Nonlinear current helicity fluxes in turbulent dynamos and alpha quenching

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Large scale dynamos produce small scale current helicity as a waste product that quenches the large scale dynamo process (alpha effect). This quenching can be catastrophic (i.e. intensify with magnetic Reynolds number) unless one has fluxes of small scale magnetic (or current) helicity out of the system. We derive the form of helicity fluxes in turbulent dynamos, taking also into account the nonlinear effects of Lorentz forces due to fluctuating fields. We confirm the form of an earlier derived magnetic helicity flux term, and also show that it is not renormalized by the small scale magnetic field, just like turbulent diffusion. Additional nonlinear fluxes are identified, which are driven by the anisotropic and antisymmetric parts of the magnetic correlations. These could provide further ways for turbulent dynamos to transport out small scale magnetic helicity, so as to avoid catastrophic quenching.

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Large scale magnetic fields in astrophysical bodies are thought to be generated by dynamo action involving helical turbulence and rotational shear [1, 2]. A particularly important driver of the mean field dynamo (MFD) is the α effect, which in the kinematic regime is proportional to the kinetic helicity of the turbulence. A question of considerable debate is how the α effect gets modified due to the backreaction of the generated mean and fluctuating fields? It is especially important to understand whether α suffers catastrophic (i.e. $R_{\rm m}$-dependent) quenching, since $R_{\rm m}$, the magnetic Reynolds number, is expected to be typically very large in astrophysical systems. Recent progress has come from realizing the importance of magnetic helicity conservation in constraining this nonlinear saturation [3, 4].

Recall that in the MFD theory, one splits the magnetic field $\boldsymbol{B}$ into a mean magnetic field $\overline{\boldsymbol{B}}$ and a small scale field $\boldsymbol{b} = \boldsymbol{B} - \overline{\boldsymbol{B}}$, and derives the mean-field dynamo equation [1]

$$\frac{\partial \overline{\boldsymbol{B}}}{\partial t} = \boldsymbol{\nabla} \times \left(\overline{\boldsymbol{U}} \times \overline{\boldsymbol{B}} + \overline{\boldsymbol{\mathcal{E}}} - \eta \overline{\boldsymbol{J}}\right). \tag{1}$$

Here $\overline{\boldsymbol{\mathcal{E}}} \equiv \overline{\boldsymbol{u} \times \boldsymbol{b}}$ is the turbulent electromotive force (emf), $\overline{\boldsymbol{J}} = \boldsymbol{\nabla} \times \overline{\boldsymbol{B}}/\mu_0$ the mean current density, μ_0 the vacuum permeability (assumed unity throughout the rest of the paper), η the microscopic resistivity and the velocity $\boldsymbol{U} = \overline{\boldsymbol{U}} + \boldsymbol{u}$ has also been split into mean $\overline{\boldsymbol{U}}$ and small scale turbulent $\boldsymbol{u} = \boldsymbol{U} - \overline{\boldsymbol{U}}$ velocities. Finding an expression for the correlator $\overline{\boldsymbol{\mathcal{E}}}$ in terms of the mean fields is a standard closure problem which is at the heart of mean field theory. In the two-scale approach [2] one assumes that $\overline{\boldsymbol{\mathcal{E}}}$ can be expanded in powers of the gradients of the mean magnetic field. For isotropic helical turbulence, this gives $\overline{\boldsymbol{\mathcal{E}}} \approx \alpha \overline{\boldsymbol{B}} - \eta_{\rm t} \overline{\boldsymbol{J}}$, where in the kinematic limit, $\alpha = \alpha_{\rm K} = -\frac{1}{3}\tau \overline{\boldsymbol{\omega} \cdot \boldsymbol{u}}$, proportional to the kinetic helicity ($\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$) and $\eta_{\rm t} = \frac{1}{3}\tau \overline{\boldsymbol{u}^2}$ is the turbulent magnetic diffusivity proportional to the specific kinetic energy of the turbulence, with τ being the velocity correlation time.

