VISCOUS UNIVERSES

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The effect of bulk viscosity on the evolution of the universe at large is investigated. It is demonstrated that bulk viscosity can lead to inflation-like solutions.

1. Introduction

It is customary to assume that the universe is homogeneous and isotropic over scales larger than ~50 Mpc. Such a universe is very well described by the Friedmann–Robertson–Walker (FRW) line element

$$ds^2 = dt^2 - S^2(t) \times \left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)\right), \tag{1}$$

where $S(t)$ is the expansion factor and $k=0, \pm 1$. The source energy–momentum tensor $T^i{}_k$ for the metric in (1) must necessarily have the form

$$T^i{}_k = \mathrm{diag}(A, B, B, B) . \tag{2}$$

it is usual to take $T^i{}_k$ to be of the form

$$T^i{}_k = (p+\rho)u^i u_k - \delta^i{}_k p , \tag{3}$$

so that

$$A+\rho, \quad B=-p , \tag{4}$$

and interpret ρ and p as the energy density and pressure respectively of an *ideal* fluid.

It is, however, possible to generalize the form of $T^i{}_k$ to include contributions from bulk viscosity, without violating the symmetries inherent in the FRW model. This is done by simply adding to (3) the term

$$(T^i{}_k)_{(\mathrm{vis})} = \mu(\delta^i{}_k - u^i u_k) u^a{}_{;a} , \tag{5}$$

where μ is the coefficient of bulk viscosity [1].

in this letter, we discuss the effect of the presence $(T^i{}_k)_{(\mathrm{vis})}$ on the cosmological evolution. (For some previous work on the role of bulk viscosity, see ref. [2].) There are many circumstances in the evolution of the universe in which bulk viscosity could arise: (i) Strings, especially superconducting strings [3], will feel a viscous drag while moving through cosmic magnetic fields. (ii) Magnetic monopoles can feel viscosity-like forces in monopole–monopole interactions. (iii) Particle creation in the early universe is known [4] to give rise to an effective energy–momentum tensor containing a term like (5). (iv) The drag arising from the radiation field contributes a photon viscosity [5]; similarly the drag contributed by neutrinos would result in neutrino viscosity [6]. Even though the microscopic details of the above phenomena can be quite involved, the major dissipative effect of these processes may be embodied in a generic term [5] containing a single parameter μ.

We note that this is the only form of dissipation consistent with homogeneity and isotropy. Several exotic processes have been thought of in the context of particle physics in the early universe. At least some of them could be envisaged as effective contributors to the coefficient of bulk viscosity μ. In what follows, we shall not be concerned with the microscopic details, but will rather concentrate on the evolutionary influence of non-zero μ.

We can also regard μ to be a free parameter similar

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to the cosmological constant Λ. We know that vacuum energy densities of quantum fields contribute a temperature-dependent part to Λ; likewise the processes outlined above may lead to a temperature-dependent part in the viscous coefficient. However, it is also conceivable to imagine the existence of a (primordial) tiny, constant part in μ, similar in nature to any small cosmological constant. This should perhaps be borne in mind while judging various cases presented in the following section. We emphasize that these solutions are similar, both in concept and in form, to the cosmological solutions with non-zero Λ.

2. Cosmological evolution with viscosity

Einstein's field equations for an FRW cosmology with a viscous term in $T^i{}_k$ take the following form:

$$(\dot{S}^2+k)/S^2=\tfrac{8}{3}\pi G\rho\,, \tag{6}$$

$$\mathrm{d}(\rho S^3)/\mathrm{d}t+p\,\mathrm{d}S^3/\mathrm{d}t=9\mu S\dot{S}^2\,. \tag{7}$$

Note that the $T^0{}_0$ component does not pick up any contribution from (5); thus (6) is the same as in the non-viscous cosmology. Eq. (7) expresses the energy balance in the presence of viscous dissipation. The term "dissipation" is somewhat counter-intuitive in the cosmological context, because the viscous term in (7) contributes positively to the time rate of change of the total energy in a comoving volume. This can be understood from the fact that the *sole* effect of bulk viscosity is replace the pressure p by p^* with

$$p^*=p-3\mu\dot{S}/S\,. \tag{8}$$

(see ref. [2], eq. (15.11.23)). Thus bulk viscosity contributes negatively to the pressure in an expanding universe, just like the cosmological constant.

The analogy with the cosmological constant prompts one to look for inflationary solutions to (6) and (7). Indeed, the presence of bulk viscosity leads to solutions of the following kind:

$$S(t)=S_0\exp(Ht)\,, \tag{9}$$

$$p=(24\pi G\rho\mu^2)^{1/2}-\rho\,, \tag{10}$$

$$H^2=\tfrac{8}{3}\pi G\rho\,, \tag{11}$$

with constant ρ, μ and $k=0$. In particular, if $p=0$ then the bulk viscosity determines both ρ and H as

$$\rho=24\pi G\mu^2,\quad H=8\pi G\mu\,. \tag{12}$$

Thus the onset of an inflationary phase can be easily realized in a viscous universe. We shall come across this feature again in our subsequent analysis.

