \to \infty$ (a feature that distinguishes it from the phantom singularities discussed in Sec. IV B). A detailed discussion of the 'new' singularities which occur in braneworld models can be found in [41].

V. CONCLUSIONS

In this paper, we have considered cosmological implications of a braneworld model described by action (1), which contains both bulk and brane curvature terms and cosmological constants. The curvature term for the brane arises naturally, as a quantum correction from the matter part of the brane action, and significantly changes the behaviour of the braneworld theory. For example, as is well known, braneworld cosmology without this term deviates from general relativity at *large* matter densities, i.e., at early cosmological times; see Eq. (24). In the model that we considered, on the other hand, early cosmological evolution remains virtually unaffected for a broad range of parameters, significant deviations from standard cosmology appearing only during *later* times.

We restrict our attention to the important case where the brane forms the boundary of the five-dimensional bulk space. This is equivalent to endowing the bulk with the Z_2 reflection isometry with respect to the brane. The presence of the curvature term in the action leads to two families of braneworld models. These two families (called BRANE1 &

BRANE2) differ in the manner in which the brane forms the boundary of the five-dimensional bulk. For example, if a spatially three-spherical brane is embedded into the three-spherically symmetric bulk, then one can discard either the interior part of the bulk, or its exterior (asymptotically flat) part. In this example, the models which discard the exterior part leaving the interior were classified as BRANE1 models, while the complementary models were called BRANE2. Similar classification arises in the case of spatially flat and open braneworld models. Alternatively, the two different families of braneworld models can be regarded as corresponding to the two possible signs of the five-dimensional Planck mass M.

Braneworld models of dark energy considered in this paper have the interesting and unusual property that their luminosity distance d_L can *exceed* that in LCDM. This is unusual since, within the general-relativistic framework, the luminosity distance has this property *only if* the equation of state of dark energy is strongly negative ($w < -1$). Caldwell [33] has shown that dark energy with $w < -1$ (phantom energy) may in fact fit supernova observations better than LCDM. However, phantom energy is beset with a host of undesirable properties which makes this model of dark energy unattractive. We show that braneworld models have all the advantages and none of the disadvantages of phantom models and therefore endow dark energy with exciting new possibilities. A recent analysis of braneworld models in [38] has demonstrated that BRANE1 models (which generically have $w \leq -1$) provide an excellent fit to supernovae observations for higher values of the matter density ($\Omega_m \gtrsim 0.3$), whereas lower values ($\Omega_m \lesssim 0.25$) are preferred by BRANE2 models (which generically have $w \geq -1$).

A distinctive feature of the braneworld scenario discussed in this paper is that it allows for a universe which is *transiently accelerating*. Recent investigations indicate that an eternally accelerating universe, which possesses a cosmological event horizon, prevents the construction of a conventional S-matrix describing particle interactions within the framework of string or M-theory [39]. We have demonstrated that braneworld models can enter into a regime of accelerated expansion at late times *even if* the brane tension and the bulk cosmological constant are tuned to satisfy the Randall–Sundrum constraint on the brane. In this case, braneworld dark energy and the acceleration of the universe are *transient* phenomena. In this class of models, the universe, after the current period of acceleration, re-enters the matter-dominated regime. We have shown that viable models realising this behaviour are those of BRANE2 type. (Since these braneworlds have a negative brane tension the question of their stability is important and needs to be examined in detail. We shall return to this issue in a companion paper.) Thus braneworld models can give rise to a *transiently accelerating phase* thereby reconciling a dark energy dominated universe with the requirements of string/M-theory. (A similar observation was previously made in [40] in the context of an anisotropic cosmological solution of Bianchi V class.)

ACKNOWLEDGMENTS

The authors acknowledge support from the Indo-Ukrainian program of cooperation in science and technology. We also thank Tarun Deep Saini for several useful discussions and for producing Figure 3.

APPENDIX: TESTING THE MODEL

In this appendix, we present a possible procedure for testing the braneworld model against observations.

There are the following four possibilities to be considered:

(i) $\Omega_{\rm m} + \Omega_\sigma + \Omega_{\Lambda_{\rm b}} \geq 0$ [Eq. (32)].

- **(a)** $\sqrt{1 - \Omega_\kappa + \Omega_{\Lambda_{\rm b}}} - \sqrt{\Omega_\ell} > 0 \implies$ {(29), (38)} and {(30), (39)} are possible.
- **(b)** $\sqrt{1 - \Omega_\kappa + \Omega_{\Lambda_{\rm b}}} - \sqrt{\Omega_\ell} < 0 \implies$ only {(29), (38)} is possible.

(ii) $\Omega_{\rm m} + \Omega_\sigma + \Omega_{\Lambda_{\rm b}} < 0$ [Eq. (35)].

- **(a)** $\sqrt{1 - \Omega_\kappa + \Omega_{\Lambda_{\rm b}}} - \sqrt{\Omega_\ell} > 0 \implies$ only {(30), (39)} is possible.
- **(b)** $\sqrt{1 - \Omega_\kappa + \Omega_{\Lambda_{\rm b}}} - \sqrt{\Omega_\ell} < 0 \implies$ only {(29), (39)} is possible.

