

ENTROPY OF HORIZONS, COMPLEX PATHS AND QUANTUM TUNNELLING

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In any spacetime, it is possible to have a family of observers following a congruence of timelike curves such that they do not have access to part of the spacetime. This lack of information suggests associating a (congruence dependent) notion of entropy with the horizon that blocks the information from these observers. While the blockage of information is absolute in classical physics, quantum mechanics will allow tunnelling across the horizon. This process can be analyzed in a simple, yet general, manner and we show that the probability for a system with energy E to tunnel across the horizon is $P(E) \propto \exp[-(2\pi/\kappa)E]$ where κ is the surface gravity of the horizon. If the surface gravity changes due to the leakage of energy through the horizon, then one can associate an entropy $S(M)$ with the horizon where $dS = [2\pi/\kappa(M)]dM$ and M is the active gravitational mass of the system. The implications are discussed.

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Since gravity can affect the trajectories of light rays, it has a strong effect on the light-cone structure of the spacetime. The extreme example of this effect occurs when a class of observers (i.e. a congruence of timelike curves) find that they cannot receive signals from a region of spacetime, due to the presence of a horizon. This blockage of information manifests as certain thermodynamic features associated with the horizon in the quantum mechanical context. The purpose of this paper is to provide a remarkably simple way of deriving the temperature of any horizon in a general manner. This is achieved by introducing and utilising the concept of a *local Rindler frame*, as an extension of the standard concept of local inertial frame.

Consider an observer moving along a timelike trajectory $X^a(s)$ in a given spacetime. Near any event \mathcal{P} one can choose a locally inertial frame and boost it so that the observer is instantaneously at rest in this frame. We will call this the local Lorentz frame (LLF). It is usual to assume that purely local measurements of the observer at \mathcal{P} will coincide with those made in this frame. For example, the quantity $\rho = T_{ab}u^a u^b$ is taken to be the energy density since, in the LLF

with $u^a = \delta_0^a$, ρ reduces to $\rho = T_{00}$. If we fill a region of spacetime with a congruence of timelike curves \mathcal{C} , then this allows us to define an energy density field $\rho(x) = T_{ab}(x)u^a(x)u^b(x)$ in that region.^a This relation $\rho(x) = T_{ab}u^a u^b$ is remarkable for allowing us to associate a generally covariant scalar field ρ with a non-covariant tensor component $T_{00}(x)$. The price we pay is twofold: (a) The resulting expression ρ is a generally covariant scalar but depends on the congruence \mathcal{C} . (b) There is tacit conceptual assumption that one can identify the measurements in LLF with physically meaningful quantities.

In the textbook example mentioned above, the velocity field $u^a(x)$ was provided by the congruence. In general, the congruence also provides other vector fields like, for example, the acceleration field $a^i = u^j \nabla_j u^i$ which can be used in a similar manner to construct generally covariant but congruence-dependent scalars. In particular, one can introduce a *local Rindler frame* at every event in which we can match the acceleration of the congruence instead of the velocity. Since the acceleration $a^j = u^i \nabla_i u^j$ is a spacelike vector, it can be mapped to a purely spatial vector $(0, \kappa, 0, 0)$ at a given event, which can be taken to be along the x -axis by a rotation of spatial axes. We now define the Local Rindler Frame (LRF) near a given event \mathcal{P} by transforming from the locally inertial frame coordinates (T, X, Y, Z) around \mathcal{P} to the Rindler coordinates (t, N, Y, Z) through the coordinate transformation:

$$\kappa X = N \cosh \kappa t, \quad \kappa T = N \sinh \kappa t. \tag{1}$$

The resulting line element near \mathcal{P} is:

$$ds^2 \simeq -N^2 dt^2 + \frac{dN^2}{\kappa^2} + dL_{\perp}^2. \tag{2}$$

An example of its utility in the classical context is provided by derivation of electromagnetic field of an *arbitrarily* moving charged particle by coordinate transformations, exactly in the manner in which the electromagnetic field of a uniformly moving particle is derived.²

