INSTABILITY OF FLATSPACE AND THE ORIGIN OF CONFORMAL FLUCTUATIONS

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It is shown that conformal fluctuations in the metric can be initiated by the vacuum fluctuations of a scalar field with mass greater than the Planck mass. Flatspace is unstable against such fluctuations.

1. Introduction. The idea that the universe was created out of a quantum fluctuation has been suggested by various authors recently (see ref. [1]). The discovery that flatspace is unstable against certain forms of quantum fluctuations has added strength to the above viewpoint [2].

In the past few years the author was investigating (along with J.V. Narlikar) an approach to quantum gravity based on quantum conformal fluctuations [3]. This approach leads to a manageable theory of quantum gravity and helps to solve the singularity, horizon and flatness problems [4]. The question arises as to how the quantum conformal fluctuations of this model are initiated, so that the spacetime can make a transition from flatspace to a space with non-trivial geometry. One way of doing this is to couple back the quantum fluctuations to the background metric, which makes flatspace unstable [5]. We present here a more general, simpler and more conventional idea. We consider a conformally coupled massless scalar field and show that the conformal fluctuations in flatspace will increase without bounds (in other words, flatspace will be unstable) if the mass of the scalar field is greater than a critical value.

2. Homogeneous case. Consider a scalar field $\eta(x, t)$ of mass m in a spacetime with the metric ($\bar{g}_{ik}$ = flat metric)

$$ds^2 = \Omega^2(x, t)\bar{g}_{ik}\, dx^i\, dx^k\,. \tag{1}$$

We take the action for the system to be

$$J = (16\pi G)^{-1} \int R(-g)^{1/2}\, d^4x + \tfrac{1}{2} \int (\eta^i\eta_i - \tfrac{1}{6}R\eta^2 - m^2\eta^2)(-g)^{1/2}\, d^4x\,. \tag{2}$$

Making a conformal transformation, and writing $\phi(x, t) = \Omega\eta$, one gets the action

$$J = \tfrac{1}{2}\int (\phi^i\phi_i - \alpha_G^2\phi^2\psi^2)\, d^4x - \tfrac{1}{2}\int \psi^i\psi_i\, d^4x\,, \tag{3}$$

where

$$\alpha_G^2 = 4\pi G m^2/3\,, \quad \psi = (3/4\pi G)^{1/2}\Omega\,. \tag{4}$$

The Euler–Lagrange equations are

$$\Box\phi + \alpha_G^2\psi^2\phi = 0\,, \tag{5}$$

$$\Box\psi - \alpha_G^2\langle\phi^2\rangle\psi = 0\,. \tag{6}$$

Normally one expects the space to be the flat vacuum, which corresponds to

$$\psi = \text{constant} = m/\alpha_G\,, \quad \langle\phi^2\rangle = 0\,. \tag{7}$$

(Of course, $\langle\phi^2\rangle$ is formally divergent even in vacuum; we assume here that this infinity has been subtracted.) Let us now perturb solution (7) in the following manner. Let us put, for $t > 0$,

$$\psi(t) = m/\alpha_G + \epsilon(t)\,, \quad \epsilon(t) = 0 \text{ for } t \leqslant 0\,. \tag{8}$$

Through eq. (5) one can find the response of the vacuum fluctuations $\langle\phi^2\rangle$ to the above perturbation. Putting it back into eq. (6) we can find out whether the fluctuations grow or not. In the present section we shall illustrate the idea, assuming $\phi(x, t)$ to be homogeneous; that is $\phi = \phi(t)$. This reduces the quantum-field-theory problem to a quantum-mechanics problem. The equation for the (Heisenberg-representation) operator $\phi(t)$ is

$$d^2\phi/dt^2 + \alpha_G^2 [m^2/\alpha_G^2 + (2m/\alpha_G)\epsilon(t)]\phi(t) = 0 . \quad (9)$$

