A unified approach to scaling solutions in a general cosmological background

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Our ignorance about the source of cosmic acceleration has stimulated study of a wide range of models and modifications to gravity. Cosmological scaling solutions in any of these theories are privileged because they represent natural backgrounds relevant to dark energy. We study scaling solutions in a generalized background $H^2 \propto \rho_T^n$ in the presence of a scalar field φ and a barotropic perfect fluid, where H is a Hubble rate and ρ_T is a total energy density. The condition for the existence of scaling solutions restricts the form of Lagrangian to be $p = X^{1/n} g(X e^{n\lambda\varphi})$, where $X = -g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi/2$ and g is an arbitrary function. This is very useful to find out scaling solutions and corresponding scalar-field potentials in a broad class of dark energy models including (coupled)-quintessence, ghost-type scalar field, tachyon and k-essence. We analytically derive the scalar-field equation of state w_φ and the fractional density Ω_φ and apply it to a number of dark energy models.

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I. INTRODUCTION

Accelerated expansion seems to have played an extremely important role in the dynamical history of our universe. The inflationary paradigm at early epoch not only solves horizon and flatness problems but predicts nearly scale-invariant and adiabatic density perturbations consistent with temperature anisotropies in Cosmic Microwave Background (CMB). The late time acceleration is supported by observations of high redshift type Ia supernovae and, more indirectly, by observations of the CMB and galaxy clustering.

Within the framework of general relativity, cosmic acceleration can be sourced by an energy-momentum tensor which has a large negative pressure called dark energy (see Refs. [1] for review). One of the well known candidates of dark energy is provided by a cosmological constant. Although the cosmological constant does not require an *adhoc* assumption for its introduction, this suffers from an extreme fine tuning problem due to its non-dynamical nature. This problem can be alleviated in models of dynamically evolving dark energy called quintessence [2]. In addition to quintessence a wide variety of scalar field dark energy models have recently been proposed, including k-essence [3, 4], ghosts (phantoms) [5] and Born-Infeld scalars (rolling tachyon [6], massive scalars [7], phantom tachyons [8]), with the last one being originally motivated by string theory.

In order to obtain viable dark energy models, it is necessary that the energy density of the scalar field remains unimportant during most of the thermal history and emerges only at late times to account for the current acceleration of universe. It is, therefore, important to investigate cosmological scenarios in which the energy density of the scalar field mimics the background energy density. These solutions are called *scaling solutions* or *trackers*. In this paper we use "scaling solutions" as a meaning that the energy density of the field decreases proportionally to that of a barotropic perfect fluid[1]. Then the equation of state of the scalar field equals to that of the fluid ($w_\varphi = w_m$) for scaling solutions in the absence of the coupling Q between them. In this case it is not possible to get an accelerated expansion at late times provided that the background fluid is dominated by a non-relativistic dark matter ($w_m = 0$). However the coupled quintessence scenario [9] provides a possibility that scaling solutions give the acceleration of the universe with a suitable fraction of dark energy ($\Omega_\varphi \simeq 0.7$).

In General Relativity (GR) steep exponential potentials give rise to scaling solutions for a minimally coupled scalar field [10] allowing the field energy density to mimic the background being sub-dominant during radiation- and matter-dominant eras. We can obtain the current accelerated expansion provided that the exponential potential becomes shallow to support the slow-roll at large values of the field [11]. Another interesting example is the model in which steep exponential potentials reduce to a particular power-law type at late times such that the universe exits from the scaling regime (e.g., $V(\phi) = V_0 \left[\cosh(\alpha\phi/M_p) - 1\right]^q, \ q > 0$ [12]). In coupled quintessence scenarios one can exploit scaling solutions at late times as well when the coupling Q grows during the transition to a scalar-field dominant era [13]. Thus scaling solutions provide us very useful information for constructing dark energy models.

The existence of scaling solutions has been extensively studied in a number of cosmological scenarios, such as standard GR, braneworlds [Randall-Sundrum (RS) and Gauss-Bonnet (GB)], and tachyon [14]-[29]. Nevertheless these works restrict the analysis to each different

[1] We use "trackers" as a meaning that the energy density of the field simply catches up that of the fluid.

scenario. In this paper we present a unified framework to investigate scaling solutions in a general cosmological background characterized by $H^2 \propto \rho_T^n$, where H is the Hubble rate and ρ_T is the total energy density. The GR, RS, GB cases correspond to $n = 1$, $n = 2$ and $n = 2/3$, respectively. Our formalism provides a generic method to study these solutions for all the known scalar field systems like quintessence, tachyon, k-essence and ghost-type field.