The kinematic theory has to be modified to take account of the backreaction due to the Lorentz forces associated with the generated large and small scale fields. Treatments of the back-reaction have typically used the quasi-linear approximation or closure schemes to derive corrections to the mean-field dynamo coefficients (cf. [4, 5] and [3] for a review). It is then found that the α effect gets "renormalized" by the addition of a term proportional to the current helicity, $\overline{\boldsymbol{j} \cdot \boldsymbol{b}}$, of the small scale fields; that is $\alpha = \alpha_{\rm K} + \alpha_{\rm M}$, where $\alpha_{\rm M} = (\tau/3\rho_0)\overline{\boldsymbol{j} \cdot \boldsymbol{b}}$ [5]. (Here ρ_0 is the density of the fluid and $\boldsymbol{j}$ the small scale current density.) At the same time there is no modification to $\eta_{\rm t}$ at lowest order [4].

In order to constrain $\alpha_{\rm M}$, in simulations in a periodic box or in systems that involve no boundaries, it has proved useful to take recourse to the evolution equation for the small scale magnetic helicity $h = \langle \boldsymbol{a} \cdot \boldsymbol{b} \rangle$, where $\boldsymbol{a}$ is the vector potential and $\langle\,\rangle$ denotes volume averaging. We have in such situations, ${\rm d}h/{\rm d}t = -2\langle \overline{\boldsymbol{\mathcal{E}}} \cdot \overline{\boldsymbol{B}} \rangle - 2\eta \langle \overline{\boldsymbol{j} \cdot \boldsymbol{b}} \rangle$. In the stationary limit, this predicts $\langle \overline{\boldsymbol{\mathcal{E}}} \cdot \overline{\boldsymbol{B}} \rangle = -\eta \langle \boldsymbol{j} \cdot \boldsymbol{b} \rangle$, which tends to zero as $\eta \to 0$, for any reasonable spectrum of current helicity. This leads to a catastrophic quenching of the turbulent emf parallel to $\overline{\boldsymbol{B}}$. Of course while evolving to this stationary state, some $\overline{\boldsymbol{B}}$ will be generated. But its value, at the end of the kinematic regime, still turns out to be small, if the scale separation is large [6]. It has been suggested that such quenching can be avoided if the system has open boundaries and is inhomogeneous, since one could then have a flux of small scale helicity out of the system which helps maintain a non-zero $\langle \overline{\boldsymbol{\mathcal{E}}} \cdot \overline{\boldsymbol{B}} \rangle$ [7, 8, 9]. It is therefore important to calculate such fluxes in a general manner, taking into account also the effect of Lorentz forces. This is the aim of the present work.

Indeed, Vishniac and Cho [7] derived an interesting flux of helicity, which arises even for nonhelical but anisotropic turbulence. We derive a generalized form of the Vishniac-Cho flux (henceforth VC flux) in the evolution equation for $\overline{\boldsymbol{j} \cdot \boldsymbol{b}}$ to include also nonlinear effects of the Lorentz force and helicity in the fluid turbulence. As we see, the VC flux can also be thought of as a generalized anisotropic turbulent diffusion. Further, due to nonlinear effects, other helicity flux contributions arise generated by the anisotropic and antisymmetric part of the magnetic correlations.

One immediate problem with previous approaches was that in open systems with boundaries, h is not gauge invariant and one has to consider instead the gauge-invariant relative magnetic helicity, say h_R, defined by subtracting the helicity of a reference vacuum field [10]. The flux of relative helicity is cumbersome to work with for arbitrarily shaped boundaries. Also the concept of a density of relative helicity is not meaningful, since h_R is defined only as a volume integral. In order to avoid these problems it is advantageous to consider instead the current helicity $\overline{\boldsymbol{j}\cdot\boldsymbol{b}}$ and its flux. The current helicity density and its flux are directly gauge invariant, locally well defined, and are in fact the observationally measured quantities in say the solar context. Furthermore it is $\overline{\boldsymbol{j}\cdot\boldsymbol{b}}$ which directly enters the expression for the nonlinear α effect [4, 5]. Also for isotropic small scale fields the spectra of small scale current (C_k) and magnetic helicities (H_k) are related by $H_k = C_k/k^2$. For these reasons we consider here directly the current helicity evolution and use the current helicity flux as a 'proxy' for the magnetic helicity flux.