To see these possibilities, one must consider Eqs. (33) and (36).

Let us introduce the quantity $F \equiv \sqrt{1-\Omega_\kappa+\Omega_{\Lambda_\mathrm{b}}}-\sqrt{\Omega_\ell}$ that enters the above conditions (a) and (b). Now we note that, when using (38) for determining Ω_σ, we have

$$\Omega_\mathrm{m}+\Omega_\sigma+\Omega_{\Lambda_\mathrm{b}}=1-\Omega_\kappa+\Omega_{\Lambda_\mathrm{b}}+2\sqrt{\Omega_\ell}\,\sqrt{1-\Omega_\kappa+\Omega_{\Lambda_\mathrm{b}}}\,, \tag{A1}$$

so that condition (i) is satisfied automatically for any sign of F. Similarly, when using (39) for determining Ω_σ, we have

$$\Omega_\mathrm{m}+\Omega_\sigma+\Omega_{\Lambda_\mathrm{b}}=1-\Omega_\kappa+\Omega_{\Lambda_\mathrm{b}}-2\sqrt{\Omega_\ell}\,\sqrt{1-\Omega_\kappa+\Omega_{\Lambda_\mathrm{b}}}\,, \tag{A2}$$

so that condition (ii) is satisfied automatically for $F<0$. This means that conditions (i) and (ii) need not be checked in the procedure.

The case of the Randall–Sundrum constraint with $\Omega_\kappa=0$ is much simpler, since it allows only BRANE2 model. Note that the condition $\Omega_\ell \le \Omega_{\Lambda_\mathrm{b}}$ must be satisfied in this case. [See the explanations above Eq. (67).] The value of Ω_ℓ can be obtained from (67):

$$2\sqrt{\Omega_\ell}=\frac{1-\Omega_\mathrm{m}}{\sqrt{1+\Omega_{\Lambda_\mathrm{b}}}-\sqrt{\Omega_{\Lambda_\mathrm{b}}}}\,. \tag{A3}$$

So that

$$\Omega_\ell \le \Omega_{\Lambda_\mathrm{b}} \quad\Longrightarrow\quad 1-\Omega_\mathrm{m}+2\Omega_{\Lambda_\mathrm{b}} \le 2\sqrt{\Omega_{\Lambda_\mathrm{b}}}\sqrt{1+\Omega_{\Lambda_\mathrm{b}}}\,. \tag{A4}$$

This inequality can be simplified by taking the square of both sides:

$$\left(1-\Omega_\mathrm{m}+2\Omega_{\Lambda_\mathrm{b}}\right)^2 \le 4\Omega_{\Lambda_\mathrm{b}}\left(1+\Omega_{\Lambda_\mathrm{b}}\right) \quad\Longrightarrow\quad \left(1-\Omega_\mathrm{m}\right)^2-4\Omega_\mathrm{m}\Omega_{\Lambda_\mathrm{b}} \le 0\,. \tag{A5}$$

Finally, we get the inequality

$$\Omega_{\Lambda_\mathrm{b}} \ge \frac{\left(1-\Omega_\mathrm{m}\right)^2}{4\Omega_\mathrm{m}}\,. \tag{A6}$$

Summarizing these results, one can suggest the following procedure for testing the braneworld models:

Specify the values of Ω_ℓ, $\Omega_{\Lambda_\mathrm{b}}$, Ω_κ, and Ω_m (initial Ω's). In doing this, note that always $\Omega_\ell > 0$ by definition, $0 < \Omega_\mathrm{m} < 1$ from observations, and $1 - \Omega_\kappa + \Omega_{\Lambda_\mathrm{b}} \geq 0$ by physical consistency [see Eq. (31)]. In addition, one can assume the universe to be spatially flat: $\Omega_\kappa = 0$.

(A) To study the BRANE1 model (29) with constraint (38):

Use (38) to calculate Ω_σ, and (29) to calculate $d_L(z)$.

(B) To study the BRANE1 model (29) with constraint (39):

In specifying the values for the initial Ω's, always choose $\Omega_\ell > 1 - \Omega_\kappa + \Omega_{\Lambda_\mathrm{b}}$. Use (39) to calculate Ω_σ, and (29) to calculate $d_L(z)$.

(C) To study the BRANE2 model (30) with constraint (39):

In specifying the values for the initial Ω's, always choose $\Omega_\ell < 1 - \Omega_\kappa + \Omega_{\Lambda_\mathrm{b}}$. Use (39) to calculate Ω_σ, and (30) to calculate $d_L(z)$.

(D) The case of *disappearing dark energy* discussed in Sec. IV C and described by BRANE2 model is studied separately as follows:

Set $\Omega_\kappa = 0$, specify the value of Ω_m in the range $0 < \Omega_\mathrm{m} < 1$, and choose $\Omega_{\Lambda_\mathrm{b}}$ in the range given by (A6). Now use (67) to find Ω_ℓ, and then (65) to find Ω_σ.

Finally, use (30) to calculate $d_L(z)$.

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