We implement the coupling Q between the field and the barotropic fluid and obtain a general form of the Lagrangian for the existence of scaling solutions, see Eq. (16). Therefore our analysis includes coupled quintessence scenario that leads to an accelerated expansion even when the background fluid is dominated by non-relativistic dark matter. Our algorithm automatically generates scalar-field potentials which give rise to scaling solutions in a general cosmological background. We recover the already known solutions in a generic way and also find new solutions in the presence of the coupling Q. We also derive the equation of state w_φ and the fractional density Ω_φ for the field φ.

II. THE LAGRANGIAN FOR SCALING SOLUTIONS

Let us consider the following general 4-dimensional action

$$\mathcal{S} = \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}\,R + p(X,\varphi)\right] + \mathcal{S}_m[\varphi, \Psi_i, g_{\mu\nu}], \quad (1)$$

where R is a scalar curvature, φ is a scalar field with X defined as $X \equiv -g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi/2$, and $p(X,\varphi)$ is a scalar-field Lagrangian that is a function in terms of X and φ. $\mathcal{S}_m$ is an action for matter fields Ψ_i, which is generally dependent on φ as well. Hereafter we set the Planck mass M_p to be unity.

We shall study cosmological scaling solutions in a spatially flat Friedmann-Robertson-Walker (FRW) background spacetime:

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2\,, \quad (2)$$

where $a(t)$ is a scale factor. We consider an effective Friedmann equation which is given by

$$H^2 = \beta_n^2\rho_T^n\,, \quad (3)$$

where β_n and n are constants, and ρ_T is a total energy density of the universe. For the background (2) the equation for φ is [30]

$$\ddot\varphi\left(p_X + \dot\varphi^2 p_{XX}\right) + 3Hp_X\,\dot\varphi + 2Xp_{X\varphi} - p_\varphi = -\sigma\,, \quad (4)$$

where a suffix X or φ denotes a partial derivative with respect to X or φ, respectively. Here the scalar charge σ corresponds to the coupling between a matter and the field φ, which is defined by the relation $\delta\mathcal{S}_m/\delta\varphi = -\sqrt{-g}\,\sigma$. We consider a cosmological scenario in which the universe is filled by the scalar field φ and by one type of barotropic perfect fluid with an equation of state $w_m = p_m/\rho_m$. Rewriting Eq. (4) in terms of the energy density, $\rho = 2Xp_X - p$, of the scalar field, we get

$$\frac{\mathrm{d}\rho}{\mathrm{d}N} + 3(1+w_\varphi)\rho = -Q\rho_m\frac{\mathrm{d}\varphi}{\mathrm{d}N}\,, \quad (5)$$

where

$$N \equiv \ln a\,, \quad w_\varphi \equiv p/\rho\,, \quad Q(\varphi) \equiv \sigma/\rho_m\,. \quad (6)$$

The energy density ρ_m of the fluid satisfies

$$\frac{\mathrm{d}\rho_m}{\mathrm{d}N} + 3(1+w_m)\rho_m = Q\rho_m\frac{\mathrm{d}\varphi}{\mathrm{d}N}\,. \quad (7)$$

We define the fractional densities of ρ and ρ_m as

$$\Omega_\varphi \equiv \frac{\rho}{(H/\beta_n)^{2/n}}\,, \quad \Omega_m \equiv \frac{\rho_m}{(H/\beta_n)^{2/n}}\,, \quad (8)$$

which satisfy $\Omega_\varphi + \Omega_m = 1$ by Eq. (3).

We are interested in asymptotic scaling solutions where both the fractional density Ω_φ and the equation of state parameter w_φ are constant, which gives $\rho/\rho_m =$ const. This translates into the condition $\mathrm{d}\log\rho/\mathrm{d}N = \mathrm{d}\log\rho_m/\mathrm{d}N$. Assuming that Q is not a time-varying function in the scaling regime we get the following relation from Eqs. (5) and (7):

$$\frac{\mathrm{d}\varphi}{\mathrm{d}N} = \frac{3\Omega_\varphi}{Q}(w_m - w_\varphi) = \mathrm{const.} \quad (9)$$

Then we find the scaling behavior of ρ and ρ_m:

$$\frac{\mathrm{dlog}\rho}{\mathrm{d}N} = \frac{\mathrm{dlog}\rho_m}{\mathrm{d}N} = -3(1+w_s)\,, \quad (10)$$

where the effective equation of state is

$$w_s \equiv w_m + \Omega_\varphi(w_\varphi - w_m)\,. \quad (11)$$

Therefore ρ and ρ_m do not scale according to w_φ and w_m in the presence of the coupling Q.