Current helicity evolution. Consider the evolution of the small scale current helicity, $\overline{\boldsymbol{j}\cdot\boldsymbol{b}} = \epsilon_{ijk}\overline{b_i\partial_j b_k}$, which is explicitly gauge invariant. We assume that the correlation tensor of fluctuating quantities ($\boldsymbol{u}$ and $\boldsymbol{b}$) vary slowly on the system scale, say $\boldsymbol{R}$. Consider the equal time, ensemble average of the product $\overline{f(\boldsymbol{x}_1)g(\boldsymbol{x}_2)}$. The common dependence of f and g on t is assumed and will not explicitly be stated. Let $\hat{f}(\boldsymbol{k}_1)$ and $\hat{g}(\boldsymbol{k}_2)$ be the Fourier transforms of f and g, respectively. We can express this correlation as $\overline{f(\boldsymbol{x}_1)g(\boldsymbol{x}_2)} = \int \Phi(\hat{f},\hat{g},\boldsymbol{k},\boldsymbol{R})\,\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{r}}\,\mathrm{d}^3k$, with

$$\Phi(\hat{f},\hat{g},\boldsymbol{k},\boldsymbol{R}) = \int \overline{\hat{f}(\boldsymbol{k}+\tfrac{1}{2}\boldsymbol{K})\hat{g}(-\boldsymbol{k}+\tfrac{1}{2}\boldsymbol{K})}\,\mathrm{e}^{i\boldsymbol{K}\cdot\boldsymbol{R}}\,\mathrm{d}^3K. \tag{2}$$

Here we have defined the difference $\boldsymbol{r} = \boldsymbol{x}_1 - \boldsymbol{x}_2$ and the mean $\boldsymbol{R} = \frac{1}{2}(\boldsymbol{x}_1 + \boldsymbol{x}_2)$, keeping in mind that all two-point correlations will vary rapidly with $\boldsymbol{r}$ but slowly with $\boldsymbol{R}$ [11]. Also $\boldsymbol{k} = \frac{1}{2}(\boldsymbol{k}_1 - \boldsymbol{k}_2)$ and $\boldsymbol{K} = \boldsymbol{k}_1 + \boldsymbol{k}_2$. In what follows we require the correlation tensors, $v_{ij}(\boldsymbol{k},\boldsymbol{R}) = \Phi(\hat{u}_i,\hat{u}_j,\boldsymbol{k},\boldsymbol{R})$, of the $\boldsymbol{u}$ field, $m_{ij}(\boldsymbol{k},\boldsymbol{R}) = \Phi(\hat{b}_i,\hat{b}_j,\boldsymbol{k},\boldsymbol{R})$ of the $\boldsymbol{b}$ field and the cross correlation $\chi_{jk}(\boldsymbol{k},\boldsymbol{R}) = \Phi(\hat{u}_j,\hat{b}_k,\boldsymbol{k},\boldsymbol{R})$ between these two fields, in Fourier space. The turbulent emf is given by $\overline{\mathcal{E}}_i(\boldsymbol{R}) = \epsilon_{ijk}\int \chi_{jk}(\boldsymbol{k},\boldsymbol{R})\,\mathrm{d}^3k$.

In order to compute the current helicity evolution of $\partial\overline{\boldsymbol{j}(\boldsymbol{x})\cdot\boldsymbol{b}(\boldsymbol{x})}/\partial t$, we use the induction equation for $\boldsymbol{b}$ in Fourier space,

$$\begin{aligned}\frac{\partial\hat{b}_k(\boldsymbol{k})}{\partial t} &= \epsilon_{kpq}\epsilon_{qlm}\mathrm{i}k_p\int \hat{u}_l(\boldsymbol{k}-\boldsymbol{k}')\hat{\overline{B}}_m(\boldsymbol{k}')\,\mathrm{d}^3k' \\ &\quad +\hat{G}_k(\boldsymbol{k}) - \eta k^2\hat{b}_k(\boldsymbol{k}).\end{aligned} \tag{3}$$