Its utility in the study of horizon thermodynamics³ stems from the fact that the LRF arises very naturally for a general class of metrics with the following properties: (i) the metric is static in the given coordinate system, $g_{0\alpha} = 0$, $g_{ab}(t, \mathbf{x}) = g_{ab}(\mathbf{x})$; (ii) the $g_{00}(\mathbf{x}) \equiv -N^2(\mathbf{x})$ vanishes on some 2-surface \mathcal{H} defined by the equation $N^2 = 0$, while (iii) $\partial_{\alpha} N$ is finite and non-zero on \mathcal{H} and (iv) all other metric components and curvature remain finite and regular on \mathcal{H} . Then the line element will be:

$$ds^2 = -N^2(x^{\alpha})dt^2 + \gamma_{\alpha\beta}(x^{\alpha})dx^{\alpha} dx^{\beta}. \tag{3}$$

^aIncidentally, the content of Einstein's equations can be stated entirely in terms of scalar quantities as follows: *The scalar projection of R_{abcd} orthogonal to the vector field u^a is $16\pi G\rho$ for all congruences.* The projection is $R_{abcd}h^{ac}h^{bd} = 2G_{ac}u^a u^c$ where $h^{ac} = (g^{ac} + u^a u^c)$ is the projection operator and the rest follows trivially; this is a far simpler statement than the one found in some textbooks.¹

The comoving observers in this frame have trajectories $\mathbf{x} = \text{const.}$, four-velocity $u_a = -N\delta_a^0$ and four-acceleration $a^i = u^j \nabla_j u^i = (0, \mathbf{a})$ which has the purely spatial components $a_\alpha = (\partial_\alpha N)/N$. The unit normal n_α to the $N = \text{const.}$ surface is given by $n_\alpha = \partial_\alpha N (g^{\mu\nu} \partial_\mu N \partial_\nu N)^{-1/2} = a_\alpha (a_\beta a^\beta)^{-1/2}$. A simple computation now shows that the normal component of the acceleration $a^i n_i = a^\alpha n_\alpha$, “redshifted” by a factor N , has the value

$$N(n_\alpha a^\alpha) = (g^{\alpha\beta} \partial_\alpha N \partial_\beta N)^{1/2} \equiv Na(\mathbf{x}), \tag{4}$$

where the last equation defines the function a . From our assumptions, it follows that on the horizon $N = 0$, this quantity has a finite limit $Na \rightarrow \kappa$; the κ is called the surface gravity of the horizon. These static spacetimes, however, have a more natural coordinate system defined in terms of the level surfaces of N . That is, we transform from the original space coordinates x^μ in Eq. (3) to the set (N, y^A) , $A = 2, 3$ by treating N as one of the spatial coordinates. Near the $N \rightarrow 0$ surface, $Na \rightarrow \kappa$, the surface gravity, and the metric reduces to the Rindler form in Eq. (2) obtained in the LRF near the horizon (for a more detailed discussion see Ref. 4).

The bad behaviour of the metric in Eq. (2) near $N = 0$ is a feature and not a bug. It is connected with the fact that the observers at constant- \mathbf{x} perceive a horizon at $N = 0$ and related *non-trivial physical phenomena*. Classically, this phenomenon is just a character of the horizon and is well understood. Quantum field theory, however, brings in new features since it permits processes which are classically forbidden, allowing, for example, tunnelling across the horizon thereby endowing the horizon with new properties.⁵

We will provide a general description of these phenomena using LRF.

To see how this comes about, let us recall that consistent formulation of quantum field theory requires analytic continuation in the time coordinate t . Consider what happens to the coordinate transformations in Eq. (1) and the metric near the horizon, when the analytic continuation $T \rightarrow T_E = Te^{i\pi/2}$ is performed. The hyperbolic trajectory of an observer stationary in the LRF at $N = 1$ (for which t measures the proper time), is given in parametric form as $\kappa T = \sinh \kappa t$, $\kappa X = \cosh \kappa t$. This becomes a circle, $\kappa T_E = \sin \kappa t_E$, $\kappa X = \cos \kappa t_E$ with, $-\infty < t_E < +\infty$ on analytically continuing in both T and t . It is now clear that the *complex t -plane probes the region which is classically inaccessible* to the family of observers on $N = \text{const.}$ trajectory.⁶ The transformations in Eq. (1) with $N > 0$, $-\infty < t < \infty$ cover *only* the right-hand wedge $[|X| > |T|, X > 0]$ of the Lorentzian sector. Nevertheless, *both* $X > 0$ and $X < 0$ are covered by different ranges of the “angular” coordinate t_E . The range $(-\pi/2) < at_E < (\pi/2)$ covers $X > 0$ while the range $(\pi/2) < at_E < (3\pi/2)$ covers $X < 0$ etc. These results have two important consequences which we will use:

