Modern Physics Letters A, Vol. 17, Nos. 15–17 (2002) 923–942

THERMODYNAMICS OF HORIZONS: A COMPARISON OF SCHWARZSCHILD, RINDLER AND de SITTER SPACETIMES

T. PADMANABHAN
IUCAA, P.O.Box 4, Ganeshkhind, Pune 411 007, Maharashtra, India

Received 21 February 2002
Revised 28 April 2002

The notions of temperature, entropy and 'evaporation', usually associated with spacetimes with horizons, are analyzed using general approach and the following results, applicable to different spacetimes, are obtained at one go. (i) The concept of temperature associated with the horizon is derived in a unified manner and is shown to arise from *purely kinematic considerations.* (ii) QFT near any horizon is mapped to a conformal field theory without introducing concepts from string theory. (iii) For spherically symmetric spacetimes (in $D = 1 + 3$) with a horizon at $r = l$, the partition function has the generic form $Z \propto \exp[S - \beta E]$, where $S = (1/4)4\pi l^2$ and $|E| = (l/2)$. This analysis reproduces the conventional result for the black hole spacetimes and provides a simple and consistent interpretation of entropy and energy $E = -(1/2)H^{-1}$ for deSitter spacetime. The classical Einstein's equations for this spacetime can be expressed as a thermodynamic identity, $TdS - dE = PdV$ with the same variables. (iv) For the Rindler spacetime the entropy per unit transverse area turns out to be (1/4) while the energy is zero. (v) In the case of a Schwarzschild black hole there exist quantum states (like Unruh vacuum) which are not invariant under time reversal and can describe black hole evaporation. There also exist quantum states (like Hartle-Hawking vacuum) in which temperature is well-defined but there is no flow of radiation to infinity. In the case of deSitter universe or Rindler patch in flat spacetime, one usually uses quantum states analogous to Hartle-Hawking vacuum and obtains a temperature without the corresponding notion of evaporation. It is, however, possible to construct the analogues of Unruh vacuum state in the other cases as well. The implications are briefly discussed.

1. Motivation

One of the remarkable features of classical gravity is that it can wrap up regions of spacetime thereby producing surfaces which act as one way membranes. The classic examples are those of Schwarzschild black hole and deSitter universe which have compact surfaces that act as horizons. The existence of one-way membranes, however, is not necessarily a feature of gravity or curved spacetime and can be induced even in flat Minkowski spacetime. It is possible to introduce coordinate charts in Minkowski spacetime such that regions are separated by horizons, a familiar example being the Rindler frame which has a non-compact horizon. (All the horizons are implicitly defined with respect to certain class of observers; for example, a suicidal observer plunging into the Schwarzschild black hole will describe the physics very differently from an observer at infinity. From this point of view, which I shall adopt, there is no need to distinguish between observer dependent and observer independent horizons.)

The study of QFT in these spacetimes suggests a natural way of associating a temperature with the spacetimes which have Killing horizons, once the normalisation of the Killing vector field is chosen.[1] The operator equations for QFT in the background metric are well defined in these spacetimes; but to make useful predictions we also need to choose a quantum state for the field. The Schwarzschild, deSitter and Rindler metrics are symmetric under time reversal and there exists a 'natural' definition of a time symmetric vacuum state in all these cases. Such a vacuum state will appear to be described a thermal density matrix in a subregion $\mathcal{R}$ of spacetime with the horizon as a boundary. The QFT based on such a state will be manifestedly time symmetric and will describe an isolated system in thermal equilibrium in the subregion $\mathcal{R}$. No time asymmetric phenomena like evaporation, outgoing radiation, irreversible changes etc can take place in this situation.

One would next ask whether one can associate an *entropy* with such spacetimes in a sensible manner, given that the notion of *temperature* arises very naturally.[2] Conventionally there are two very different ways of defining the entropy, given the notion of temperature: (1) In statistical mechanics, the partition function $Z(\beta)$ of the canonical ensemble of systems with constant temperature β^{-1} is related to the entropy S and energy E by $Z(\beta) \propto \exp(S - \beta E)$. (2) In classical thermodynamics, on the other hand, it is the *change in* the entropy, which can be operationally defined via $dS = dE/T(E)$. Integrating this equation will lead to the function $S(E)$ except for an additive constant which needs to be determined from additional considerations. Proving the equality of these two concepts was nontrivial and — historically — led to the unification of thermodynamics with mechanics.

In the case of time symmetric vacuum state, there will be no change of entropy dS and the thermodynamic route is blocked. I will show, however, that it is possible to construct a canonical ensemble of a class of spacetimes and evaluate the partition function $Z(\beta)$. For spherically symmetric spacetimes with a horizon at $r = l$, the partition function has the generic form $Z \propto \exp[S - \beta E]$, where $S = (1/4)4\pi l^2$ and $|E| = (l/2)$. This analysis reproduces the conventional result for the black hole spacetimes and provides a simple and consistent interpretation of entropy and energy for deSitter spacetime, with the latter being given by $E = -(1/2)H^{-1}$. For the Rindler spacetime the entropy per unit transverse area turns out to be (1/4) while the energy is zero. In fact, it is possible to write Einstein's equations for a spherically symmetric spacetime as a thermodynamic identity $TdS - dE = PdV$ with T, S and E determined as above and the PdV term arising from the source.[3]

I will also show how to construct radiating states in all these spacetimes such that the thermodynamic approach to entropy can also be realised. It will turn out that there is no mathematical distinction between the horizons which arise in the Schwarzschild, deSitter and Rindler spacetimes. There is no simple way one can associate entropy with black holes *without* associating entropy with Rindler or deSitter. It is all or none.

2. A Unified Approach to Spacetimes with Horizons

Consider a (D+1) dimensional flat Lorentzian manifold $\mathcal{S}$ with the signature $(+,-,-,\ldots)$ and Cartesian coordinates Z^A where $A=(0,1,2,\ldots,D)$. A four dimensional sub-manifold $\mathcal{D}$ in this (D+1) dimensional space can be defined through a mapping $Z^A = Z^A(x^a)$ where x^a with $a=(0,1,2,3)$ are the four dimensional coordinates on the surface. The flat Lorentzian metric in the (D+1) dimensional space induces a metric $g_{ab}(x^a)$ on the four dimensional space which — for a wide variety of the mappings $Z^A = Z^A(x^a)$ — will have the signature $(+,-,-,-)$ and will represent, in general, a curved four geometry. The quantum theory of a free scalar field in $\mathcal{S}$ is well defined in terms of the, say, plane wave modes which satisfy the wave equation in $\mathcal{S}$. A subset of these modes, which does not depend on the 'transverse' directions, will satisfy the corresponding wave equation in $\mathcal{D}$ and will depend only on x^a. These modes induce a natural QFT in $\mathcal{D}$. We are interested in the mappings $Z^A = Z^A(x^a)$ which leads to a horizon in $\mathcal{D}$ so that we can investigate the QFT in spacetimes with horizons using the free, flat spacetime, QFT in $\mathcal{S}$. [This approach was used earlier by Deser and Levin[4] for some special cases. But I use this technique with a different motivation, in a more general context, to draw some important conclusions.]

For this purpose, I will restrict attention to a class of surfaces defined by the mappings $Z^A = Z^A(x^a)$ which ensures the following properties for $\mathcal{D}$: (i) The induced metric g_{ab} has the signature $(+,-,-,-)$. (ii) The induced metric is static in the sense that $g_{0\alpha}=0$ and all g_{ab}s are independent of x^0. [The Greek indices run over 1,2,3.] (iii) Under the transformation $x^0 \to x^0 \pm i(\pi/g)$, where g is a non zero, positive constant, the mapping of the coordinates changes as $Z^0 \to -Z^0$, $Z^1 \to -Z^1$ and $Z^A \to Z^A$ for $A=2,\ldots,D$. It will turn out that the four dimensional manifolds defined by such mappings possess a horizon and most of the interesting features of the thermodynamics related to the horizon can be obtained from the above characterization. Let us first determine the nature of the mapping $Z^A = Z^A(x^a) = Z^A(t,\mathbf{x})$ such that the above conditions are satisfied.