From the definition of X one finds (in the FRW background)

$$2X = H^2\left(\frac{\mathrm{d}\varphi}{\mathrm{d}N}\right)^2 \propto H^2 \propto \rho_T^n\,, \quad (12)$$

which shows that the scaling property of X is the same as ρ^n and ρ_m^n. Then we obtain

$$\frac{\mathrm{dlog}X}{\mathrm{d}N} = -3n(1+w_s)\,. \quad (13)$$

Since $p = w_\varphi\rho$ scales in the same way as ρ, one has $\mathrm{d}\log p/\mathrm{d}N = -3(1+w_s)$. Here the pressure density p corresponds to the Lagrangian of the scalar field and is a function of X and φ. Therefore we obtain the following relation by using Eqs. (9) and (13):

$$n\frac{\partial \log p}{\partial \log X} - \frac{1}{\lambda}\frac{\partial \log p}{\partial \varphi} = 1\,, \tag{14}$$

where

$$\lambda \equiv Q\frac{1+w_m-\Omega_\varphi(w_m-w_\varphi)}{\Omega_\varphi(w_m-w_\varphi)}\,. \tag{15}$$

This equation gives a constraint on the functional form of $p(X,\varphi)$ for the existence of scaling solutions:

$$p(X,\varphi) = X^{1/n}\, g\left(Xe^{n\lambda\varphi}\right)\,, \tag{16}$$

where g is any function in terms of $Y \equiv Xe^{n\lambda\varphi}$. This coincides with what was obtained in Ref. [30] in the GR case ($n=1$). One can easily show that Y is constant along the scaling solution, i.e.,

$$Xe^{n\lambda\varphi} = Y_0 = \mathrm{const}\,. \tag{17}$$

This property tells us that p is proportional to $X^{1/n}$ by Eq. (16). This could be a defining property of scaling solutions which means that the Lagrangian or the pressure density depends upon the kinetic energy alone in the scaling regime. For an ordinary scalar field it leads to constancy of the ratio of kinetic to potential energy which is often taken to be a definition of scaling solutions.

From the pressure density (16) we obtain the energy density ρ as $\rho = X^{1/n}(2/n-1+2Yg'/g)g$, where a prime denotes a derivative in terms of Y. Then the equation of state $w_\varphi = p/\rho$ reads

$$w_\varphi = \left(\frac{2}{n} - 1 + 2\alpha\right)^{-1}\,, \tag{18}$$

where

$$\alpha \equiv \left.\frac{\mathrm{d}\log g(Y)}{\mathrm{d}\log Y}\right|_{Y=Y_0}\,. \tag{19}$$

Making use of Eqs. (9), (12) and (15), we get

$$3H^2 = \frac{2(Q+\lambda)^2}{3(1+w_m)^2}X\,. \tag{20}$$

Then the fractional density (8) of the field φ yields

$$\Omega_\varphi = \left(\frac{9\beta_n^2(1+w_m)^2}{2(Q+\lambda)^2}\right)^{1/n}\frac{g(Y_0)}{w_\varphi}\,. \tag{21}$$

By combining Eq. (9) with Eq. (8) together with the relation $w_\varphi = p/\rho$, we find that g in Eq. (16) can be written as

$$g(Y_0) = -Q\left(\frac{2}{9\beta_n^2}\right)^{1/n}\frac{w_\varphi}{w_\varphi - w_m}\left(\frac{1+w_m}{Q+\lambda}\right)^{(n-2)/n}\,. \tag{22}$$

Then Eq. (21) yields

$$\Omega_\varphi = \frac{Q}{Q+\lambda}\frac{1+w_m}{w_m-w_\varphi}\,. \tag{23}$$

Once the functional form of $g(Y)$ is known, the equation of state w_φ is determined by Eq. (18) with Eq. (19). Then we get the fractional density Ω_φ from Eq. (23).