Here, $\boldsymbol{G} = \boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{b} - \overline{\boldsymbol{u}\times\boldsymbol{b}})$ is the nonlinear term. (We also neglect the velocity shear due to $\boldsymbol{U}$ compared to that due to $\boldsymbol{u}$.) A tedious but straightforward calculation gives [3]

$$\begin{aligned}\frac{\partial}{\partial t}\overline{\boldsymbol{j}\cdot\boldsymbol{b}} &= \epsilon_{ljk}\int\Big[2\chi_{lk}\,k_j(\boldsymbol{k}\cdot\overline{\boldsymbol{B}}) - \chi_{lk}\nabla_j(\mathrm{i}\boldsymbol{k}\cdot\overline{\boldsymbol{B}}) \\ &\quad -\mathrm{i}k_j\,\overline{\boldsymbol{B}}\cdot\boldsymbol{\nabla}\chi_{lk} + 2\mathrm{i}k_j\chi_{pk}\nabla_p\overline{B}_l\Big]\,\mathrm{d}^3k + T_C.\end{aligned} \tag{4}$$

We have written out explicitly only that part of the helicity evolution driven by the coupling of the turbulent emf to the mean magnetic field. This is because we are particularly interested in turbulent helicity fluxes driven by an inhomogeneous $\overline{\boldsymbol{\mathcal{E}}}$ and $\overline{\boldsymbol{B}}$. The T_C term represents the triple correlations of the small scale $\boldsymbol{u}$ and $\boldsymbol{b}$ fields and the microscopic diffusion terms that one gets on using Eq. (3). The handling of the triple correlations needs a closure approximation. But we will not need to explicitly evaluate these terms to identify the helicity fluxes we are interested in; i.e. those which couple $\overline{\boldsymbol{\mathcal{E}}}$ and $\overline{\boldsymbol{B}}$, and so continue to write this term as T_C.

In order to calculate the current helicity evolution, using Eq. (4), we have to calculate also χ_{lk}. This has been done in detail by [3, 12]. (One adopts a closure approximation whereby triple correlations, T_{lk}, which arise now in the evolution equation for $\partial\chi_{lk}/\partial t$, are assumed to provide relaxation of the turbulent emf or χ_{lk} and one takes $T_{lk} = -\chi_{lk}/\tau$, where τ is the relaxation time (cf. also [13])). We concentrate below on nonrotating but helical turbulence. For such turbulence we have from [3, 12], $\chi_{lk} = \tau\mathsf{I}_{lk}$, where I_{lk} is given by; see Eq. (10.30) in [3].

$$\begin{aligned}\mathsf{I}_{lk} &= -\mathrm{i}\boldsymbol{k}\cdot\overline{\boldsymbol{B}}(v_{lk} - m_{lk}) + \tfrac{1}{2}\overline{\boldsymbol{B}}\cdot\boldsymbol{\nabla}(v_{lk} + m_{lk}) \\ &\quad +\overline{B}_{l,s}m_{sk} - \overline{B}_{k,s}v_{ls} - \tfrac{1}{2}k_m\,\overline{B}_{m,s}\left(\frac{\partial v_{lk}}{\partial k_s} + \frac{\partial m_{lk}}{\partial k_s}\right) \\ &\quad -2\frac{k_lk_s}{k^2}\overline{B}_{s,p}m_{pk}.\end{aligned} \tag{5}$$

We use this in what follows.

Let us denote the four terms under the integral in Eq. (4) by A_1, A_2, A_3 and A_4, respectively. In A_1, due to the presence of ϵ_{ljk}, only the antisymmetric parts of the tensors v_{lk} and m_{lk} survive, and these are denoted by v^{A}_{lk} and m^{A}_{lk}, respectively. Also note that the last term above vanishes because it involves the product $\epsilon_{ljk}k_lk_j = 0$. All the other terms of Eq. (4) already have one R derivative, and so one only needs to retain the term in $\chi_{lk} = \tau\mathsf{I}_{lk}$ which does not contain R derivatives. In A_3 one can then use the fact that $\boldsymbol{\nabla}\cdot\overline{\boldsymbol{B}} = 0$ to write it as a total divergence. We now turn to specific cases.