Eqs. (6) and (7) can be recast in a different form by taking

$$\rho=f(t)/S^4(t)+g(t)/S^3(t) \tag{13}$$

and

$$p=\tfrac{1}{3}f(t)/S^4(t)\,. \tag{14}$$

Physically the two terms on the right-hand side of (13) represent respectively the radiation and matter components of the energy density. In the absence of viscosity, f and g would have been independent of time. In the present context f and g satisfy the equation

$$\dot{f}+s\dot{g}=9\mu S^2\dot{S}^2\,. \tag{15}$$

A straightforward algebraic manipulation yields

$$\frac{\mathrm{d}g}{\mathrm{d}S}+\frac{1}{S}\frac{\mathrm{d}f}{\mathrm{d}S}=3\mu[24\pi G(f+gS)-9kS^2]^{1/2}\,. \tag{16}$$

We have now eqs. (6) and (16) to determine three unknown quantities: $S(t)$, $f(t)$ and $g(t)$. One more equation expressing the coupling (if any) between matter and radiation is needed to describe the system completely. We shall investigate the various possible choices in what follows:

(i) *Radiation dominated phase.* In the early phase of the universe we may set $k=0$ and $g=0$, and regard radiation as the source of expansion. Then (16) gives

$$\mathrm{d}f/\mathrm{d}S=3\mu(S)(24\pi G)^{1/2}f^{1/2}S\,, \tag{17}$$

which can be integrated to give

$$f^{1/2}(S)=f_0^{1/2}+\tfrac{3}{2}(24\pi G)^{1/2}\int_0^S x\mu(x)\,\mathrm{d}x\,. \tag{18}$$

In (17), (18) we have explicitly incorporated a time dependence in μ. This is because – broadly speaking – various processes which go to make up μ may operate differently at different epochs. From (18), (13) we get,

$$\rho_{\rm rad}(S)=\left(\frac{f_0^{1/2}}{S^2}+\frac{3}{2}\frac{(24\pi G)^{1/2}}{S^2}\int_0^S x\mu(x)\,\mathrm{d}x\right). \tag{19}$$

The evolution depends on the form assumed for $\mu(x)$. In the simplest case of constant μ ($=\mu_0$) we get

$$\rho_{\rm rad}=\tfrac{27}{2}\pi G\mu_0^2+\alpha/S^2+\beta/S^4\,. \tag{20}$$

Substituting into (6) we find that an inflationary solution exists for large times with

$$S=S_0\exp(Ht),\quad H=6\pi G\mu_0\,. \tag{21}$$

As a second possibility let us consider the case in which $\mu(x)$ is non-zero only for a short interval of time. (For example, viscosity may be operational during a phase transition.) Taking

$$\mu(S)=\mu_0 S_0\delta(S-S_0) \tag{22}$$

we get

$$f^{1/2}(S)=\tfrac{3}{2}(24\pi G)^{1/2}\mu_0 S_0^2+f_0^{1/2},\quad S>S_0\,, \tag{23}$$

$$=f_0^{1/2},\qquad S<S_0\,. \tag{24}$$

In other words, $f^{1/2}$ jumps discontinuously by an amount $\frac{3}{2}(24\pi G)^{1/2}\mu_0 S_0^2$ across this epoch. If $\mu(S)$ was smoothed out over a time scale T, then $f^{1/2}$ will also change smoothly over the same time scale. The viscosity can, thus, add to the creation of entropy in a phase transition.

It should be noted that the conventional radiative viscosity is too small to provide significant entropy during a phase transition. The processes which we have considered will be relevant only if the phase transition takes place at high enough energies, i.e. at GUTS scale or higher. Under these circumstances the presence of exotic particle interactions may be able to contribute to a larger value of μ_0.

Lastly, one may assume $\mu(S)$ to be a power law in S:

$$\mu(S)=\mu_0(S/S_0)^n\,. \tag{25}$$

Repeating the integrations one finds that

$$\rho_{\rm rad}(S)=\alpha S^{2n}+\beta S^{n-2}+f_0/S^4\,, \tag{26}$$

where α, β and f_0 are constants with

$$\alpha=\frac{54\pi G\mu_0^2}{S_0^{2n}(n+2)^2},\quad \beta=\frac{3(24\pi G)^{1/2}\mu_0 f_0^{1/2}}{S_0^n(n+2)}, \tag{27}$$

and we assume $n\neq-2$. (*For* $n=-2$, f increases logarithmically.)