The condition (iii) above singles out the spatial coordinate Z^1 from the others. To satisfy this condition we can take the mapping $Z^A = Z^A(t,r,\theta,\phi)$ to be of the form $Z^0 = Z^0(t,r)$, $Z^1 = Z^1(t,r)$, $Z^\perp = Z^\perp(r,\theta,\phi)$ where $Z^\perp$ denotes the transverse coordinates Z^A with $A=(2,\ldots,D)$. To impose the condition (ii) above, one can make use of the fact that $\mathcal{S}$ possesses invariance under translations, rotations and Lorentz boosts which are characterized by the existence of a set of $N=(1/2)(D+1)(D+2)$ Killing vector fields $\xi^A(Z^A)$. Consider any linear combination V^A of these Killing vector fields which is timelike in a region of $\mathcal{S}$. The integral curves to this vector field V^A will define time like curves in $\mathcal{S}$. If one treats these curves as the trajectories of a hypothetical observer, then one can set up the proper Fermi-Walker transported coordinate system for this observer. Since the four velocity of the observer is along the Killing vector field, it is obvious that the metric in this coordinate system will be static.[5] In particular, there exists a Killing vector which corresponds to Lorentz boosts along the Z^1 direction that can be interpreted as rotation in imaginary time coordinate allowing a natural realization of (iii) above. Using the property of Lorentz boosts, it is easy to see that the transformations of the form $Z^0 = lf(r)^{1/2}\sinh gt$; $Z^1 = \pm lf(r)^{1/2}\cosh gt$ will satisfy both conditions (ii) and (iii) where (l, g) are constants introduced for dimensional reasons and $f(r)$ is a given function. This map covers only the two quadrants with $|Z^1| > |Z^0|$ with positive sign for the right quadrant and negative sign for the left. To cover the entire (Z^0, Z^1) plane, we will use the full set

$$Z^0 = lf(r)^{1/2}\sinh gt\,; \quad Z^1 = \pm lf(r)^{1/2}\cosh gt \quad (\text{for } |Z^1| > |Z^0|) \tag{1}$$

$$Z^0 = l[-f(r)]^{1/2}\cosh gt\,; \quad Z^1 = \pm l[-f(r)]^{1/2}\sinh gt \quad (\text{for } |Z^1| < |Z^0|) \tag{2}$$

The inverse transformations corresponding to (1) are

$$l^2 f(r) = (Z^1)^2 - (Z^0)^2\,; \quad gt = \tanh^{-1}(Z^0/Z^1) \tag{3}$$

Clearly, to cover the entire two dimensional plane of $-\infty < (Z^0, Z^1) < +\infty$, it is necessary to have both $f(r) > 0$ and $f(r) < 0$. The pair of points (Z^0, Z^1) and $(-Z^0, -Z^1)$ are mapped to the same (t, r) making this a 2-to-1 mapping. The null surface $Z^0 = \pm Z^1$ is mapped to the surface $f(r) = 0$.

The transformations given above with any arbitrary mapping for the transverse coordinate $Z^\perp = Z^\perp(r, \theta, \phi)$ will give rise to an induced metric on $\mathcal{D}$ of the form

$$ds^2 = f(r)(lg)^2 dt^2 - \frac{l^2}{4}\left(\frac{f'^2}{f}\right) dr^2 - dL_\perp^2 \tag{4}$$

where $dL_\perp^2$ depends on the form of the mapping $Z^\perp = Z^\perp(r, \theta, \phi)$. This form of the metric is valid in all the quadrants even though we will continue to work in the right quadrant and will comment on the behaviour in other quadrants only when necessary. It is obvious that the $\mathcal{D}$, in general, is curved and has a horizon at $f(r) = 0$.

As a specific example, let us consider the case of (D+1)=6 with the coordinates $(Z^0, Z^1, Z^2, Z^3, Z^4, Z^5) = (Z^0, Z^1, Z^2, R, \Theta, \Phi)$ and consider a mapping to 4-dimensional subspace in which: (i) The (Z^0, Z^1) are mapped to (t, r) as before; (ii) the spherical coordinates (R, Θ, Φ) in $\mathcal{S}$ are mapped to standard spherical polar coordinates in $\mathcal{D}$: (r, θ, φ) *and* (iii) we take Z^2 to be an arbitrary function of r: $Z^2 = q(r)$. This leads to the metric

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2 d\Omega^2_{2-\text{sphere}}\,; \tag{5}$$

with

$$A(r) = (lg)^2 f\,; \qquad B(r) = 1 + q'^2 + \frac{l^2}{4}\frac{f'^2}{f} \tag{6}$$

Equation (5) is the form of a general, spherically symmetric, static metric in 4-dimension with two arbitrary functions $f(r)$, $q(r)$. Given any specific metric with $A(r)$ and $B(r)$, equations (6) can be solved to determine $f(r)$, $q(r)$. As an example, let us consider the Schwarzschild solution for which we will take $f = 4[1 - (l/r)]$; the condition $g_{00} = (1/g_{11})$ now determines $q(r)$ through the equation

$$(q')^2 = \left(1 + \frac{l^2}{r^2}\right)\left(1 + \frac{l}{r}\right) - 1 = \left(\frac{l}{r}\right)^3 + \left(\frac{l}{r}\right)^2 + \frac{l}{r} \tag{7}$$

That is

$$q(r) = \int^r \left[\left(\frac{l}{r}\right)^3 + \left(\frac{l}{r}\right)^2 + \frac{l}{r}\right]^{1/2} dr \tag{8}$$

Though the integral cannot be expressed in terms of elementary functions, it is obvious that $q(r)$ is well behaved everywhere including at $r = l$. The transformations $(Z^0, Z^1) \to (t, r)$; $Z^2 \to q(r)$; $(Z^3, Z^4, Z^5) \to (r, \theta, \varphi)$ thus provide the embedding of Schwarzschild metric in a 6-dimensional space. [This result was originally obtained by Frondsal;[6] but the derivation in that paper is somewhat obscure and does not bring out the generality of the situation]. As a corollary, we may note that this procedure leads to a spherically symmetric Schwarzschild-like metric in arbitrary dimension, with the 2-sphere in (5) replaced any N-sphere.

Incidentally, the choice $lg = 1, f(r) = [1 - (r/l)^2]$ will provide an embedding of the deSitter spacetime in 6-dimensional space with $Z^2 = r$, $(Z^3, Z^4, Z^5) \to (r, \theta, \phi)$. Of course, in this case, one of the coordinates is actually redundant and we can achieve the embedding in 5-dimensional space. A still more trivial case is that of Rindler metric which can be obtained with D=3, $lg = 1$, $f(r) = 1+2gr$; in this case, the "embedding" is just a reparametrization within four dimensional spacetime and — in this case — r runs in the range $(-\infty, \infty)$. The key point is that the metric in (4) is fairly generic and can describe a host of spacetimes with horizons located at $f = 0$.

3. Dimensional Reduction and Emergence of CFT Near a Horizon

To investigate the nature of spacetimes with horizons, we only need to assume that $f(r)$ vanishes at some $r = l$ with $f'(l)$ being finite; such spacetimes have a horizon at $r = l$. Let us consider the example of a QFT for a self-interacting scalar field with a potential $V(\phi)$, in a spacetime with the metric of the form in equation (4):

$$ds^2 = f(r)(lg)^2 dt^2 - \frac{l^2}{4}\left(\frac{f'^2}{f}\right) dr^2 - g_{\alpha\beta}dx^\alpha dx^\beta\,; \quad g_{\alpha\beta} = g_{\alpha\beta}(r, \mathbf{x}_\perp) \tag{9}$$

where the line element $g_{\alpha\beta}dx^\alpha dx^\beta$ denotes the irrelevant transverse part corresponding to the transverse coordinates $\mathbf{x}_\perp$, *as well as* any regular part of the metric corresponding to dr^2. For example, in the case of metric in (5) it is convenient to combine the $(1 + q'^2)dr^2$ term with the transverse part since it is regular at the horizon. The field equation for a scalar field in this metric $\nabla_a \nabla^a \phi = -(\partial V/\partial\phi)$ can be expanded as

$$\ddot{\phi} - \frac{4g^2}{Q}\frac{f}{f'}\frac{\partial}{\partial r}\left[Q\frac{f}{f'}\frac{\partial\phi}{\partial r}\right] - (lg)^2 f\left[(\nabla_\perp^2\phi) + \frac{\partial V}{\partial\phi}\right] = 0 \tag{10}$$

where we have set $\sqrt{-g} = (l^2g/2)f'(r)Q(r,\mathbf{x}_\perp)$ with $Q(r,\mathbf{x}_\perp)$ giving the square root of the determinant of transverse metric. It is assumed that f has a simple zero at some $r = l$ while Q is regular at this point. We see from (10) that something drastic will happen if f has a simple zero at some point $r = l$, signalling a horizon in the spacetime. Then the factor f will kill the dependence of the solution on the transverse coordinates as well as on the potential and we will be left with essentially a 2−dimensional scalar field theory, which – as is well known – acquires an extra symmetry of conformal invariance. Introducing the coordinate $\xi = (1/2g)\ln f$, equation (10) can be rewritten as

