

Neutrino Interactions in a Magnetized Medium

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Neutrino-photon processes, forbidden in vacuum, can take place in presence of a thermal medium or an external electro-magnetic field, mediated by the corresponding charged leptons (real or virtual). The effect of a medium or an electromagnetic field is two-fold - to induce an effective ν - γ vertex and to modify the dispersion relations of all the particles involved to render the processes kinematically viable. It has already been noted that in presence of a thermal medium such an electromagnetic interaction translates into the neutrino acquiring a small effective charge. In this work, we extend this concept to the case of a thermal medium in presence of an external magnetic field and calculate the effective charge of a neutrino in the limit of a weak magnetic field. We find that the effective charge is direction dependent which is a direct effect of magnetic field breaking the isotropy of the space.

I. INTRODUCTION

Processes involving neutrinos and photons are of great importance in astrophysics and cosmology [1]. In particular, reactions which are forbidden (or are highly suppressed) in vacuum, notably plasmon decay ($\gamma \rightarrow \nu\nu$) or the Cherenkov process ($\nu \rightarrow \nu\gamma$) and the cross-processes (e.g. $\gamma\gamma \rightarrow \nu\bar{\nu}$, $\nu\bar{\nu} \rightarrow e^+e^-$ etc.) play a significant role in regions pervaded by dense plasma and/or large-scale external magnetic fields. White Dwarfs, Neutron Stars or the final phases of stellar evolution (Supernovae) are particular examples where such processes become important by virtue of the large material density and the presence of strong magnetic fields. Prompted by these objectives we calculate the effective neutrino charge in an external magnetic field in presence of a thermal medium.

It has already been shown that the ν - γ interaction in presence of a thermal medium induces a small effective charge to the neutrino and that the neutrino electromagnetic vertex is related to the photon self-energy in the medium [2,3]. We re-investigate this problem considering not only a thermal medium but also an external magnetic field for a neutrino coupled to a dynamical photon having $q_0 = 0$ and $|\vec{q}| \rightarrow 0$. This calculation is pertinent, for example, in the case of a type-II supernovae collapse [4]. It is conjectured that the neutrinos, produced deep inside the stellar core, deposit some fraction of their energy to the surrounding medium through different kinds of electromagnetic interactions, e.g. $\nu \rightarrow \nu\gamma$, $\nu\bar{\nu} \rightarrow e^+e^-$ [23]. But it is important to note that the amplitudes of such processes are proportional to G_F^2 and the amount of energy transferred to the mantle of the proto-neutron star is barely sufficient to blow the stellar envelope out. Recently in a series of papers, it has been argued that the freely streaming neutrinos from the supernova core interact with the non-relativistic electrons present in the outer part of the core through collective interactions (known as the ‘two-stream instability’ in the context of plasma physics) and is responsible for blowing

up the mantle of the supernova progenitor [6].

In this work, we show that the effective charge acquired by a neutrino in a magnetised medium is, in fact, *direction dependent* which should affect these processes significantly. Though it should be noted here that except for in the very early universe or in the recently discovered ‘magnetars’ (newly born neutron stars with magnetic fields in excess of 10^{15} Gauss) [7] or perhaps in the central region of core-collapse supernovae, the magnetic fields are smaller than the QED limit ($eB < m_e^2$ i.e. $B \leq 10^{13}$ Gauss). This allows for a weak-field treatment of the plasma processes relevant to almost all physical situations. Moreover, this treatment is also valid for compact astrophysical objects (viz. white dwarfs or neutron stars) for which the Landau level spacings are quite small compared to the electron Fermi energy [8]. This ensures that the magnetic field does not introduce any spatial anisotropy in the collective plasma behaviour.

In the standard model, the above mentioned ν - γ processes appear at the one-loop level. They do not occur in vacuum because they are kinematically forbidden and also because the neutrinos do not couple to the photons at the tree-level. In the presence of a medium or a magnetic field, it is the charged particle running in the loop which, when integrated out, confers its electromagnetic properties to the neutrino. Therefore, processes involving two neutrinos and one photon which are forbidden in vacuum can become important in the presence of a medium and/or external fields [9–11]. These charged particles could be the electrons of the background thermal medium or the virtual electrons and positrons in presence of an external magnetic field or both. The processes also become kinematically allowed since the photons acquire an effective mass in a thermal medium. This, for example, opens up the phase space for the Cherenkov process $\nu \rightarrow \nu\gamma$ [10,12,13]. The presence of a magnetic field would also modify the photon dispersion relation and then the Cherenkov condition would be satisfied for significant ranges of photon frequencies [14–16]. A ther-

mal medium and an external magnetic field, thus, fulfill the dual purpose of inducing an effective neutrino-photon vertex and of modifying the photon dispersion relation such that the phase-space for various neutrino-photon processes is opened up (see [11] and references therein for a detailed review).