- The light cones of the inertial frame $X^2 = T^2$ are mapped to the origin of the $(T_E - X)$ -plane, (The region “inside” the horizon $|T| > |X|$ simply *disappears* in the Euclidean sector.) making physics near the horizon to be localized near

the origin of (T_E, X) -plane. *It is this fact which allows one to use a local Rindler frame to capture the relevant physics.*

- Eq. (1) shows that $\kappa t \rightarrow \kappa t - i\pi$ changes X to $-X$, ie. trajectory with complex value for t coordinate can describe the tunnelling across the horizon, from $-X$ to X . (Performing this operation twice shows that $\kappa t \rightarrow \kappa t - 2i\pi$ is an identity transformation implying periodicity in the imaginary time $i\kappa t = \kappa t_E$. This fact is used in one way or another in several derivations of the temperature associated with the horizon.⁷⁾

We shall now provide a simple but quite general derivation of the thermodynamic properties of the horizon using these facts. Consider a physical system (it could be a particle but our description is fairly general) described by a wave function $\Psi(t, N, \mathbf{X}_\perp; E) = \exp[iA(t, N, \mathbf{X}_\perp; E)]$ in the WKB approximation where A is the solution of Hamilton–Jacobi equation for the system corresponding to energy E . The dependence of the quantum mechanical probability $P(E) = |\Psi|^2$ on the energy E can be quantified in terms of the derivative

$$\frac{\partial \ln P}{\partial E} \approx -\frac{\partial}{\partial E} 2(\text{Im } A) = -2 \text{Im} \left(\frac{\partial A}{\partial E} \right) \quad (5)$$

in which the dependence on the coordinates is suppressed. Under normal circumstances, action will be real in the leading order approximation and the imaginary part will vanish. (One well-known counterexample is in the case of tunnelling in which the action acquires an imaginary part; Eq. (5) correctly describes the dependence of tunnelling probability on the energy in the case of radioactive decay, for example.) For any Hamiltonian system, one can set $(\partial A / \partial E) = -t_0 = \text{const.}$ thereby determining the trajectory of the system. Once the trajectory is known, this equation determines t_0 as a function of E . Hence we can write

$$\frac{\partial \ln P}{\partial E} \approx 2 \text{Im}[t_0(E)]. \quad (6)$$

Now we only need to note that $t_0(E)$ can pick up an imaginary part if the trajectory of the system crosses the horizon. In fact, since $\kappa t \rightarrow \kappa t - i\pi$ changes X to $-X$ the imaginary part of t_0 for trajectories that cross the horizon is given by $(-\pi/\kappa)$ leading to $(\partial \ln P / \partial E) = -2\pi/\kappa$. Integrating, we find that the probability for the trajectory of any system to cross the horizon, with the energy E will be given by the Boltzmann factor

$$P(E) \propto \exp \left[-\frac{2\pi}{\kappa} E \right] = P_0 \exp[-\beta E] \quad (7)$$

with temperature $T = \kappa/2\pi$. (For special cases of this general result see Ref. 8 and references cited therein. The sign of imaginary part in $\kappa t \rightarrow \kappa t - i\pi$ is decided by the sign of E and is chosen such that we are considering the tunnelling of a positive energy particle.)

In obtaining the above result, we have treated κ (which is determined by the background geometry) as a constant independent of E . A more interesting situation

develops if the surface gravity of the horizon changes when some amount of energy crosses it. Suppose that $\kappa(M)$ denotes the surface gravity when the horizon is generated by an active gravitational mass M . When the energy E tunnels through the horizon, the surface gravity becomes $\kappa(M - E)$ and the above result generalises to

$$P(E) \propto \exp - \int \frac{2\pi dE}{\kappa(M - E)} \propto \exp S(E), \tag{8}$$

where

$$dS \equiv (2\pi/\kappa(M))dM = dM/T(M) \tag{9}$$

suggests a natural definition for an entropy function. An explicit example in which this situation arises is in the case of a spherical shell of energy E escaping from a black hole of mass M . This changes the mass of the black hole to $(M - E)$ with the corresponding change in the surface gravity. Clearly, Eq. (9) gives an entropy that is proportional to the area of the horizon in the spherically symmetric spacetimes.