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial\xi^2} = \left(\frac{2g}{f'}\right)^2\frac{f^2}{Q}\left(\frac{\partial\phi}{\partial r}\right)\left(\frac{\partial Q}{\partial r}\right) + (lg)^2 f\left[(\nabla_\perp^2\phi) + \frac{\partial V}{\partial\phi}\right] \tag{11}$$

The right hand side vanishes as $r \to l$ because f vanishes faster than all other terms for well behaved solutions and mode functions $\phi(r)$. [This will exclude singular solutions for which the term in the square bracket behaves, for example, as f^{-2} near the horizon. For the modes we obtain below, this can be explicitly verified.] It follows that near the horizon we are dealing with a (1+1) dimensional field theory governed by

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial\xi^2} \approx 0 \tag{12}$$

which has an extra symmetry of conformal invariance.

If we take $f(r) \approx 2g(r-l)$ near $r = l$ and separate the time dependence by $\phi = \phi_\omega e^{-i\omega t}$, it is easy to see that near $r = l$, the solution has the universal form:

$$\phi_\omega \cong |r-l|^{\pm(i\omega/2g)} \cong \exp\left[\pm\frac{i\omega}{2g}\ln|\frac{r}{l}-1|\right] \tag{13}$$

The fundamental wave modes are now

$$\phi = e^{-i\omega t \pm i\omega\xi} = e^{-i\omega(t\pm\xi)} = \left(e^{-i\omega z}, e^{-i\omega\bar{z}}\right) \tag{14}$$

where $\tau = it$ and $z \equiv (\xi + i\tau)$ is the standard complex coordinate of the conformal field theory. The boundary condition on the horizon can be expressed most naturally in terms of z and $\bar{z}$. For example, purely in-going modes are characterized by $(\partial f/\partial\bar{z}) = 0$; $(\partial f/\partial z) \neq 0$. Since the system is periodic in τ, the coordinate z is on a cylinder $(R^1 \times S^1)$ with τ being the angular coordinate (S^1) and ξ being the R^1 coordinate. The periodicity in τ is clearer if we introduce the related complex variable ρ by the definition $\rho = \exp g(\xi + i\tau) = \exp(gz)$. The coordinate ρ respects the periodicity in τ and is essentially a mapping from a cylinder to a plane.[7] It follows that the modes $(e^{-i\omega z}, e^{-i\omega \bar{z}})$ become $(\rho^{-i\omega/g}, \bar{\rho}^{-i\omega/g})$ in terms of ρ.

The situation is simpler in the case of a free field with $V = 0$. Then the general solution to the wave equation can be expanded in terms of the modes $\phi(t, r, \mathbf{x}_\perp) = F_\lambda(\mathbf{x}_\perp)\phi_{\lambda\omega}(r)e^{-i\omega t}$ where the function F is the eigenfunction of transverse Laplacian with (set of) eigenvalue(s) λ; that is $\nabla^2_{D-1}F = -\lambda^2 F$. In general, the radial part of the solution $\phi_{\lambda\omega}(r)$ will depend both on ω and λ and will have a complicated r dependence. But if the spacetime has a horizon, then the mode functions have a unique behaviour [given by (13)] near the horizon. The corresponding two point function $G(t-t'; r, r'; \mathbf{x}_\perp, \mathbf{x}'_\perp) = \langle 0|\phi(t, r, \mathbf{x}_\perp)\phi(t', r', \mathbf{x}'_\perp)|0\rangle$ will have the limit

$$G(t - t'; r, r'; \mathbf{x}_\perp, \mathbf{x}'_\perp) \cong \left\{\sum_\lambda f_\lambda(\mathbf{x}_\perp) f_\lambda(\mathbf{x}'_\perp\right\} \left\{\sum_\omega e^{-i\omega[(t-t')\pm(\xi-\xi')]}\right\} \qquad (15)$$

near $r \simeq l$, $r' \simeq l$. That is, the two point function factorises into a transverse and radial part with the radial part being that of a two dimensional massless scalar field. The latter is the same as the Green function of the standard conformal filed theory. Similar results exist for other two-point functions, like Feynman Green function etc. [The CFT structure near the horizon and its relation to Virasoro algebra was used earlier by many people after the seminal work by Carlip and Strominger.[8] Some of these approaches,[9] attempt to treat the radial degree of freedom r as a "scalar field". Note that the approach taken here is different.]

The fact that the field modes have a unique behaviour near the event horizon has implications for the QFT in the bulk of the spacetime. A free QFT is uniquely determined by the two-point function, say, the Feynman Green function which satisfies a local differential equation. Since the metrics we are studying are all static, time translation invariance implies that the Green function G only depends on the time difference $t - t'$. The spatial dependence, however, can be quite non trivial especially if we have to deal with surfaces like $r = l$ on which the metric is singular. Since this equation is hyperbolic, the boundary conditions required to specify the solution are nontrivial in general even after imposing time-translation invariance. The existence of a CFT near the horizon, however, allows us to specify a natural boundary condition on the $[t = \text{constant}, r = l + \epsilon]$ surface, where ϵ is an infinitesimal quantity. In other words, the QFT in the bulk of the spacetime is uniquely determined once we specified the boundary conditions on the *surface* of the event horizon (plus at infinity or origin depending on the location of bulk region wrt the horizon).

4. All Horizons Lead to Temperature

There exists a natural definition of QFT in the original (D+1)-dimensional space; in particular, we can define a vacuum state for the quantum field on the $Z^0 = 0$ surface, which coincides with the $t = 0$ surface. By restricting the field modes (or the field configurations in the Schrodinger picture) to depend only on the coordinates in $\mathcal{D}$, we will obtain a quantum field theory in $\mathcal{D}$ in the sense that these modes will satisfy the relevant field equation defined in $\mathcal{D}$. In general, this is a complicated problem and it is not easy to have a choice of modes in $\mathcal{S}$ which will lead to a natural set of modes in $\mathcal{D}$. We can, however, take advantage of the arguments given in the last section — that all the interesting physics arises from the (Z^0, Z^1) plane and the other transverse dimensions are irrelevant near the horizon. In particular, solutions to the wave equation in $\mathcal{S}$ which depends only on the coordinates Z^0 and Z^1 will satisfy the wave equation in $\mathcal{D}$ and will depend only on (t, r). Such modes will define a natural $s-$wave QFT in $\mathcal{D}$. I will now show that the positive frequency modes of the above kind will be a specific superposition of negative and positive frequency modes in $\mathcal{D}$ leading to a temperature $T = (g/2\pi)$ in the 4-dimensional subspace on one side of the horizon. There are several ways of proving this result, all of which depend essentially on the property that under the transformation $t \to t \pm (i\pi/g)$ the two coordinates Z^0 and Z^1 reverses sign.