Isotropic, helical, nonrotating turbulence. Let us first reconsider the simple case of isotropic, helical, nonrotating, and weakly inhomogeneous turbulence. For such turbulence, the form of the velocity and magnetic correlation tensors is given by [11, 12]. In evaluating the k-integrals, only terms which involve integration over an even number of k_i survive. Also, in terms which already involve one R_i derivative, one needs to keep only the homogeneous terms in v_{lk} and m_{lk}. Further in the presence of ϵ_{ljk} all terms symmetric in any pair of the indices vanish. Taking account of these considerations, it turns out that only the homogeneous part of the velocity and magnetic correlation tensors given in [11, 12] survive. The homogeneous part of these correlations is

$$v_{ij} = \left[\delta_{ij} - \frac{k_i k_j}{k^2}\right] E(k,\boldsymbol{R}) - \frac{\epsilon_{ijk}\mathrm{i}k_k}{k^2} F(k,\boldsymbol{R}), \quad (6)$$

and a similar expression for m_{ij} with functions say $M(k,\boldsymbol{R})$ and $N(k,\boldsymbol{R})$ replacing E and F, respectively. Here $4\pi k^2 E$ and $4\pi k^2 M$ are the kinetic and magnetic energy spectra, respectively, and $4\pi k^2 F$ and $4\pi k^2 N$ are the corresponding helicity spectra. They obey the relations $\overline{\boldsymbol{u}^2} = 2\int E \mathrm{d}^3k$, $\overline{\boldsymbol{u}\cdot\boldsymbol{\nabla}\times\boldsymbol{u}} = 2\int F\mathrm{d}^3k$, $\overline{\boldsymbol{b}^2} = 2\int M\mathrm{d}^3k$, and $\overline{\boldsymbol{j}\cdot\boldsymbol{b}} = 2\int N\mathrm{d}^3k$. With these simplifications we have, after carrying out the angular integrals over the unit vectors $\hat{k}_i = k_i/k$,

$$A_1 = \tfrac{4}{3}\overline{\boldsymbol{B}}^2 \int k^2(F-N)\,\mathrm{d}^3k + \tfrac{2}{3}\overline{\boldsymbol{B}}\cdot\overline{\boldsymbol{J}} \int k^2(M+E)\,\mathrm{d}^3k. \quad (7)$$

In the case of isotropic turbulence, the second and third terms, A_2 and A_3, are zero because, to leading order in R derivatives, the integrands determining A_2 and A_3 have an odd number (3) of $\hat{k}_i$'s. The fourth term is given by

$$A_4 = \tfrac{2}{3}\overline{\boldsymbol{J}}\cdot\overline{\boldsymbol{B}}\tau \int k^2[E-M]\,\mathrm{d}^3k. \quad (8)$$

Adding all the contributions, $A_1 + A_2 + A_3 + A_4$, we get for the isotropic, helical, weakly inhomogeneous turbulence,

$$\frac{\partial}{\partial t}\overline{\boldsymbol{j}\cdot\boldsymbol{b}} = \tfrac{4}{3}\overline{\boldsymbol{B}}^2\tau \int k^2(F-N)\,\mathrm{d}^3k + \tfrac{4}{3}\overline{\boldsymbol{J}}\cdot\overline{\boldsymbol{B}}\tau \int k^2 E\,\mathrm{d}^3k + T_C. \quad (9)$$

We see that there is a nonlinear correction due to the small scale helical part of the magnetic correlation to the term $\propto \overline{\boldsymbol{B}}^2$. But the nonlinear correction to the term $\propto \overline{\boldsymbol{J}}\cdot\overline{\boldsymbol{B}}$ has canceled out, just as there is no such correction to turbulent diffusion [4].

As pointed out above, in the isotropic case the magnetic helicity spectrum is $H_k = C_k/k^2$. So the first two terms of the current helicity evolution equation Eq. (9) can be interpreted as representing the effects of exactly the source term $-2\overline{\boldsymbol{\mathcal{E}}}\cdot\overline{\boldsymbol{B}}$ which is obtained for the magnetic helicity evolution. Also for this isotropic, but weakly inhomogeneous case, one sees that there is no flux which explicitly depends on the mean magnetic field.