For all $n>-2$, the term S^{2n} will dominate over S^{n-2} for sufficiently large S. In this limit it is easily seen that S has the form

$$S(t)=(a-bnt)^{-1/n}\quad (a,b>0)\,. \tag{28}$$

Note that for $n>0$, the quantity $(a-bnt)$ decreases with increasing t, while for $n<0$, $(a-bnt)$ increases with t.

(ii) *Matter dominated phase.* Here we shall take $k=0$ and $f=0$. From (16) we immediately get

$$\mathrm{d}g/\mathrm{d}S=3\mu(S)(24\pi G)^{1/2}g^{1/2}S^{1/2}\,, \tag{29}$$

which integrates to give

$$g^{1/2}(S)=g_0^{1/2}+\tfrac{3}{2}(24\pi G)^{1/2}\int_0^S\mu(x)x^{1/2}\,\mathrm{d}x\,. \tag{30}$$

We can examine the various special cases like before. In particular, if μ is a constant ($=\mu_0$), (30) gives

$$g^{1/2}(S)=g_0^{1/2}+(24\pi G)^{1/2}\mu_0 S\,, \tag{31}$$

corresponding to a density evolution of the form

$$\rho(S)=\frac{g_0}{S^3}+\frac{2(24\pi G)^{1/2}\mu_0 g_0^{1/2}}{S}+24\pi G\mu_0^2\,. \tag{32}$$

It is interesting to note the regardless of how small μ_0 is, the term $(24\pi G\mu_0^2)$ in (32) will make the dominant contribution to ρ at sufficiently late epochs, and direct the universe into an inflationary mode. This inflation would occur at very late stages of the universe, in contrast to the conventional inflation.

In view of the novel aspects of the above solution it is worthwhile to work out the explicit time-dependent form of $S(t)$. From (15) and (16) we get

$$\dot g=24\pi G\mu g\,, \tag{33}$$

which integrates to give the density

$$\rho(t)=\frac{g(t)}{S^3(t)}=\frac{g_0}{S^3(t)}\exp\left(24\pi G\int_0^t\mu(\bar t)\,\mathrm{d}\bar t\right). \tag{34}$$

Substituting into (6) we get

$$\frac{dS}{dt} = \frac{8\pi G g_0}{3S(t)} \exp\left(24\pi G \int_0^t \mu(\bar{t})\, d\bar{t}\right). \quad (35)$$

For constant $\mu(t) = \mu_0$, (35) simplifies to yield

$$S(t) = \frac{g_0}{24\pi G \mu_0^2} \left(e^{12\pi G \mu_0 t} - 1\right). \quad (36)$$

If $12\pi G\mu_0 t \ll 1$, then $S(t)$ has the conventional matter-dominated form,

$$S(t) \approx (6\pi G g_0)^{1/3} t^{2/3}, \quad (37)$$

which is independent of μ_0, as it should be. Eq. (37) can represent our present day universe in which the viscosity effects are not significant. If $\mu_0 = 0$ one would have expected (37) to be valid uniformly in future. But if $\mu_0 \neq 0$ then at sufficiently late times $12\pi G\mu_0 t \gtrsim 1$ and the form of $S(t)$ switches to an exponential

$$S(t) \approx \text{const} \times \exp(8\pi G \mu_0 t). \quad (38)$$

Thus the late epochs of the universe will be viscosity dominated. The time scale t_c after which viscosity dominates can be easily obtained:

$$t_c = c^2/12\pi G\mu_0$$

$$\approx (15\ G\ \text{years}) \left(\frac{\mu_0}{10^9\ \text{g cm}^{-1}\ \text{s}^{-1}}\right).$$

(In the spirit of the discussion in this paper we have not specified any single value for μ_0. Clearly the value 10^9 g cm^{-1} s^{-1} for μ_0 will initiate the inflationary phase around the present epoch.)

Solutions similar to those found in the radiation dominated universe exist for other choices (power law, delta function etc.) of $\mu(x)$. For example, if $\mu(S) \propto S^n$ ($n > -1.5$), the form of $S(t)$ for large t is given by $S(t) \approx (a - bnt)^{-1/n}$, with a and b positive constants, which is identical in form to eq. (28).

3. Conclusions

We have investigated the effect of bulk viscosity on the evolution of the universe. It turns out that the presence of the viscous term generically leads to an inflation-like behaviour.

In ordinary fluid mechanical consideration, the bulk viscosity comes into play when the fluid is compressed and

$$\text{div}\, \boldsymbol{u} = -\frac{1}{\rho}\frac{d\rho}{dt} < 0. \quad (40)$$

On the other hand, in the cosmological context, an *expanding* universe will have div $\boldsymbol{u} > 0$. This is mainly responsible for generating the "counter-intuitive" results encountered in this paper.

In the contracting phase of the universe or in the late stages of stellar collapse the bulk viscosity may play a somewhat different but crucial role since div $\boldsymbol{u} < 0$ in such a situation. This effect is under investigation.

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