Consider a positive frequency mode of the form $F(Z^0, Z^1) \propto \exp[-i\Omega Z^0 + iPZ^1]$ with $\Omega > 0$. This mode can be expressed in the 4-dimensional sector in the form $\Phi = F(t, r) = F[Z^0(t, r), Z^1(t, r)]$. The Fourier transform of $F(t, r)$ with respect to t will be:,

$$K(\omega, r) = \int_{-\infty}^{\infty} dt\, e^{+i\omega t} F[Z^0(t, r), Z^1(t, r)]\,; \qquad (-\infty < \omega < \infty) \tag{16}$$

Thus a positive frequency mode in the higher dimension can only be expressed as an integral over ω with ω ranging over both positive and negative values. However, using the fact that $t \to t - (i\pi/g)$ leads to $Z^0 \to -Z^1$, it is easy to show that $K(-\omega, r) = e^{-(\pi\omega/g)} K^*(\omega, r)$. This allows us to write the inverse relation to (16) as

$$F(t, r) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} K(\omega, r) e^{-i\omega t} = \int_0^{\infty} \frac{d\omega}{2\pi} [K(\omega, r) e^{-i\omega t} + e^{-\pi\omega/g} K^*(\omega, r) e^{i\omega t}] \tag{17}$$

The term with K^* represents the contribution of negative frequency modes in the the 4-D spacetime to the pure positive frequency mode in the embedding spacetime. A field mode of the embedding spacetime containing creation and annihilation operators $(A, A^\dagger)$ can now be represented in terms of the creation and annihilation operators $(a, a^\dagger)$ appropriate to the (t, r) coordinates as

$$\begin{aligned} AF + A^\dagger F^* &= \int_0^{\infty} \frac{d\omega}{2\pi} \left[\left(A + A^\dagger e^{-\pi\omega/g} \right) K e^{-i\omega t} + \text{h.c.} \right] \\ &= \int_0^{\infty} \frac{d\omega}{2\pi} \frac{1}{N} [aK e^{-i\omega t} + \text{h.c.}] \end{aligned} \tag{18}$$

where N is a normalization constant. Identifying $a = N(A + e^{-\pi\omega/g} A^\dagger)$ and using the conditions $[a, a^\dagger] = 1$, $[A, A^\dagger] = 1$ etc., we get $N = [1 - \exp(-2\pi\omega/g)]^{-1/2}$. It follows that the number of $a-$particles in the vacuum defined by $A|\text{vac}>=0$ is given by

$$< \text{vac}|a^\dagger a|\text{vac} >= N^2 e^{-2\pi\omega/g} = (e^{2\pi\omega/g} - 1)^{-1} \tag{19}$$

This is a Planckian spectrum with temperature $T = g/2\pi$ in normal units.

There is a more elegant way of obtaining this result which is due to Lee[10] and I give here an adaptation of the same. The basic theme is illustrated in figure 1 in which the origin of r is chosen such that $f(r=0)=0$ for simplicity. On the $Z^0 = t = 0$ hyper-surface one can define a vacuum state $|\text{vac}>$ of the theory by giving the field configuration for the whole of $-\infty < Z^1 < +\infty$. This field configuration, however, separates into two disjoint sectors when one uses the (t,r) coordinate system. Concentrating on the (Z^0, Z^1) plane, we now need to specify the field configuration $\phi_R(Z^1)$ for $Z^1 > 0$ and $\phi_L(Z^1)$ for $Z^1 < 0$ to match the initial data in the global coordinates; given this data, the vacuum state is specified by the functional $< \text{vac}|\phi_L, \phi_R >$.

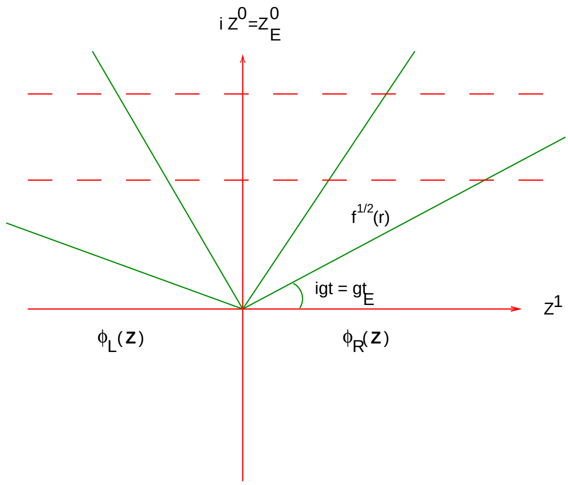

Fig. 1. Thermal effects due to a horizon; see text for a discussion.

Let us next consider the *Euclidean* sector corresponding to the (Z^0_E, Z^1) plane where $Z^0_E = iZ^0$. The QFT in this plane can be defined along standard lines. The analytic continuation in t, however, is a different matter; the coordinates $(gt_E = igt, f(r)^{1/2})$ are like polar coordinates in this (T, Z^1) plane with t_E having a periodicity of $(2\pi/g)$. Figure 1 now shows that evolution in gt_E from 0 to π will take the system configuration from $Z^1 > 0$ to $Z^1 < 0$. This allows one to prove that $\langle \text{vac}|\phi_L, \phi_R\rangle \propto \langle \phi_L|e^{-\pi H/g}|\phi_R\rangle$; normalization now fixes the proportionality constant, giving

$$\langle \text{vac}|\phi_L, \phi_R\rangle = \frac{\langle \phi_L|e^{-\pi H/g}|\phi_R\rangle}{Tr(e^{-2\pi H/g})^{1/2}} \tag{20}$$

To provide a simple proof of this relation, let us consider the ground state wave functional $< \text{vac}|\phi_L, \phi_R >$ in the embedding spacetime expressed as a path integral. As is well known, ground state wave functional can be represented as a Euclidean path integral of the form

$$\langle \text{vac}|\phi_L, \phi_R\rangle \propto \int_{Z_E^0=0;\phi=(\phi_L,\phi_R)}^{Z_E^0=\infty;\phi=(0,0)} \mathcal{D}\phi e^{-A} \tag{21}$$

where $Z_E^0 = iZ^0$ is the Euclidean time coordinate. From figure 1 it is obvious that this path integral could also be evaluated in the polar coordinates by varying the angle $\theta = gt_E$ from 0 to π. When $\theta = 0$ the field configuration corresponds to $\phi = \phi_R$ and when $\theta = \pi$ the field configuration corresponds to $\phi = \phi_L$. Therefore

$$\langle \text{vac}|\phi_L, \phi_R\rangle \propto \int_{gt_E=0;\phi=\phi_R}^{gt_E=\pi;\phi=\phi_L} \mathcal{D}\phi e^{-A} \tag{22}$$

In Heisenberg picture this quantity can be expressed as a matrix element of the Hamiltonian H_R in the (t, r) coordinates giving us the result:

$$\langle \text{vac}|\phi_L, \phi_R\rangle \propto \int_{gt_E=0;\phi=\phi_R}^{gt_E=\pi;\phi=\phi_L} \mathcal{D}\phi e^{-A} = \langle \phi_L|e^{-(\pi/g)H_R}|\phi_R\rangle \tag{23}$$

Normalizing the result properly gives equation (20).

This result, in turn, implies that for operators $\mathcal{O}$ made out of field variables belonging to the right wedge, the vacuum expectation values become thermal expectation values. This arises from straightforward algebra of inserting a complete set of states appropriately:

$$\langle \text{vac}|\,\mathcal{O}(\phi_R)|\text{vac}\rangle = \sum_{\phi_L}\sum_{\phi_R^1,\phi_R^2} \langle \text{vac}|\phi_L, \phi_R^1\rangle\langle\phi_R^1|\mathcal{O}(\phi_R)|\phi_R^2\rangle\langle\phi_R^2, \phi_L|\text{vac}\rangle$$

$$= \sum_{\phi_L}\sum_{\phi_R^1,\phi_R^2} \frac{\langle\phi_L|e^{-\pi H/g}|\phi_R^1\rangle\langle\phi_R^1|\mathcal{O}|\phi_R^2\rangle\langle\phi_R^2|e^{-\pi H/g}|\phi_L\rangle}{Tr(e^{-2\pi H/g})} = \frac{Tr(e^{-2\pi H/g}\mathcal{O})}{Tr(e^{-2\pi H/g})} \tag{24}$$

The most important conclusion which follows from all these is that the existence of the temperature is a *purely kinematic effect* arising from the coordinate system we have used. Dynamical evolution has no role to play. The main ingredients which have gone into this result are the following. (i) The singular behaviour of the (t, r) coordinate system near $r = l$ separates out the $Z^0 = 0$ hyper-surface into two separate regions. (ii) In terms of real (t, r) coordinates, it is not possible to distinguish between the points (Z^0, Z^1) and $(-Z^0, -Z^1)$ but the transformation $t \to t \pm i\pi$ maps the point (Z^0, Z^1) to the point $(-Z^0, -Z^1)$. Thus a rotation in the complex time plane encodes the information contained in the full $Z^0 = 0$ plane.