For scaling solutions an acceleration parameter yields

$$-q \equiv \frac{\ddot{a}a}{\dot{a}^2} = 1 - \frac{3n(1+w_m)\lambda}{2(\lambda+Q)}\,. \tag{24}$$

When $Q=0$ the condition $-q>0$ gives $w_m < 2/(3n)-1$. For example $w_m < -1/3$ for $n=1$. For non-relativistic dark matter ($w_m=0$), an accelerated expansion occurs only for $n<2/3$. If we account for the coupling Q, it is possible to get an acceleration even for $n \geq 2/3$. The condition for acceleration corresponds to

$$\frac{Q}{\lambda} > \frac{3n(1+w_m)-2}{2}\,, \tag{25}$$

which is useful for the construction of realistic dark energy scenarios.

III. APPLICATION TO DARK ENERGY MODELS

A. Ordinary scalar fields

Let us first obtain the form of lagrangian when p is written in the form

$$p(X,\varphi) = f(X) - V(\varphi)\,. \tag{26}$$

Then by Eq. (14) one gets

$$nX\frac{\mathrm{d}f}{\mathrm{d}X} - f(X) = -\frac{1}{\lambda}\frac{\mathrm{d}V}{\mathrm{d}\varphi} - V \equiv C\,, \tag{27}$$

where C is a constant. Integrating this relation gives $f = c_1X^{1/n} - C$ and $V = c_2e^{-\lambda\varphi} - C$, which restricts the form of the Lagrangian to be

$$p = c_1X^{1/n} - c_2e^{-\lambda\varphi}\,. \tag{28}$$

In the case of GR ($n=1$), this corresponds to a standard canonical scalar field with an exponential potential [10]. When $n \neq 1$ the Lagrangian does not take a canonical form, so the exponential potential does not correspond to scaling solutions.

One may look for scaling solutions that give a standard kinematic term by an appropriate transformation to a new variable in Eq. (16). Introducing a new variable $\phi \equiv e^{\beta\lambda\varphi}$, we obtain $Y_0 = \tilde{X}\phi^{(n-2\beta)/\beta}/\beta^2\lambda^2 = \mathrm{const}$, where $\tilde{X} \equiv -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi/2$. Then the Lagrangian (16) can be written as

$$p = \frac{Y_0^{1/n}}{\phi^{1/\beta}}g(Y_0) = Y_0^{1/n}\left(\frac{\tilde{X}}{\beta^2\lambda^2Y_0}\right)^{1/(n-2\beta)}g(Y_0)\,. \tag{29}$$

Since p is proportional to $\tilde{X}^{1/(n-2\beta)}$, the transformation that gives $p \propto \tilde{X}$ corresponds to $\beta = (n-1)/2$, i.e., $\phi = e^{(n-1)\lambda\varphi/2}$. In this case we have $p \propto \phi^{-2/(n-1)}$ by Eq. (29), which means that the potential of the field ϕ corresponding to scaling solutions is

$$V(\phi) = V_0 \phi^{-2/(n-1)}\,, \tag{30}$$

where V_0 is a constant. For example one has an inverse square potential $V(\phi) = V_0\phi^{-2}$ for $n = 2$, which agrees with what was obtained in Ref. [23]. The Gauss-Bonnet braneworld ($n = 2/3$) gives the potential $V(\phi) = V_0\phi^6$, as shown in Ref. [27].

If we choose the function $g(Y)$ as

$$g(Y) = c_1 Y^{1-1/n} - c_2 Y^{-1/n}\,, \tag{31}$$

then we get the Lagrangian

$$\begin{aligned} p &= c_1 X e^{(n-1)\lambda\varphi} - c_2 e^{-\lambda\varphi} && (32)\\ &= \frac{4c_1}{(n-1)^2\lambda^2}\tilde{X} - c_2\phi^{-2/(n-1)}\,, && (33)\end{aligned}$$

where the last equality is valid for $n \neq 1$. Therefore the choice (31) gives the canonical Lagrangian with potential given by (30) for $n \neq 1$. When $n = 1$ the lagrangian (32) itself is canonical. We can obtain w_φ and Ω_φ in these cases by using the function (31). Note that a normal scalar field corresponds to $\epsilon \equiv 4c_1/((n-1)^2\lambda^2) > 0$, whereas a ghost field to $\epsilon < 0$.