(We are interested in the linearized stability and hence have restricted ourselves to first order in $\epsilon(t)$.) The solution to eq. (9) can be obtained using the Green function for the harmonic oscillator, and is given by

$$\begin{aligned}\phi(t) &= (2m)^{-1/2}Q(t) \qquad (t \leqslant 0), \\ &= (2m)^{-1/2} \left\{ Q(t) - 2\alpha_G \int_0^t dt' \, \epsilon(t')Q(t') \times \sin[m(t-t')] \right\} \qquad (t \geqslant 0), \quad (10)\end{aligned}$$

where a, a^+ are the standard annihilation–creation operators and

$$Q(t) = a e^{imt} + a^+ e^{-imt} . \quad (11)$$

Using this explicit form, one can compute the vacuum expectation value $\langle 0|\phi^2|0\rangle$; to be consistent with our background solution one must subtract the zero-point value. Simple algebra leads to

$$\langle\phi^2(t)\rangle_{\text{ren}} = m\alpha_G \int_0^t dt' \, \dot\epsilon(t') \cos[2m(t-t')] , \quad (12)$$

where the dot denotes differentiation with respect to time. After substitution into eq. (6) one has, to first order in $\epsilon(t)$,

$$\ddot\epsilon(t) = m^2\alpha_G^2 \int_0^t dt' \, \dot\epsilon(t') \cos[2m(t-t')] . \quad (13)$$

This integral equation can be solved by Laplace transforms. It is convenient to write

$$\begin{aligned}\ddot\epsilon(t) &= m^2\alpha_G^2 \int_0^t dt' \, \dot\epsilon(t') \frac{-1}{2m}\frac{d}{dt'} \{\sin[2m(t-t')]\} \\ &= \tfrac{1}{2}m\alpha_G^2 \, \dot\epsilon(0)\sin(2mt) \\ &+ \tfrac{1}{2}m\alpha_G^2 \int_0^t dt' \, \ddot\epsilon(t') \sin[2m(t-t')] , \quad (14)\end{aligned}$$

where we have used an integration by parts (and the initial condition $\epsilon(0) = 0$) to produce an $\ddot\epsilon$ inside the integral. Taking the Laplace transform of the whole equation we get

$$\ddot\epsilon(s) = \tfrac{1}{2}m\alpha_G^2 \, \dot\epsilon(0)L(s) + \tfrac{1}{2}m\alpha_G^2 \, \ddot\epsilon(s)L(s) , \quad (15)$$

where $L(s)$ is the Laplace transform of $\sin(2mt)$,

$$L(s) = \int_0^\infty dt \, e^{-st} \sin(2mt) = 2m/(s^2 + 4m^2) . \quad (16)$$

Solving this algebraic equation for $\ddot\epsilon(s)$, we get

$$\ddot\epsilon(s) = \tfrac{1}{2}m\alpha_G^2 \, \dot\epsilon(0)L(s)/[1 - \tfrac{1}{2}m\alpha_G^2 L(s)] . \quad (17)$$

From the theory of Laplace transforms (see ref. [6]) we know that $\ddot\epsilon(t)$ is determined by the poles of the function $\ddot\epsilon(s)$. The fluctuations will increase without bounds if $\ddot\epsilon(s)$ has poles for real s. From eq. (16) we see that $L(s)$ has its maximum value at $s = 0$, which is

$$L_{\max} = L(0) = 1/2m . \quad (18)$$

Real poles can exist – and produce instability – only if

$$\tfrac{1}{2}m\alpha_G^2 L_{\max} = \tfrac{1}{4}\alpha_G^2 \geqslant 1 . \quad (19)$$

Using eq. (4), we see that flatspace is unstable for the values

$$m^2 \geqslant (3/\pi)(c\hbar/G) . \quad (20)$$

Thus if the conformal fluctuations originate, it can originate with the creation of superheavy scalar bosons.