Anisotropic turbulence. Let us now consider anisotropic turbulence. In the first term A_1 in Eq. (4), one cannot now assume the isotropic form for the velocity and magnetic correlations. But again, due to the presence of ϵ_{ljk}, only the antisymmetric parts of the tensors v_{lk} and m_{lk} survive. Also the last term in Eq. (5) does not contribute to A_1 because it involves the product $\epsilon_{ljk}k_l k_j = 0$. One can further simplify the term involving k derivatives by integrating it by parts. Straightforward algebra, and a judicious combination of the terms then gives

$$\begin{aligned} A_1 &= \tau\epsilon_{ljk}\Big\{ -2\mathrm{i}\overline{B}_p\overline{B}_s \int k_j k_p k_s (v^{\mathrm{A}}_{lk} - m^{\mathrm{A}}_{lk})\,\mathrm{d}^3k \\ &\quad + 2\overline{B}_p \int k_j k_p (\overline{B}_{l,s} m_{sk} - \overline{B}_{k,s} v_{ls})\,\mathrm{d}^3k \\ &\quad + \overline{B}_p \overline{B}_{m,j} \int k_m k_p (v^{\mathrm{A}}_{lk} + m^{\mathrm{A}}_{lk})\,\mathrm{d}^3k \\ &\quad + \nabla_s \left[\overline{B}_p\overline{B}_s \int k_j k_p (v^{\mathrm{A}}_{lk} + m^{\mathrm{A}}_{lk})\right] \Big\}\,\mathrm{d}^3k. \end{aligned} \quad (10)$$

All the other terms, A_2, A_3 and A_4, cannot be further simplified. They are explicitly given by

$$A_2 = -\tau\epsilon_{ljk}\overline{B}_p\overline{B}_{s,j} \int k_s k_p (v^{\mathrm{A}}_{lk} - m^{\mathrm{A}}_{lk})\,\mathrm{d}^3k, \quad (11)$$

$$A_3 = -\nabla_s \left[\tau\epsilon_{ljk}\overline{B}_p\overline{B}_s \int k_j k_p (v^{\mathrm{A}}_{lk} - m^{\mathrm{A}}_{lk})\right]\,\mathrm{d}^3k, \quad (12)$$

$$A_4 = 2\tau\epsilon_{ljk}\overline{B}_p\overline{B}_{l,s} \int k_j k_p (v_{sk} - m_{sk})\,\mathrm{d}^3k. \quad (13)$$

Adding all the contributions, $A_1 + A_2 + A_3 + A_4$, we get

$$\begin{aligned} \frac{\partial}{\partial t}\overline{\boldsymbol{j}\cdot\boldsymbol{b}} &= 2\epsilon_{jlk}\tau\Big[\overline{B}_p\overline{B}_s \int \mathrm{i}k_j k_p k_s (v^{\mathrm{A}}_{lk} - m^{\mathrm{A}}_{lk})\,\mathrm{d}^3k \\ &+ 2\overline{B}_p\overline{B}_{k,s} \int k_j k_p v^{\mathrm{S}}_{ls}\,\mathrm{d}^3k - \overline{B}_p\overline{B}_{s,j} \int k_s k_p m^{\mathrm{A}}_{lk}\,\mathrm{d}^3k \\ &- \nabla_s \left(\overline{B}_p\overline{B}_s \int k_j k_p m^{\mathrm{A}}_{lk}\right)\,\mathrm{d}^3k\Big] + T_C. \end{aligned} \quad (14)$$

Here $v^{\mathrm{S}}_{ls} = \frac{1}{2}(v_{ls} + v_{sl})$ is the symmetric part of the velocity correlation function.

Let us discuss the various effects contained in Eq. (14) for current helicity evolution. The first term in Eq. (14) represents the anisotropic version of helicity generation due to the full nonlinear α effect. In fact, for isotropic turbulence it exactly will match the first term in Eq. (9). The second term in Eq. (14) gives the effects on helicity evolution due to a generalized anisotropic turbulent diffusion. This is the term which contains the VC flux. To see this, rewrite this term as

$$\begin{aligned} \left.\frac{\partial\overline{\boldsymbol{j}\cdot\boldsymbol{b}}}{\partial t}\right|_V &= 4\tau\epsilon_{jlk}\overline{B}_p\overline{B}_{k,s} \int k_j k_p v^{\mathrm{S}}_{ls}\,\mathrm{d}^3k \\ &= -\boldsymbol{\nabla}\cdot\overline{\boldsymbol{\mathcal{F}}}^V + 4\tau\overline{B}_k\epsilon_{klj}\overline{B}_{p,s} \int k_j k_p v^{\mathrm{S}}_{ls}\,\mathrm{d}^3k. \end{aligned} \quad (15)$$