The analysis, so far, did not require and special assumptions and is completely general. We shall now discuss the conditions under which one can prove that the entropy is related to the area of the horizon. One possible line of attack is the following: In any static spacetime, it is possible to prove the relation:

$$\int_{\partial V} \sqrt{\sigma} d^2x (N n_\mu a^\mu) = 4\pi GM \tag{10}$$

which relates the flux of surface gravity to the Tolman-Komar⁹ active gravitational mass

$$M = 2 \int_V d^3x \sqrt{\gamma} N \left(T_{ab} - \frac{1}{2} T g_{ab} \right) u^a u^b \tag{11}$$

contained inside the compact surface. When the surface is a horizon, the left-hand side of Eq. (10) reduces to κA_\perp where A_\perp is the horizon area. If some amount of energy tunnels through the horizon, then all the quantities κ, A_\perp and M will change and we get the relation $A_\perp \Delta\kappa + \kappa \Delta A_\perp = 4\pi G \Delta M$. Using this relation and Eq. (9) and rearranging the terms, we get the expression for the entropy to be:

$$S = \frac{1}{2} A_\perp + \frac{1}{2} \int A_\perp \frac{dT}{T}. \tag{12}$$

To proceed further we need a relation between the horizon area and horizon temperature so that the integral in the second term can be evaluated. This is difficult to obtain since we have no way of relating the horizon area to horizon temperature in the *general* case. (This issue is under investigation.) In the simplest case of spherically symmetric metrics with $-g_{tt} = g^{rr} = f(r)$, the area of horizon will be $4\pi a^2$ where a is a root of the equation $f(r) = 0$ while the temperature is related to $f'(a)$; in general, the function $f(r)$ and its derivative are independent and any relation between them is deeply embedded in the constraint equations of gravity. However, one can show by a detailed investigation (see the first work cited in

Ref. 11) that the correct results are indeed obtained in this case. Of course, it is straightforward to check that Eq. (12) gives the correct result for the familiar cases like Schwarzschild, de Sitter etc. In both Schwarzschild and de Sitter spacetimes, the second term in Eq. (12) — which arises from the variation of surface gravity — contributes $(-1/4)A_{\perp}$, thereby leading to the correct final result of $S = (1/4)A_{\perp}$. Clearly, Eq. (12) deserves further study.

Finally, motivated by these considerations, one would like to explore the possibility of associating an entropy with area of the horizon in a more general context.¹⁰ A natural definition for static horizons is provided by:

$$S = \frac{\beta}{8\pi G} \int_{\partial\mathcal{V}} \sqrt{\sigma} d^2x (Nn_{\mu}a^{\mu}). \quad (13)$$

The integration is over a three-dimensional region \mathcal{V} with a boundary $\partial\mathcal{V}$ and $\beta = 2\pi/\kappa$ where κ is the surface gravity of the horizon. (This is generally covariant but congruence-dependent.) We take this quantity to be the definition of gravitational entropy for any static spacetime with a horizon, based on the following considerations: First, if the boundary $\partial\mathcal{V}$ is a standard black hole horizon, $(Nn_{\mu}a^{\mu})$ will tend to a constant surface gravity κ and using $\beta\kappa = 2\pi$ we get $S = \mathcal{A}/4G$ where \mathcal{A} is the area of the horizon. Thus, in the familiar cases, this does reduce to the standard expression for entropy. Second, if the boundary $\partial\mathcal{V}$ is a compact surface enclosing a compact horizon \mathcal{H} and if the region between $\partial\mathcal{V}$ and \mathcal{H} is empty, then again we get the entropy $S = \mathcal{A}/4G$ because the flux through the two surfaces is the same when the in between region has $T_{ab} = 0$. Similar considerations apply to each piece of *any* area element when it acts as a horizon for some Rindler observer. Finally, the results obtained in a series of previous papers¹¹ showed that the bulk action for gravity can be obtained from a surface term in the action, if we take the entropy of any horizon to be proportional to its area with an elemental area $\sqrt{\sigma} d^2x$ contributing an entropy $dS = (Nn_{\mu}a^{\mu})\sqrt{\sigma}d^2x$. Our definition is the integral expression of the same.

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