3. General case. The previous analysis can be repeated even when the field ϕ depends on space coordinates. We shall briefly indicate the procedure. The field $\phi(x, t)$ satisfies the equation (operator equation, Heisenberg picture)

$$\Box\phi + \alpha_G^2 [m^2/\alpha_G^2 + (2m/\alpha_G)\epsilon(t)]\phi = 0 . \quad (21)$$

Introducing the Fourier decomposition

$$\phi(x, t) = (2\pi)^{-3} \int d^3k \, q_k(t) \exp(ik \cdot x) , \quad (22)$$

each mode satisfies the equation ($\omega_k^2 = k^2 + m^2$)

$$\ddot q_k + \omega_k^2 q_k = -2m\alpha_G \epsilon(t) q_k(t) . \quad (23)$$

We shall assume, as before, that the field was in the vacuum state for $t \leqslant 0$. By repeating the analysis of the previous section, one can arrive at the renormalised value for $\langle\phi^2\rangle$ to be,

$$\langle\phi^2\rangle_{\text{ren}} = \frac{m\alpha_G}{(2\pi)^3}\int\frac{\mathrm{d}^3k}{2\omega_k^3}\int_0^t \mathrm{d}t'\ \dot{\epsilon}(t')$$
$$\times \cos[2\omega_k(t-t')]\ . \tag{24}$$

One major difference between this and the case in section 2 is that here the vacuum background term is divergent. It is an open question as to whether the gravitational effect of such a term is negligible. However, it has become a standard practice to omit this term. Substituting eq. (24) into (6), one finally arrives at

$$\ddot{\epsilon}(t) = (m^2\alpha_G^2/8\pi^2)\dot{\epsilon}(0)A(t)$$
$$+ (m^2\alpha_G^2/8\pi^2)\int_0^t \mathrm{d}t'\ \ddot{\epsilon}(t')A(t-t')\ , \tag{25}$$

where

$$A(t) = \int_0^\infty \frac{k^2\mathrm{d}k}{\omega_k^4}\sin(2\omega_k t)\ . \tag{26}$$

Taking the Laplace transform,

$$\ddot{\epsilon}(s) = \frac{m^2\alpha_G^2}{8\pi^2}\dot{\epsilon}(0)\frac{L(s)}{1-(m^2\alpha_G^2/8\pi^2)L(s)}\ , \tag{27}$$

where $L(s)$ is the Laplace transform of $A(t)$,

$$L(s) = \int_0^\infty \frac{k^2\mathrm{d}k}{\omega_k^4}\int_0^\infty \mathrm{d}t\ \mathrm{e}^{-st}\sin(2\omega_k t)$$
$$= \int_0^\infty \frac{2k^2\mathrm{d}k}{\omega_k^3(s^2+4\omega_k^2)}\ . \tag{28}$$

While $L(s)$ can be explicitly calculated we do not need that expression. It is clear from eq. (28) that $L(s)$ decreases for increasing $|s|$ and has the maximum value for $s = 0$, given by

$$L(0) = \int_0^\infty \frac{k^2\mathrm{d}k}{2(k^2+m^2)^{5/2}} = \frac{1}{6m^2}\ . \tag{29}$$

The condition for instability reads

$$(m^2\alpha_G^2/8\pi^2)L(0) = \alpha_G^2/48\pi^2 > 1\ . \tag{30}$$

In other words,

$$m^2 \geqslant 36\pi(\hbar c/G)\ . \tag{31}$$

This limit is different from the one in eq. (20), but is mostly in the same range (that is larger than the Planck mass).

4. Conclusion. It is surprising that there exists a threshold mass above which spacetime seems to undergo a "phase transition". Tentatively, one may consider the universe to have originated with such massive bosons, which probably later lead to other particles via standard interactions. It is likely that the universe makes a transition to one of the "quantum stationary geometries" [3] because of the fluctuations. It is of interest to see which geometries are stable against these fluctuations, when various types of matter fields are present. This and related issues are under investigation.

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