The organization of the paper is as follows. In section-II we discuss the basic formalism for calculating the effective charge of a neutrino. Section-III contains the details of the calculation of the 1-loop diagram in presence of a magnetised medium. In section-IV we discuss the generic form of the neutrino effective charge in different background conditions. Finally, in section-V we present the expression for the effective charge of a neutrino. And we conclude with a discussion on the possible implications of our result in section-VI.

II. FORMALISM

The off-shell electromagnetic vertex function Γ_λ is defined in such a way that, for on-shell neutrinos, the $\nu\nu\gamma$ amplitude is given by:

$$\mathcal{M} = -i\bar{u}(k')\Gamma_\lambda u(k)A^\lambda(q), \quad (1)$$

where, q, k, k' are the momentum carried by the photon and the neutrinos respectively and $q = k - k'$. Here, $u(k)$ is the neutrino wave-function and A^μ stands for the electromagnetic vector potential. In general, Γ_λ would depend on k, q , the characteristic of the medium and the external electromagnetic field. We shall, in this work, consider neutrino momenta that are small compared to the masses of the W and Z bosons. We can, therefore, neglect the momentum dependence in the W and Z propagators, which is justified if we are performing a calculation to the leading order in the Fermi constant, G_F . In this limit four-fermion interaction is given by the following effective Lagrangian:

$$\mathcal{L}_{\text{eff}} = \frac{1}{\sqrt{2}}G_F\bar{\nu}\gamma^\mu(1 + \gamma_5)\nu\bar{l}_\nu\gamma_\mu(g_V + g_A\gamma_5)l_\nu, \quad (2)$$

where, ν and l_ν are the neutrino and the corresponding lepton field respectively. For electron neutrinos,

$$g_V = 1 - (1 - 4\sin^2\theta_W)/2, \quad (3)$$

$$g_A = 1 - 1/2; \quad (4)$$

where the first terms in g_V and g_A are the contributions from the W exchange diagram and the second one from the Z exchange diagram. Then the amplitude effectively reduces to that of a purely photonic case with one of the photons replaced by the neutrino current, as seen in the diagram in fig. 1. Therefore, Γ_λ is given by:

$$\Gamma_\mu = -\frac{1}{\sqrt{2}e}G_F\gamma^\nu(1 + \gamma_5)(g_V\Pi_{\mu\nu} + g_A\Pi_{\mu\nu}^5), \quad (5)$$

where, $\Pi_{\mu\nu}^5$ represents the vector-axial vector coupling and $\Pi_{\mu\nu}$ is the polarisation tensor arising from the diagram in fig. 2. In an earlier paper [17] (paper-I henceforth) we have analysed the structure of $\Pi^{\mu\nu}$ and calculated the photon dispersion relation, in a background medium in presence of a uniform external magnetic field, in the weak-field limit by retaining terms up-to $\mathcal{O}(\mathcal{B})$, calculated at the 1-loop level. We shall use the results of paper-I here to obtain the total effective charge of the neutrinos under equivalent conditions. Because of the electromagnetic current conservation, for the polarisation tensor, we have the following gauge invariance condition:

$$q^\mu\Pi_{\mu\nu} = 0 = \Pi_{\mu\nu}q^\nu. \quad (6)$$

Same is true for the photon vertex of fig. 1 and we have

$$\Pi_{\mu\nu}^5 q^\nu = 0. \quad (7)$$

Therefore, the effective charge of the neutrinos is defined in terms of the vertex function by the following relation [2]:

$$e_{\text{eff}} = \frac{1}{2k_0}\bar{u}(k)\Gamma_0(q_0 = 0, \mathbf{q} \rightarrow 0)u(k). \quad (8)$$

For massless Weyl spinors this definition can be rendered into the form:

$$e_{\text{eff}} = \frac{1}{2k_0}\text{tr}[\Gamma_0(q_0 = 0, \mathbf{q} \rightarrow 0)(1 + \lambda\gamma^5)\not{k}] \quad (9)$$

where $\lambda = \pm 1$ is the helicity of the spinors.