5. All Horizons Lead to Entropy

The next logical question will be whether one can associate other thermodynamic quantities, especially the entropy, with such spacetimes. Given that the temperature can be introduced very naturally, using only the behaviour of metric near the horizon, one would look for a similarly elegant and natural derivation of the entropy. Such a derivation should not depend on the introduction of external degrees of freedom (like a scalar field) since we want to associate the entropy with the spacetime and not with an external field. Further, the thermodynamical description should depend only on the behaviour of the metric near the horizon. I will show[3] that it is indeed possible to provide such a description for spacetimes of the form in

$$ds^2 = f(r)dt^2 - f(r)^{-1}dr^2 - dL_\perp^2 \tag{25}$$

where $f(r)$ vanishes at some surface $r = l$, say, with $f'(l) \equiv B$ remaining finite. When $dL_\perp^2 = r^2 dS_2^2$ with $[0 \le r \le \infty]$, equation (25) covers a variety of spherically symmetric spacetimes with a compact horizon at $r = l$. If r is interpreted as one of the Cartesian coordinates x with $(-\infty \le x \le \infty)$ and $dL_\perp^2 = dy^2 + dz^2$, $f(x) = 1 + 2gx$, equation (25) can describe the Rindler frame in flat spacetime. We shall first concentrate on compact horizons with r interpreted as radial coordinate, and comment on the Rindler frame at the end.

Since the metric is static, Euclidean continuation is trivially effected by $t \to \tau = it$ and an examination of the conical singularity near $r = a$ [where $f(r) \approx B(r-a)$] shows that τ should be interpreted as periodic with period $\beta = 4\pi/|B|$ corresponding to the temperature $T = |B|/4\pi$. Let us consider a set $\mathcal{S}$ of such metrics in (25) with the restriction that $[f(a) = 0, f'(a) = B]$ but $f(r)$ is otherwise arbitrary and has no zeros. The partition function for this set of metrics $\mathcal{S}$ is given by the path integral sum

$$Z(\beta) = \sum_{g\epsilon\mathcal{S}} \exp(-A_E(g)) = \sum_{g\epsilon\mathcal{S}} \exp\left(-\frac{1}{16\pi}\int_0^\beta d\tau \int d^3x\sqrt{g_E}R_E[f(r)]\right) \tag{26}$$

where I have made the Euclidean continuation of the Einstein action and imposed the periodicity in τ with period $\beta = 4\pi/|B|$. The sum is restricted to the set $\mathcal{S}$ of all metrics of the form in (25) with the behaviour $[f(a) = 0, f'(a) = B]$ and the Euclidean Lagrangian is a functional of $f(r)$. The spatial integration will be restricted to a region bounded by the 2-spheres $r = a$ and $r = b$, where the choice of b is arbitrary except for the requirement that within the region of integration the Lorentzian metric must have the proper signature with t being a time coordinate. The remarkable feature is the form of the Euclidean action for this class of spacetimes. Using the result $R = \nabla_r^2 f - (2/r^2)(d/dr)[r(1-f)]$ valid for metrics of the form in (25), a straight forward calculation shows that

$$-A_E = \frac{\beta}{4}\int_a^b dr\left[-[r^2 f']' + 2[r(1-f)]'\right] = \frac{\beta}{4}[a^2 B - 2a] + Q[f(b), f'(b)] \tag{27}$$

where Q depends on the behaviour of the metric near $r = b$ and we have used the conditions $[f(a) = 0, f'(a) = B]$. The sum in (26) now reduces to summing over the values of $[f(b), f'(b)]$ with a suitable (but unknown) measure. This sum, however, will only lead to a factor which we can ignore in deciding about the dependence of

$Z(\beta)$ on the form of the metric near $r = a$. Using $\beta = 4\pi/B$ (and taking $B > 0$, for the moment) the final result can be written in a very suggestive form:

$$Z(\beta) = Z_0 \exp\left[\frac{1}{4}(4\pi a^2) - \beta(\frac{a}{2})\right] \propto \exp\left[S(a) - \beta E(a)\right] \tag{28}$$

with the identifications for the entropy and energy being given by:

$$S = \frac{1}{4}(4\pi a^2) = \frac{1}{4}A_{\text{horizon}}\,; \quad E = \frac{1}{2}a = \left(\frac{A_{\text{horizon}}}{16\pi}\right)^{1/2} \tag{29}$$

In addition to the simplicity, the following features are noteworthy regarding this result:

(i) Conceptually, a canonical ensemble for a minisuperspace of metrics of the form in (25) should be constructed by keeping the temperature constant *without* assuming the metrics to be the solutions of Einstein's equation; this is what I do and exploit the form of R. I managed to get a closed result for the path integral because of the choice of the minisuperspace of metrics in (25). This allows me to sum over a class of spherically symmetric spacetimes at one go rather than deal with, say, black hole spacetimes and deSitter spacetime separately. [This is different from the conventional approaches.[11]]

(ii) The ideas also work in the case of $(1 + 2)$ dimensional gravity which has attracted fair amount of attention.[12] In $D = (1+2)$, metrics of the type in (25) with $dL_\perp^2 = r^2 d\theta^2$ will give $S = (1/4)(2\pi a) = (1/4)A_{\text{horizon}}$ with $E = 0$. The vanishing of energy is consistent with the fact that at the level of the metric, Einstein's equations are vacuous in (1+2) and we have not incorporated any topological effects [like deficit angles corresponding to point masses in (1+2) dimensions] in our approach.

(iv) In the case of the Schwarzschild black hole with $a = 2M$, the energy turns out to be $E = (a/2) = M$ which is as expected. (More generally, $E = (A_{\text{horizon}}/16\pi)^{1/2}$ corresponds to the so called 'irreducible mass' in BH spacetimes.[13]) Of course, the identifications $S = (4\pi M^2)$, $E = M$, $T = (1/8\pi M)$ are consistent with the result $dE = TdS$ in this particular case.

Most importantly, *the above analysis provides an interpretation of entropy and energy in the case of deSitter universe* which is gaining in popularity. In this case, $f(r) = (1 - H^2r^2)$, $a = H^{-1}, B = -2H$. Since the region where t is timelike is "inside" the horizon, the integral for A_E in (27) should be taken from some arbitrary value $r = b$ to $r = a$ with $a > b$. So the horizon contributes in the upper limit of the integral introducing a change of sign in (27). Further, since $B < 0$, there is another negative sign in the area term from $\beta B \propto B/|B|$. Taking all these into account we get, in this case,

$$Z(\beta) = Z_0 \exp\left[\frac{1}{4}(4\pi a^2) + \beta\left(\frac{a}{2}\right)\right] \propto \exp\left[S(a) - \beta E(a)\right] \tag{30}$$

giving $S = (1/4)(4\pi a^2) = (1/4)A_{\text{horizon}}$ and $E = -(1/2)H^{-1}$. These definitions do satisfy the relation $TdS - PdV = dE$ when it is noted that the deSitter universe has a nonzero pressure $P = -\rho_\Lambda = -E/V$ associated with the cosmological constant. In fact, if we use the "reasonable" assumptions $S = (1/4)(4\pi H^{-2})$, $V \propto H^{-3}$ and $E = -PV$ in the equation $TdS - PdV = dE$ and treat E as an unknown function of H, we get the equation $H^2(dE/dH) = -(3EH + 1)$ which integrates to give precisely $E = -(1/2)H^{-1}$. (Note that I only needed the proportionality, $V \propto H^{-3}$ in this argument since $PdV \propto (dV/V)$. The ambiguity between the coordinate and proper volume is irrelevant.)