1. The GR case

Let us consider the GR case ($n = 1$). Since $c_2/Y_0 = c_1 - g(Y_0)$ by Eq. (31), one has $\alpha = c_1/g(Y_0) - 1$ by using Eq. (19). From Eq. (22) we find that $g(Y_0)$ can be written as

$$g(Y_0) = \frac{1}{1+w_m}\left[2c_1 w_m - \frac{2Q(Q+\lambda)}{3(1+w_m)}\right]. \tag{34}$$

Then we obtain the equation of state

$$w_\varphi = \frac{3c_1 w_m(1+w_m) - Q(Q+\lambda)}{3c_1(1+w_m) + Q(Q+\lambda)}\,. \tag{35}$$

Inserting this into Eq. (23) gives

$$\Omega_\varphi = \frac{3c_1(1+w_m) + Q(Q+\lambda)}{(Q+\lambda)^2}\,. \tag{36}$$

When $Q = 0$ the above results reduce to $w_\varphi = w_m$ and $\Omega_\varphi = 3c_1(1+w_m)/\lambda^2$. This coincides with the scaling solution for an exponential potential obtained in Ref. [10]. In this case although the energy density of the field φ contributes to some portion of the total energy density, we can not obtain an acceleration of the universe for a normal fluid satisfying $w_\varphi > -1/3$. However the presence of the coupling Q opens up a possibility of an accelerated expansion. In the case of non-relativistic dark matter ($w_m = 0$), we obtain $w_\varphi = -Q(Q+\lambda)/(3c_1 + Q(Q+\lambda))$ and $\Omega_\varphi = (3c_1 + Q(Q+\lambda))/(Q+\lambda)^2$, which agrees with the coupled quintessence scenario in Ref. [9].

2. The RS case

In the RS case ($n = 2$) we have $c_2/\sqrt{Y_0} = c_1\sqrt{Y_0} - g(Y_0)$, $\alpha = -1/2 + c_1\sqrt{Y_0}/g(Y_0)$ and $w_\varphi = g(Y_0)/(2c_1\sqrt{Y_0})$. Hereafter we shall use the parameter $\epsilon = 4c_1/\lambda^2$ that is positive for an ordinary scalar field. Then by Eq. (22) one gets

$$g(Y_0) = \frac{\epsilon\lambda^2}{2}\sqrt{Y_0}w_m - \frac{\sqrt{2}Q}{3\beta_2}\,, \tag{37}$$

where

$$\frac{\epsilon\lambda^2}{2}\sqrt{Y_0} = \frac{-\sqrt{2}Q/(3\beta_2) + \sqrt{2Q^2/(9\beta_2^2) + \epsilon\lambda^2 c_2(1-2w_m)}}{1-2w_m}\,. \tag{38}$$

Here the conditions $\epsilon > 0$ and $c_2 > 0$ are assumed.

The equation of state for the field φ is

$$w_\varphi = w_m - \frac{(1-2w_m)Q}{-Q + \sqrt{Q^2 + (9/2)\beta_2^2\epsilon\lambda^2 c_2(1-2w_m)}}\,. \tag{39}$$

By Eq. (21) we find

$$\Omega_\varphi = \frac{1+w_m}{1-2w_m}\frac{-Q + \sqrt{Q^2 + (9/2)\beta_2^2\epsilon\lambda^2 c_2(1-2w_m)}}{|Q+\lambda|}\,. \tag{40}$$

When $Q = 0$ one has $w_\varphi = w_m$ with a nonzero value of Ω_φ, as is similar to the case of $n = 1$. If we include the coupling Q, we have $w_\varphi = -Q/(\sqrt{Q^2 + (9/2)\beta_2^2\epsilon\lambda^2 c_2} - Q)$ and $\Omega_\varphi = (\sqrt{Q^2 + (9/2)\beta_2^2\epsilon\lambda^2 c_2} - Q)/|Q+\lambda|$ for $w_m = 0$. Therefore it is possible to have an accelerated expansion in the presence of the coupling Q.

B. Phantoms and ghost condensates

A ghost (phantom) scalar field corresponds to a negative sign of c_1 in Eq. (31). In the GR case with $Q = 0$ one has $\Omega_\varphi = 3c_1(1+w_m)/\lambda^2 < 0$ for $w_m > -1$, which means the absence of viable scaling solutions. In the presence of the coupling Q there exist scaling solutions that satisfy the condition for an accelerated expansion, see Eqs. (35) and (36). In the RS case since $g(Y_0)/w_\varphi = 2c_1\sqrt{Y_0}$ is negative in Eq. (21) for $c_1 < 0$, one can not obtain viable scaling solutions.