It can be seen from Eq.(9) that, in general, the effective neutrino charge depends on $\Pi_{\mu\nu}(q)$ as well as on $\Pi_{\mu\nu}^5(q)$. Now, in a magnetised medium, the dispersive part of $\Pi_{\mu\nu}$ to liner order in \mathcal{B} has the following form [paper-I]:

$$\Pi_{\mu\nu} \propto \varepsilon_{\mu\nu\alpha\parallel\beta}q^\beta, \quad (10)$$

where α_\parallel stands for either 0 or 3 (to be explained in detail in the next section). This evidently vanishes in the limit $q_0 = 0, \mathbf{q} \rightarrow 0$. Therefore, in a magnetised medium, the non-zero contribution to the effective charge of the neutrinos come solely from $\Pi_{\mu\nu}^5(q)$. In section-IV we shall discuss, from a more general point of view, why the effective charge of a neutrino, in a magnetised medium, comes only from $\Pi_{\mu\nu}^5(q)$ to linear order in \mathcal{B} . Therefore, the effective charge of the neutrinos is given by:

$$e_{\text{eff}} = -\frac{1}{2k_0}\frac{G_F}{\sqrt{2}e}g_A\Pi_{\mu 0}^5(q_0 = 0, \mathbf{q} \rightarrow 0) \times \text{tr}\{\gamma^\mu(1 + \gamma_5)(1 + \lambda\gamma^5)\not{k}\}. \quad (11)$$

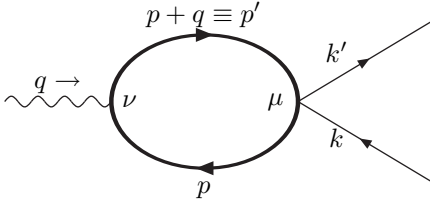


FIG. 1. One-loop diagram for the effective electromagnetic vertex of the neutrino in the limit of infinitely heavy W and Z masses.

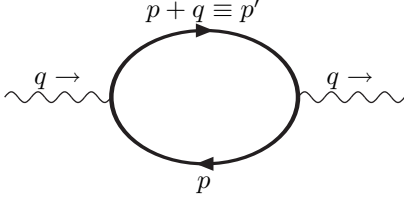


FIG. 2. One-loop diagram for the polarisation tensor.

III. CALCULATION OF THE 1-LOOP DIAGRAM

A. The Propagator

Since we investigate the case of a purely magnetic field, it can be taken in the z -direction without any further loss of generality. We denote the magnitude of this field by \mathcal{B} . Ignoring at first the presence of the medium, the electron propagator in such a field can be written down following Schwinger's approach [18–20]:

$$iS_B^V(p) = \int_0^\infty ds e^{\Phi(p,s)} C(p,s), \quad (12)$$

where Φ and C are defined below. To write these in a compact notation, we decompose the metric tensor into two parts:

$$g_{\mu\nu} = g_{\mu\nu}^{\parallel} + g_{\mu\nu}^{\perp}, \quad (13)$$

where

$$\begin{aligned} g_{\mu\nu}^{\parallel} &= \text{diag}(1, 0, 0, -1) \\ g_{\mu\nu}^{\perp} &= \text{diag}(0, -1, -1, 0). \end{aligned} \quad (14)$$

This allows us to use the following definitions,

$$p_{\parallel}^2 = p_0^2 - p_3^2 \quad (15)$$

$$p_{\perp}^2 = p_1^2 + p_2^2. \quad (16)$$

Using these notations we can write:

$$\Phi(p,s) \equiv is \left(p_{\parallel}^2 - \frac{\tan(e\mathcal{B}s)}{e\mathcal{B}s} p_{\perp}^2 - m^2 \right) - \epsilon|s|, \quad (17)$$