Here the first term is the VC flux, $\overline{\mathcal{F}}^V_s = \phi_{spk}\overline{B}_p\overline{B}_k$, where ϕ_{spk} is a new turbulent transport tensor with

$$\phi_{spk} = -4\tau\epsilon_{jlk} \int k_j k_p v^{\mathrm{S}}_{ls}\,\mathrm{d}^3k = -4\tau\overline{\omega_k \nabla_p u_s}. \quad (16)$$

Obviously, only components of ϕ_{spk} symmetric in p and k enters in the flux $\overline{\mathcal{F}}^V_s$. The second term in Eq. (15) is the effect on helicity due to 'anisotropic turbulent diffusion'. (We have not included the large scale derivative of v_{ls} to the leading order.) Strictly speaking, $\overline{\mathcal{F}}^V$ is a current helicity flux, but if we define the spectrum of the magnetic helicity flux by dividing the spectrum of the current helicity flux by a k^2 factor, Eq. (16) for $\overline{\mathcal{F}}^V$ leads exactly to the magnetic helicity flux given in Eqs. (18) and (20) of Vishniac and Cho [7].

This split into helicity flux and anisotropic diffusion may seem arbitrary; some support for its usefulness comes from the fact that, for isotropic turbulence, $\overline{\mathcal{F}}^V$ vanishes, while the second term exactly matches with the corresponding helicity generation due to turbulent diffusion, i.e. the $\overline{\boldsymbol{J}} \cdot \overline{\boldsymbol{B}}$ term in Eq. (9). Of course, we could have just retained the non-split expression in Eq. (15), which can then be looked at as an effect of anisotropic turbulent diffusion on helicity evolution. Also, interestingly, there is no nonlinear correction to the VC flux from the small scale magnetic field in the form of, say, a term proportional to $m^{\rm S}_{ls}$; just as previously, there was no nonlinear correction to turbulent diffusion in lowest order!

Finally, Eq. (14) also contains terms (the last two) involving only the antisymmetric parts of the magnetic correlations. These terms vanish for isotropic turbulence, but contribute to helicity evolution for nonisotropic turbulence. The last term gives a purely magnetic contribution to the helicity flux, but one that depends only on the antisymmetric part of m_{lk}. Note that such magnetic correlations, even if initially small, may spontaneously develop due to the kinematic α effect or anisotropic turbulent diffusion and may again provide a helicity flux. More work is needed to understand this last flux term better [14, 15]. Preliminary simulations of helical turbulence with shear and open boundaries suggest that the sign of the VC flux agrees with that of the small scale current helicity flux, but that its magnitude may only account for about 25% of the actual flux [16]. The existence of the VC flux has also been verified in simulations of nonhelically driven shear flow turbulence [17], but its magnitude was too small to produce dynamo action.

In conclusion, we have derived helicity fluxes in turbulent dynamos, taking also into account the nonlinear effects of Lorentz forces due to the fluctuating field. To avoid gauge ambiguities, we have followed the current helicity evolution. We confirm the form of the helicity flux found by Vishniac and Cho, who used the first order smoothing approximation. We note however that it is more correctly interpreted as a current helicity flux and not as a flux of relative magnetic helicity. In addition we have found that the corresponding turbulent coefficient does not get renormalized due to nonlinear effects, just as is the case of turbulent diffusion. Additional nonlinear fluxes have been identified as being driven by the anisotropic and antisymmetric parts of the magnetic correlations. These could provide further ways for turbulent dynamos to transport out small scale magnetic helicity so as to avoid catastrophic $R_{\rm m}$-dependent quenching. It remains to calculate these fluxes in specific circumstances and also verify their presence in direct numerical simulations of turbulent dynamos.

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[15] Note that our helicity flux is distinct from that discussed by Kleeorin and coworkers [9]. In the original treatment involving path integral methods, a flux proportional to the anisotropic part of the α tensor and the anisotropic part of the magnetic helicity tensor were derived [14]. However, the subsequent use of this flux in [9] involved adopting a phenomenological term proportional to the mean field; such a term should have arisen automatically in our treatment.

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