We need to keep in mind that phantoms are generally plagued by severe ultraviolet quantum instabilities [31]. However it was shown in Ref. [32] that a scalar field with a negative sign kinematic term does not necessarily lead to inconsistencies, provided that a suitable structure of higher-order kinematics terms are present in the effective theory. Let us consider the Lagrangian of the form

$$p = \epsilon X + c e^{\lambda\varphi} X^2\,, \tag{41}$$

where negative ϵ corresponds to the phantom. This is motivated by dilatonic higher-order corrections in low energy effective string theory [30]. Since the function $g(Y)$

is $g(Y) = \epsilon + cY$ in the GR case ($n = 1$), we obtain $\alpha = cY_0/(\epsilon + cY_0)$, $w_\varphi = (\epsilon + cY_0)/(\epsilon + 3cY_0)$ and

$$cY_0 = -\frac{2Q(Q+\lambda) + 3\epsilon(1-w_m^2)}{3(1+w_m)(1-3w_m)}\,. \quad (42)$$

Then we get

$$w_\varphi = \frac{3\epsilon(1+w_m)w_m + Q(Q+\lambda)}{3\epsilon(1+w_m) + 3Q(Q+\lambda)}\,, \quad (43)$$

$$\Omega_\varphi = \frac{3(1+w_m)\left[-\epsilon(1+w_m) - Q(Q+\lambda)\right]}{(Q+\lambda)^2(1-3w_m)}\,. \quad (44)$$

This agrees with those obtained in Ref. [30] for $w_m = 0$ and $\epsilon = -1$. The stability of quantum fluctuations is ensured for $p_X + 2Xp_{XX} \geq 0$ and $p_X \geq 0$, which corresponds to the condition $cY_0 \geq 1/2$ in our case. This translates into the relation $Q(Q+\lambda) \leq 3(1+w_m)^2/4$. On the other hand the condition for acceleration requires $Q/\lambda > (1+3w_m)/2$. The values of the coupling Q satisfying both of these conditions provide viable scaling solutions.

C. Tachyon fields

The Lagrangian for a tachyon field is given by [6]

$$p = -V(\phi)\sqrt{1-\dot{\phi}^2}\,. \quad (45)$$

Apparently the general form (16) does not seem to include this case, but one can rewrite the Lagrangian (16) by introducing a new field $\phi = e^{\beta\lambda\varphi}/(\beta\lambda)$. Since the quantity Y is written as $Y = \tilde{X}(\beta\lambda\phi)^{n/\beta-2}$ with $\tilde{X} \equiv -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi/2$ under this transformation, we obtain $Y = \tilde{X}$ for $\beta = n/2$. Then the Lagrangian (16) yields

$$p = \left(\frac{n\lambda\phi}{2}\right)^{-2/n}\tilde{X}^{1/n}g(\tilde{X})\,. \quad (46)$$

This is a system $p(\tilde{X},\phi) = V(\phi)f(\tilde{X})$ with potential

$$V(\phi) = V_0\phi^{-2/n}\,, \quad (47)$$

and $f(\tilde{X}) = \tilde{X}^{1/n}g(\tilde{X})$.

We get the tachyon system (45) by the choice

$$g(Y) = -cY^{-1/n}\sqrt{1-2\epsilon Y}, \quad \text{with} \quad \epsilon = 1\,, \quad (48)$$

where c is positive. We introduce a parameter ϵ so that the system includes a phantom tachyon field with $\epsilon = -1$ [8]. By Eq. (47) one has an inverse square potential $V(\phi) = V_0\phi^{-2}$ for $n = 1$ (GR) for the existence of scaling solutions [24]. We have $V(\phi) = V_0\phi^{-1}$ for $n = 2$ (RS) and $V(\phi) = V_0\phi^{-3}$ for $n = 2/3$ (GB). Hereafter we shall derive w_φ and Ω_φ for the function (48) in GR and RS cases.