$$\begin{aligned} C(p,s) &\equiv \frac{e^{ie\mathcal{B}s\sigma_z}}{\cos(e\mathcal{B}s)} \left(\not{p}_{\parallel} + \frac{e^{-ie\mathcal{B}s\sigma_z}}{\cos(e\mathcal{B}s)} \not{p}_{\perp} + m \right) \\ &= \left[(1 + i\sigma_z \tan e\mathcal{B}s)(\not{p}_{\parallel} + m) + (\sec^2 e\mathcal{B}s)\not{p}_{\perp} \right], \end{aligned} \quad (18)$$

where

$$\sigma_z = i\gamma_1\gamma_2 = -\gamma_0\gamma_3\gamma_5, \quad (19)$$

and we have used,

$$e^{ie\mathcal{B}s\sigma_z} = \cos e\mathcal{B}s + i\sigma_z \sin e\mathcal{B}s. \quad (20)$$

Of course in the range of integration indicated in Eq. (12) s is never negative and hence $|s|$ equals s . It should be mentioned here that we follow the notation adopted in paper-I to ensure continuity. In the presence of a background medium, the above propagator is modified to [21]:

$$iS(p) = iS_B^V(p) + S_B^\eta(p), \quad (21)$$

where

$$S_B^\eta(p) \equiv -\eta_F(p) \left[iS_B^V(p) - i\overline{S}_B^V(p) \right], \quad (22)$$

and

$$\overline{S}_B^V(p) \equiv \gamma_0 S_B^{V\dagger}(p) \gamma_0, \quad (23)$$

for a fermion propagator, such that

$$S_B^\eta(p) = -\eta_F(p) \int_{-\infty}^{\infty} ds e^{\Phi(p,s)} C(p,s). \quad (24)$$

And $\eta_F(p)$ contains the distribution function for the fermions and the anti-fermions:

$$\begin{aligned} \eta_F(p) &= \Theta(p \cdot u) f_F(p, \mu, \beta) \\ &\quad + \Theta(-p \cdot u) f_F(-p, -\mu, \beta). \end{aligned} \quad (25)$$

Here, f_F denotes the Fermi-Dirac distribution function:

$$f_F(p, \mu, \beta) = \frac{1}{e^{\beta(p \cdot u - \mu)} + 1}, \quad (26)$$

and Θ is the step function given by:

$$\begin{aligned} \Theta(x) &= 1, \text{ for } x > 0, \\ &= 0, \text{ for } x < 0. \end{aligned}$$

B. Identifying the Relevant Terms

The amplitude of the 1-loop diagram of fig. 1 can be written as:

$$i\Pi_{\mu\nu}^5(q, \beta) = - \int \frac{d^4 p}{(2\pi)^4} (ie)^2 \text{tr} [\gamma_\mu \gamma_5 iS(p) \gamma_\nu iS(p')], \quad (27)$$

where, for the sake of notational simplicity, we have used

$$p' = p + q. \quad (28)$$

The minus sign on the right side is for a closed fermion loop and $S(p)$ is the propagator given by Eq. (21). This implies:

$$\Pi_{\mu\nu}^5(q, \beta) = -ie^2 \int \frac{d^4p}{(2\pi)^4} \text{tr} [\gamma_\mu \gamma_5 iS(p) \gamma_\nu iS(p')] . \quad (29)$$

Now using Eq.(24) we have the terms containing the effects of medium and the external field (non-absorptive)

$$\begin{aligned} \Pi_{\mu\nu}^5(q, \beta) = & -ie^2 \int \frac{d^4p}{(2\pi)^4} \text{tr} \left[\gamma_\mu \gamma_5 S_B^\eta(p) \gamma_\nu iS_B^V(p') \right. \\ & \left. + \gamma_\mu \gamma_5 iS_B^V(p) \gamma_\nu S_B^\eta(p') \right] . \quad (30) \end{aligned}$$

Substituting p by $-p'$ in the second term and using the cyclic property of traces, we can write Eq. (30) as

$$\begin{aligned} \Pi_{\mu\nu}^5(q, \beta) = & -ie^2 \int \frac{d^4p}{(2\pi)^4} \text{tr} \left[\gamma_\mu \gamma_5 S_B^\eta(p) \gamma_\nu iS_B^V(p') \right. \\ & \left. + \gamma_\nu S_B^\eta(-p) \gamma_\mu \gamma_5 iS_B^V(-p') \right] . \quad (31) \end{aligned}$$