Let us now consider the spacetimes with planar symmetry for which (25) is still applicable with $r = x$ being a Cartesian coordinate and $dL_\perp^2 = dy^2 + dz^2$. In this case $R = f''(x)$ and the action becomes

$$-A_E = \frac{1}{16\pi}\int_0^\beta d\tau \int dydz \int_a^b dx f''(x) = \frac{\beta}{16\pi}A_\perp f'(a) + Q[f'(b)] \tag{31}$$

where we have confined the transverse integrations to a surface of area $A_\perp$. If we now sum over all the metrics with $f(a) = 0$, $f'(a) = B$ and $f'(b)$ arbitrary, the partition function will become $Z(\beta) = Z_0 \exp[(1/4)A_\perp]$ which suggests that planar horizons have an entropy of (1/4) per unit transverse area but zero energy. This includes Rindler frame as a special case. Note that if we freeze f to its Rindler form $f = 1 + 2gx$, (by demanding the validity of Einstein's equations in the WKB approach, say) then $R = f'' = 0$ as it should. In the action in (31), $f'(a) - f'(b)$ will give zero. It is only because I am *not* doing a WKB analysis — but varying $f'(b)$ with fixed $f'(a)$ — that I obtain an entropy for these spacetimes.

Finally, I want to comment on a peculiar feature of the metrics in (25). This metric will satisfy Einstein's equations provided the the source stress tensor has the form $T^t_t = T^r_r \equiv (\epsilon(r)/8\pi)$; $T^\theta_\theta = T^\phi_\phi \equiv (\mu(r)/8\pi)$. The Einstein's equations now reduce to:

$$\frac{1}{r^2}(1 - f) - \frac{f'}{r} = \epsilon\,; \quad \nabla^2 f = -2\mu \tag{32}$$

The remarkable feature about the metric in (25) is that the Einstein's equations become linear in $f(r)$ so that solutions for different $\epsilon(r)$ can be superposed. Given any $\epsilon(r)$ the solution becomes

$$f(r) = 1 - \frac{a}{r} - \frac{1}{r}\int_a^r \epsilon(r) r^2\, dr \tag{33}$$

with a being an integration constant and $\mu(r)$ is fixed by $\epsilon(r)$ through: $\mu(r) = \epsilon + (1/2)r\epsilon'(r)$. I have chosen the integration constant a in (33) such that $f(r) = 0$ at $r = a$ so that this surface is a horizon. Let us now assume that the solution (33) is such that $f(r) = 0$ at $r = a$ with $f'(a) = B$ finite leading to leading to a notion of temperature with $\beta = (4\pi/|B|)$. From the first of the equations (32) evaluated at $r = a$, we get

$$\frac{1}{2}Ba - \frac{1}{2} = -\frac{1}{2}\epsilon(a)a^2 \tag{34}$$

It is possible to provide an interesting interpretation of this equation which throws light on the notion of entropy and energy. Multiplying the above equation by da and using $\epsilon = 8\pi T^r_r$, it is trivial to rewrite equation (34) in the form

$$\frac{B}{4\pi} d\left(\frac{1}{4} 4\pi a^2\right) - \frac{1}{2} da = -T^r_r(a) d\left(\frac{4\pi}{3} a^3\right) = -T^r_r(a)[4\pi a^2] da \tag{35}$$

Let us first consider the case in which a particular horizon has $f'(a) = B > 0$ so that the temperature is $T = B/4\pi$. Since $f(a) = 0, f'(a) > 0$, it follows that $f > 0$ for $r > a$ and $f < 0$ for $r < a$; that is, the "normal region" in which t is time like is outside the horizon as in the case of, for example, the Schwarzschild metric. The first term in the left hand side of (35) clearly has the form of TdS since we have an independent identification of temperature from the periodicity argument in the local Rindler coordinates. Since the pressure is $P = -T^r_r$, the right hand side has the structure of PdV or — more relevantly — is the product of the radial pressure times the transverse area times the radial displacement. This is important because, for the metrics in the form (25), the proper transverse area is just that of a 2-sphere though the proper volumes and coordinate volumes differ. In the case of horizons with $B = f'(a) > 0$ which we are considering (with $da > 0$), the volume of the region where $f < 0$ will increase and the volume of the region where $f > 0$ will decrease. Since the entropy is due to the existence of an inaccessible region, dV must refer to the change in the volume of the inaccessible region where $f < 0$. We can now identify T in TdS and P in PdV without any difficulty and interpret the remaining term (second term in the left hand side) as $dE = da/2$. We thus get the expressions for the entropy S and energy E (when $B > 0$) to be the same as in (29).

Using (35), we can again provide an interpretation of entropy and energy in the case of deSitter universe. In this case, $f(r) = (1 - H^2r^2)$, $a = H^{-1}$, $B = -2H < 0$ so that the temperature — which should be positive — is $T = |f'(a)|/(4\pi) = (-B)/4\pi$. For horizons with $B = f'(a) < 0$ (like the deSitter horizon) which we are now considering, $f(a) = 0$, $f'(a) < 0$, and it follows that $f > 0$ for $r < a$ and $f < 0$ for $r > a$; that is, the "normal region" in which t is time like is inside the horizon as in the case of, for example, the deSitter metric. Multiplying equation (35) by (-1), we get

$$\frac{-B}{4\pi} d\left(\frac{1}{4} 4\pi a^2\right) + \frac{1}{2} da = T^r_r(a) d\left(\frac{4\pi}{3} a^3\right) = P(-dV) \tag{36}$$

The first term on the left hand side is again of the form TdS (with positive temperature and entropy). The term on the right hand side has the correct sign since the inaccessible region (where $f < 0$) is now outside the horizon and the volume of this region changes by $(-dV)$. Once again, we can use (36) to identify[3] the entropy and the energy: $S = (1/4)(4\pi a^2) = (1/4)A_{\text{horizon}}$; $E = -(1/2)H^{-1}$. These results agree with the previous analysis.

The connection is far from accidental since the ideas work in (1+2) dimensional gravity as well. For the metrics in (25) with $dL_\perp^2 = r^2 d\theta^2$, Einstein's equations demand that the stress tensor has the form $8\pi T^a_b = \text{dia}(\epsilon(r), \epsilon(r), \mu(r))$. The Einstein's equations are $(f'/r) = -2\epsilon(r)$; $f''(r) = -2\mu(r)$. These are also linear in the source term and can be integrated for a given function $\epsilon(r)$ in order to produce a solution with $f = 0$ at $r = a$. The relation $f' = -2r\epsilon$ evaluated at $r = a$ gives $B = -2a\epsilon(a)$. Multiplying by da and rearranging terms, this relation can be written in the form

$$\left(\frac{B}{4\pi}\right) d\left(\frac{1}{4}(2\pi a)\right) = (-T^r_r)(2\pi a)da = (-T^r_r)d(\pi a^2) \tag{37}$$

Since the temperature is $T = (B/4\pi)$ when $B > 0$ and the pressure is $P = -T^r_r$, we can immediately identify the TdS and PdV terms. The entropy is still one quarter of the "area", $S = (1/4)2\pi a$, and the energy vanishes identically: $E = 0$. We see that our interpretation carries through in this case as well.

6. QFT in Spacetimes with Asymptotic Horizons

The analysis so far was based on a strictly *static* 4-dimensional spacetime obtained as a subspace of the higher dimensional flat manifold. The black hole metric, for example, corresponds to an eternal black hole and the vacuum state which we constructed corresponds to the Hartle-Hawking vacuum of the Schwarzschild spacetime, describing a black hole in thermal equilibrium.[14] There is no net radiation flowing to infinity and the entropy and temperature obtained in the previous sections were based on equilibrium considerations.

Physically, one would like to have a situation which is asymmetric in time so that one can consider an irreversible flow of energy providing us with a dE that can be used to define a $dS = dE/T$. Once it is realized that only the asymptotic form of the metric matters, we can simplify the above analysis by just choosing a time *asymmetric* vacuum and working with the asymptotic form of the metric with the understanding that the asymptotic form arose due to a time asymmetric process (like gravitational collapse). In the case of black hole spacetimes this is accomplished — for example — by choosing the Unruh vacuum.[15] The question arises as to how our unified approach fares in handling such a situation which is not time symmetric and the horizon forms only asymptotically as $t \to \infty$.