1. The GR case

When $n = 1$ we obtain $g(Y_0) = -c\sqrt{1-2\epsilon Y_0}/Y_0$, $\alpha = (\epsilon Y_0 - 1)/(1-2\epsilon Y_0)$ and $w_\varphi = 2\epsilon Y_0 - 1$. Since $1 - 2\epsilon Y_0 \geq 0$ the equation of state ranges in $w_\varphi \leq 0$. Making use of Eq. (22) and $g(Y_0) = -2c\sqrt{-w_\varphi}/(w_\varphi + 1)$, one gets the following equation

$$\frac{x^2 + w_m}{x(1-x^2)} = \frac{Q(Q+\lambda)}{3c\epsilon(1+w_m)}\,, \quad (49)$$

where $x \equiv \sqrt{-w_\varphi}$. The solution in the limit $Q \to 0$ corresponds to $x = \sqrt{-w_m}$, i.e., $w_\varphi = w_m$. Note that the existence of scaling solutions requires the condition $w_m < 0$, as was pointed out in Ref. [24].

One can approximately obtain the solution for Eq. (49) when the coupling Q is small. Let us write the solution as $x = \sqrt{-w_m} + \delta$, where δ is small relative to $\sqrt{-w_m}$. Substituting this for Eq. (49), we find that δ is given by $\delta = \lambda Q/(6c\epsilon)$. Then we get

$$w_\varphi = w_m - \frac{\lambda\sqrt{-w_m}}{3c\epsilon}Q\,, \quad (50)$$

$$\Omega_\varphi = \frac{3c\epsilon}{(Q+\lambda)\lambda}\frac{1+w_m}{\sqrt{-w_m}}\,, \quad (51)$$

which are valid when the coupling Q is small. In the limit $Q \to 0$ one has $\Omega_\varphi = 3c\epsilon(1+w_m)/(\lambda^2\sqrt{-w_m})$, which agrees with the result in Ref. [24] for $\epsilon = 1$. The phantom tachyon ($\epsilon = -1$) corresponds to $\Omega_\varphi < 0$ for $-1 < w_m < 0$ under the acceleration condition (25), which means that viable scaling solutions do not exist.

Even if scaling solutions do not exist for a fluid satisfying $w_m \geq 0$, this does not mean that we do not have a plausible dark energy scenario in the tachyon system. In fact there is a stable critical point that approaches $\Omega_\varphi = 1$ and $\Omega_m = 0$ with an accelerated expansion for the potential (47) [24]. One can construct a viable dark energy model that evolves toward this critical point in the future with the present value $\Omega_\varphi \sim 0.7$ provided that initial conditions of the field are appropriately chosen [33]. In addition an inverse power-law potential $V(\phi) \propto \phi^{-q}$ with $q < 2$ leads to an acceleration of the universe at late times, while it does not for $q > 2$ [34]. Thus the potential corresponding to scaling solutions marks the border between accelerated and decelerated expansions, which provides a useful information for the construction of dark energy models.

2. The RS case

When $n = 2$ we have $g(Y_0) = -c\sqrt{1/Y_0 - 2\epsilon}$, $\alpha = -1/(2(1-2\epsilon Y_0))$ and $w_\varphi = 1 - 1/(2\epsilon Y_0)$, which means that $g(Y_0) = -c\sqrt{-2\epsilon w_\varphi}$. Then $w_\varphi < 0$ for $\epsilon > 0$ and $w_\varphi > 0$ for $\epsilon < 0$. By using these relations together with Eq. (22), we find that

$$\frac{x^2 + \epsilon w_m}{x} = \frac{Q}{3c\beta_2}\,, \quad (52)$$

where $x \equiv \sqrt{-\epsilon w_\varphi}$. In this case one can obtain w_φ and Ω_φ for any values of Q, i.e.,

$$w_\varphi = -\frac{1}{4\epsilon}\left[\frac{Q}{3c\beta_2} + \sqrt{\frac{Q^2}{9c^2\beta_2^2} - 4\epsilon w_m}\right]^2, \qquad (53)$$

$$\Omega_\varphi = 6\epsilon c\beta_2 \left|\frac{1+w_m}{Q+\lambda}\right| \left[\frac{Q}{3c\beta_2} + \sqrt{\frac{Q^2}{9c^2\beta_2^2} - 4\epsilon w_m}\right]^{-1}. \qquad (54)$$

When $\epsilon = 1$ and $Q = 0$ scaling solutions exist only for $w_m < 0$, but the presence of the coupling Q allows a possibility of their existence even for $w_m > 0$. When $\epsilon = -1$ one has $w_\varphi > 0$ and $\Omega_\varphi < 0$ by Eqs. (53) and (54), which means the absence of ideal scaling solutions.

D. K-essence

K-essence scenario is characterized by the pressure density [3]

$$p = V(\phi) f(\tilde{X}), \qquad (55)$$

where $\tilde{X} = -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi/2$. Since the Lagrangian (16) can be written as a decoupled form (45) under a transformation $\phi = 2e^{n\lambda\varphi/2}/(n\lambda)$, the k-essence Lagrangian (55) has a scaling solution for

$$V(\phi) \propto \phi^{-2/n}, \quad \text{and} \quad \tilde{X} = \text{const}. \qquad (56)$$

Note that the function $f(\tilde{X})$ can be chosen arbitrary provided that the conditions (56) are satisfied. These conditions mean that the field ϕ evolves with a constant velocity along the potential $V(\phi) = V_0\phi^{-2/n}$. This is a general property of scaling solutions in k-essence scenario.

The pressure density of the form,

$$p(X,\varphi) = K(\varphi)X + L(\varphi)X^2, \qquad (57)$$

is transformed to the Lagrangian (55) with $V(\phi) = K^2/L$ and $f(\tilde{X}) = -\tilde{X} + \tilde{X}^2$ by the field redefinitions [4]:

$$\phi = \int^\varphi \mathrm{d}\varphi \sqrt{\frac{L}{|K|}}, \quad \tilde{X} = \frac{L}{|K|}X. \qquad (58)$$

Therefore the dilatonic ghost condensate considered in Sec. III B belongs to a class of k-essence. In fact the Lagrangian (41) corresponds to $K = \epsilon$ and $L = ce^{\lambda\varphi}$, which gives $\phi \propto e^{\lambda\varphi/2}$ and $V \propto e^{-\lambda\varphi} \propto \phi^{-2}$. Therefore scaling solutions (43) and (44) can be viewed as the system (56) with $n = 1$.

In this section we dealt with a variety of dynamical systems in GR and RS backgrounds; a comment on the GB dynamics is in order. We find that the fundamental features of scaling solutions which appear in GR and RS cases also persist in the GB background. Since the algebra gets merely cumbersome, we have not shown the results in the GB case.

IV. CONCLUSIONS

In this paper we discussed cosmological scaling solutions in a general cosmological background $H^2 \propto \rho_T^n$ including General Relativity, Randall-Sundrum braneworld and Gauss-Bonnet braneworld. The condition for the existence of scaling solutions restricts the form of the Lagrangian to be Eq. (16). Since the starting action (1) is very general, the formula (16) is applicable for a wide variety of dark energy models such as (coupled)-quintessence, ghost-type scalar field, tachyon and k-essence. This is a powerful tool to find out scaling solutions and corresponding effective potentials in *any* scalar-field system.

We analytically derived the scalar-field equation of state w_φ and the fractional density Ω_φ in general, see Eqs. (18) and (23) with (19). We applied these formula to a number of dark energy models and discussed the existence of viable scaling solutions. In the absence of the coupling Q between a scalar field and a perfect barotropic fluid, it is not possible to get an acceleration of the universe since the energy density of the field φ decreases in proportional to that of the background fluid for scaling solutions. However the presence of the coupling Q allows to have an accelerated expansion with an effective equation of state given by (11).

We have reproduced previous results about the form of the scalar field potential in the GR case ($n = 1$) in the context of quintessence and tachyon. We derived the potentials (30) and (47) for scaling solutions from the Lagrangian (16) by redefining a new field ϕ. These may be obtained by considering the condition for power-law inflation together with slow-roll parameters [27], but out treatment is more general since we did not use any approximations.

We also applied our formula to a ghost-type scalar field and k-essence. We accounted for dilatonic higher-order terms $e^{\lambda\varphi}X^2$ in order to avoid severe ultraviolet instabilities present for a phantom field with a negative sign of X. This scenario is closely linked with k-essence, since the Lagrangian (41) is transformed to (55) by a field redefinition. By using Eq. (16), we showed that the Lagrangian (55) has a scaling solution when $V(\phi) \propto \phi^{-2/n}$ and $\tilde{X} = \text{const}$. Provided that these conditions are satisfied, scaling solutions exist for any arbitrary function $f(\tilde{X})$ in Eq. (55) including the tachyon case.

If the scalar-field potential is not steeper than the one for scaling solutions, this leads to an accelerated expansion since scaling solutions give the border of acceleration and deceleration. Therefore our formalism is useful to construct realistic dark energy models. It is also of interest to place constraints on scalar-field potentials using supernova and CMB data and discriminate between a host of dark energy models from future high-precision observations.

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