There exist analogues of the collapsing black hole in the case of deSitter and even Rindler. The analogue in the case of deSitter spacetime will be an FRW universe which behaves like a deSitter universe only at late times. Emboldened by the analogy with black hole spacetime one can also directly construct quantum states (similar to Unruh vacuum of black hole spacetime) which are time asymmetric, even in the exact deSitter spacetime, with the understanding that the deSitter universe came about at late times through a time asymmetric evolution.

The analogy also works for Rindler spacetime which is also time symmetric. The coordinate system for an observer with *time dependent* acceleration will generalize the standard Rindler spacetime in a time dependent manner. For an observer who was inertial at early times and is uniformly accelerating at late times, an event horizon forms at late times exactly in analogy with a collapsing black hole. It is now possible to choose quantum states which are analogous to Unruh vacuum - that will correspond to an inertial vacuum state at early times and will appear as a thermal state at late times.

It is easy to demonstrate these results in 2D using the emergence of CFT near horizon and calculating expectation values of the stress tensor especially near the horizon. Concentrating on the (t, r) plane, one can introduce a series of natural coordinates of the form

$$ds^2 = f(r)dt^2 - \frac{dr^2}{f(r)} = f(\xi)(dt^2 - d\xi^2) = f[(v-u)]dudv \tag{38}$$

where

$$\xi = \int \frac{dr}{f(r)}; \qquad u = t - \xi; \qquad v = t + \xi \tag{39}$$

These coordinate systems are singular near the horizon $r \approx l$. Assuming $f(r) \approx B(r-l)$ near $r = l$ [with $B = f'(l)$], it is easy to see that near the horizon the line element has the form $ds^2 \approx Ble^{(B/2)(v-u)}dvdu$. This form suggests a natural coordinate transformation from (u, v) to a non singular coordinate system (U, V) with $V = (2/B)\exp[(B/2)v]$; $U = -(2/B)\exp[(-B/2)u]$ in terms of which the metric becomes $ds^2 = -(4f/B^2UV)dUdV$. By construction, the (U, V) coordinate system is regular on the horizon, since the combination (f/UV) is finite on the horizon. (This will lead to the familiar Kruskal coordinate system in the case of Schwarzschild manifold.)

Since the mode functions are plane waves in conformally flat (1+1) spacetime, we can immediately identify two very natural set of modes [and corresponding vacuum states] in the spacetime. The outgoing and ingoing modes of the kind $(4\pi\omega)^{-1/2}[\exp(-i\omega u), \exp(-i\omega v)]$ defines a static vacuum state (called Boulware vacuum in the case of Schwarzschild black hole). The modes of the kind $(4\pi\omega)^{-1/2}[\exp(-i\omega U), \exp(-i\omega V)]$ defines another vacuum state (called Hartle-Hawking vacuum in the case of Schwarzschild black hole). Finally, the modes of the kind $(4\pi\omega)^{-1/2}[\exp(-i\omega U), \exp(-i\omega v)]$ define the analogue of Unruh vacuum.

In any conformally flat coordinate system of the form $ds^2 = C(x^+, x^-)dx^+dx^-$, there is a natural definition of vacuum based on the modes $\exp[-i\omega x^{\pm}]$. The expectation values of the stress-tensor component in this state are given by[5]

$$< T_{\pm\pm} >= -\frac{1}{12\pi}C^{1/2}\partial_{\pm}^2 C^{-1/2}; \quad < T_{+-} >= \frac{C}{96\pi}R; \quad R = 4C^{-1}\partial_+\partial_- \ln C \tag{40}$$

Using this formula (with $x^{\pm}$ identified with (u, v) coordinates and the relation $\partial_u = (\partial\xi/\partial u)(\partial r/\partial\xi)\partial_r = -(1/2)f(r)\partial_r$) gives the expectation values in the Boulware vacuum:

$$< B|T_{--}|B >=< B|T_{++}|B >= \frac{1}{96\pi}\left[ff'' - \frac{1}{2}(f')^2\right] \tag{41}$$

where $f' = df/dr$ etc. Identifying $x^{\pm}$ with (U, V) coordinates gives the expectation values in the Hartle-Hawking vacuum:

$$< HH|T_{--}|HH >=< HH|T_{++}|HH >=< B|T_{--}|B > +\frac{f'(l)^2}{192\pi} \tag{42}$$

In both these cases, there is no flux since $< T_{rt} >= 0$. Near the horizon, we have

$$< B|T_{\pm\pm}|B >\approx -\frac{f'(l)^2}{192\pi}\,; \qquad < HH|T_{\pm\pm}|HH >\approx 0 \tag{43}$$

But since (u, v) are bad coordinates near the horizon, while (U, V) are non singular, we should consider T_{UU} etc. rather than T_{uu} etc.. The former will vary as $< B|T_{uu}|B > (du/dU)^2$ and will diverge near the horizon in the Boulware vacuum.

A more interesting situation arises in the case of Unruh vacuum which differs from the Boulware vacuum only in the outgoing modes. If the coordinate x^- is replaced by $X^- \equiv F(x^-)$, the conformally flat nature of the line element is maintained and the only stress tensor component which changes is

$$< T_{--} > \to < T_{--} > +\frac{1}{24\pi}\left[\left(\frac{Q''}{Q}\right) - \frac{1}{2}\left(\frac{Q'}{Q}\right)^2\right] \tag{44}$$

where $Q = (1/F')$. Using this we find that

$$< U|T_{--}|U >=< HH|T_{--}|HH >; \quad < U|T_{++}|U >=< B|T_{++}|B > \tag{45}$$

thereby making $< U|T_{--}|U >\neq< U|T_{++}|U >$. This leads to a flux of radiation with $< U|T_{r^*t}|U >= -(B^2/192\pi)$. It is also clear that the energy density, as measured by inertial observers, is finite near the horizon. We thus conclude that we have two acceptable vacuum states in *all these spacetimes*: (i) the time symmetric Hartle-Hawking state describing thermal equilibrium and zero flux and (ii) the time-asymmetric Unruh vacuum with a flux of radiation.

We cannot, of course, use this trick in 4D away from the horizon and a formal analysis of this problem will involve setting up the in and out vacua of the theory, evolving the modes from $t = -\infty$ to $t = +\infty$, and computing the Bogolibov coefficients. It is, however, not necessary to perform the details of such an analysis because all the three spacetimes (Schwarzschild, deSitter and Rindler) have virtually identical kinematical structure. In the case of Schwarzschild metric, it is well known that the thermal spectrum at late times arises because the modes which reach spatial infinity at late times propagate from near the event horizon at early times and undergo exponential redshift. The corresponding result occurs in all the three spacetimes (and a host of other spacetimes).

Consider the propagation of a wave packet centered around a radial null ray in a spherically symmetric (or Rindler) spacetime which has the form in equation (5) or (25). The trajectory of the null ray which goes from the initial position r_{in} at t_{in} to a final position r at t is determined by the equation

$$t - t_{in} = \pm\left(\frac{1}{2g}\right)\int_{r_{in}}^{r}\left(\frac{f'}{f}\right)(1+\cdots)^{1/2}dr \tag{46}$$

where the $\cdots$ denotes terms arising from the transverse part containing dr^2 (if any). Consider now a ray which was close to the horizon initially so that $(r_{in} - l) \ll l$ and propagates to a region far away from the horizon at late times. (In a black hole metric $r \gg r_{in}$ and the propagation will be outward directed; in the deSitter metric we will have $r \ll r_{in}$ with rays propagating towards the origin.) Since we have $f(r) \to 0$ as $r \to l$, the integral will be dominated by a logarithmic singularity near the horizon and the regular term denoted by $\cdots$ will not contribute. [This can be verified directly from (5) or (25).] Then we get

$$t - t_{in} = \pm \left(\frac{1}{2g}\right) \int_{r_{in}}^{r} \left(\frac{f'}{f}\right) (1 + \cdots)^{1/2} dr \approx \pm \left(\frac{1}{2g}\right) \ln |f(r_{in})| + \text{constant} \tag{47}$$

As the wave propagates away from the horizon its frequency will be redshifted by the factor $\omega \propto (1/\sqrt{g_{00}})$ so that

$$\frac{\omega(t)}{\omega(t_{in})} = \left(\frac{g_{00}(r_{in})}{g_{00}(r)}\right)^{1/2} = \left[\frac{f(r_{\rm in})}{f(r)}\right]^{1/2} \approx K e^{\pm gt} \tag{48}$$

where K is an unimportant constant. It is obvious that the dominant behaviour of $\omega(t)$ will be exponential for any null geodesic starting near the horizon and proceeding away since all the transverse factors will be subdominant to the diverging logarithmic singularity arising from the integral of $(1/f(r))$ near the horizon. Thus $\omega(t) \propto \exp[\pm gt]$ and the phase $\theta(t)$ of the wave will be vary with time as $\theta(t) = \int \omega(t) dt \propto \exp[\pm gt]$. An observer at a fixed r will see the wave to have the time dependence $\exp[i\theta(t)]$ which, of course, is not monochromatic. If this wave is decomposed into different Fourier components with respect to t, then the amplitude at frequency ν is given by the Fourier transform

$$f(\nu) \propto \int e^{i\theta(t) - i\nu t} dt \propto \int_{-\infty}^{\infty} dt e^{-i(\nu t - Q \exp[\pm gt])} \tag{49}$$

where Q is some constant. Changing the variables from t to τ by $Qe^{\pm gt} = \tau$, evaluating the integral by analytic continuation to Im τ and taking the modulus one finds that the result is a thermal spectrum:

$$|f(\nu)|^2 \propto \frac{1}{e^{\beta\nu} - 1}; \quad \beta = \frac{2\pi}{g} \tag{50}$$

The standard expressions for the temperature are reproduced for Schwarzschild, deSitter and Rindler spacetimes. This analysis stresses the fact that the origin of thermal spectrum lies in the Fourier transforming of an exponentially redshifted spectrum.

An interesting question is whether similar results hold for Rindler spacetime in 4D which, of course, is flat. To study this case we need to work with the metric for an observer who is moving with variable acceleration. The transformation from the flat inertial coordinates $(\bar{t}, \bar{x}, \bar{y}, \bar{z})$ to the proper coordinates (t, x, y, z) of an observer with variable acceleration is effected by $\bar{y} = y$, $\bar{z} = z$ and

$$\bar{x} = \int^{t} \sinh\Theta(t)dt + x\cosh\Theta(t)\,; \qquad \bar{t} = \int^{t} \cosh\Theta(t)dt + x\sinh\Theta(t) \tag{51}$$

where the function $\Theta(t)$ is related to the time dependent acceleration $g(t)$ by $g(t) = (d\Theta/dt)$. The form of the metric in the accelerated frame is remarkably simple:

$$ds^2 = (1 + g(t)x)^2 dt^2 - dx^2 - dy^2 - dz^2 \tag{52}$$

We will treat $g(t)$ to be an arbitrary function except for the limiting behaviour $g(t) \to 0$ for $t \to -\infty$ and $g(t) \to g_0 = \text{constant}$ for $t \to +\infty$. Hence, at early times, the line element in (52) represent the standard inertial coordinates and the positive frequency modes define the standard Minkowski vacuum. At late times, the metric goes over to the Rindler coordinates and we are interested in knowing how the initial vacuum state will be interpreted at late times. The wave equation $(\Box + m^2)\phi = 0$ for a massive scalar field can be separated in the transverse coordinates as $\phi(t, x, y, z) = f(t, x)e^{ik_y y}e^{ik_z z}$ where f satisfies the equation

$$-\frac{1}{(1 + g(t)x)}\frac{\partial}{\partial t}\left(\frac{1}{(1 + g(t)x)}\frac{\partial f}{\partial x}\right) = \chi^2 f \tag{53}$$

with $\chi^2 \equiv m^2 + k_x^2 + k_y^2$. It is possible to solve this partial differential equation exactly and obtain the solution

$$f(x,t) = f_{k_y k_z \eta}(x,t) = \exp -i\chi\left[\int \cosh(\Theta - \eta)dt + x\sinh(\Theta - \eta)\right] \tag{54}$$

where η is another constant. For the limiting behaviour we have assumed for $g(t)$, we see that $\Theta(t)$ vanishes at early times and varies as $\Theta(t) \approx (g_0 t + \text{constant})$ at late times. Correspondingly, the mode f will behave as

$$f(x,t) \to \exp -i\chi\,[t\cosh(\eta) - x\sinh(\eta)] \tag{55}$$

at early times $(t \to -\infty)$ which is just the standard Minkowski positive frequency mode with $\omega = \chi\cosh\eta$, $k_x = \chi\sinh\eta$. At late times the mode evolves to

$$f(x,t) \to \exp -i\left[(\chi/2g_0)(1 + g_0 x)e^{g_0 t}\right] \tag{56}$$

We are once again led to a wave mode with exponential redshift at any given x. The metric is static in t at late times and the out-vacuum will be defined in terms of modes which are positive frequency with respect to t. The Bogolibov transformations between the mode in (56) and modes which vary as $\exp(-i\nu t)$ will involve exactly the same mathematics as in equation (49). We will get a thermal spectrum at late times.

How do we interpret these results? It may seem correct to conclude that the horizons *always* have temperature but it may not be conceptually straight forward to associate an entropy with the horizon in all cases. Unfortunately, there is no clear mathematical reason for such a dichotomous approach since: (i) The temperature and entropy for these spacetimes arise in identical manner due to identical mathematical formalism. It will be surprising if one has entropy while the other does not. (ii) Just as collapsing black hole leads to an asymptotic event horizon, a universe which is dominated by cosmological constant at late times will also lead to a horizon. Just as we can mimic the time dependent effects in a collapsing black hole by a time asymmetric quantum state (say, Unruh vacuum), we can mimic the late time behaviour of an asymptotically deSitter universe by a corresponding time asymmetric quantum state. Both these states will lead to stress tensor expectation values in which there will be a flux of radiation. One is almost forced to the conclusion that in such a scenario the cosmological constant is evaporating.

Acknowledgments

I thank Apoorva Patel for useful discussions.

References

1. Birrell N.D and Davies P.C.W, *Quantum fields in curved space*, (Cambridge University Press, Cambridge, 1982);
2. Bekenstein, J.D., *Phys. Rev. D* **7**, 2333 (1973); Hawking S.W., *Comm.Math.Phys.*, **43**, 199 (1975) Gerlach, U.H., *Phys. Rev. D* **15**, 1479 (1976); For a review, see R.D.Sorkin, gr-qc/9705006.
3. Padmanabhan T., gr-qc/0204019 (2002).
4. Deser S and Levin O., *Phys. Rev. D*, **59**, 064004 (1999).
5. Sriramkumar L. and Padmanabhan T., *Int. Jour. Mod. Phys.*, **D11**, 1 (2002).
6. Frondsal C., *Phys. Rev.*, **116**, 778 (1959).
7. Di Francesco P., Mathieu P. and Senechal D., *Conformal field theory*, (Springer, New York, 1997).
8. Carlip, S Phys. Rev **D 15 51**, 631 (1995); A.Strominger, JHEP **2**, 009 (1998)
9. See e.g., Solodukhin, S.N., Phys.Lett.B, **454**, 213 (1999).
10. Lee T.D., *Nucl. Phys. B*, **264**, 437 (1986).
11. The partition function approach has a long history. The earliest attempt was Gibbons G.W. and Hawking S.W., *Phys. Rev. D* **15**, 2738 (1977); for extensive references, see J.D. Brown, S.R. Lau, J.W. York, gr-qc/0010024;
12. Carlip S., *Quantum Gravity in 2+1 dimension*, Cambridge University Press, (1998).
13. Miser, C.W., Thorne K.S., Wheeler, J.A. , *Gravitation* (Freeman and co., 1973), pp.889.
14. Hartle J. B. and Hawking S.W., *Phys. Rev. D*, **13**, 2188 (1976).
15. Unruh W.G., *Phys. Rev. D*, **14**, 870 (1976).