Using now the form of the propagators from Eqs.(12) and (24), we obtain

$$\begin{aligned} \Pi_{\mu\nu}^5(q, \beta) = & ie^2 \int \frac{d^4p}{(2\pi)^4} \int_{-\infty}^{\infty} ds e^{\Phi(p,s)} \int_0^{\infty} ds' e^{\Phi(p',s')} \\ & \times \mathcal{G}_{\mu\nu}(p, p', s, s', B) , \quad (32) \end{aligned}$$

with,

$$\begin{aligned} \mathcal{G}_{\mu\nu} = & \eta_F(-p) \text{tr} \left[\gamma_\nu C(-p, s) \gamma_\mu \gamma_5 C(-p', s') \right] \\ & + \eta_F(p) \text{tr} \left[\gamma_\mu \gamma_5 C(p, s) \gamma_\nu C(p', s') \right] . \quad (33) \end{aligned}$$

It should be mentioned here that the effective charge of the neutrinos come from the dispersive part of the axial polarisation tensor. Therefore, we work with the real part of the 11-component of the axial polarisation tensor throughout and for notational simplicity suppress the 11-index everywhere.

C. Extracting the Gauge Invariant Piece

$\Pi_{\mu\nu}^5(k, \beta)$ **in odd powers of \mathcal{B}** - Notice that the phase factors appearing in Eq. (32) are even in \mathcal{B} . Thus, we need consider only the odd terms from the traces. Performing the traces the gauge invariant expression, odd in powers of \mathcal{B} , comes out to be:

$$\begin{aligned} \Pi_{\mu\nu}^5(q, \beta) = & -4e^2 \int \frac{d^4p}{(2\pi)^4} \eta_+(p) \int_{-\infty}^{\infty} ds e^{\Phi(p,s)} \\ & \times \int_{-\infty}^{\infty} ds' e^{\Phi(p',s')} R_{\mu\nu} ; \quad (34) \end{aligned}$$

where

$$\eta_+(p) = \eta_F(p) + \eta_F(-p) \quad (35)$$

and

$$\begin{aligned} \mathcal{R}_{\mu\nu} = & -\varepsilon_{\mu\nu 12} \frac{\sec^2 e\mathcal{B}s \tan^2 e\mathcal{B}s'}{\tan e\mathcal{B}s + \tan e\mathcal{B}s'} q_\perp^2 \\ & - \varepsilon_{\mu\nu 12} (q \cdot p) (\tan e\mathcal{B}s + \tan e\mathcal{B}s') \\ & + 2\varepsilon_{\mu 12 \alpha \parallel} (p'_\nu p^{\alpha \parallel} \tan e\mathcal{B}s + p_{\nu \parallel} p'^{\alpha \parallel} \tan e\mathcal{B}s') \\ & + g_{\mu\alpha \parallel} q_{\nu \perp} \tilde{p}^{\alpha \parallel} (\tan e\mathcal{B}s - \tan e\mathcal{B}s') \\ & - g_{\mu\alpha \parallel} q_{\nu \perp} \tilde{q}^{\alpha \parallel} \frac{\sec^2 e\mathcal{B}s \tan^2 e\mathcal{B}s'}{\tan e\mathcal{B}s + \tan e\mathcal{B}s'} \\ & + \left\{ g_{\mu\alpha \parallel} (p \cdot \tilde{q})_\parallel + g_{\nu\alpha \parallel} \tilde{p}^{\alpha \parallel} q_{\mu \perp} \right\} (\tan e\mathcal{B}s - \tan e\mathcal{B}s') \\ & + g_{\nu\alpha \parallel} \tilde{q}^{\alpha \parallel} p_{\mu \perp} \sec^2 e\mathcal{B}s \tan e\mathcal{B}s' . \quad (36) \end{aligned}$$

In writing this expression, we have used the notation $\tilde{p}^{\alpha \parallel}$, for example. This signifies that $\tilde{\alpha}_\parallel$ can take only the 'parallel' indices, i.e., 0 and 3, and is moreover different from the index α appearing elsewhere in the expression. We perform the calculations in the rest frame of the medium where $p \cdot u = p_0$. Thus the distribution function does not depend on the spatial components of p and is given simply by $\eta_+(p_0)$.

IV. TENSORIAL STRUCTURES OF $\Pi_{\mu\nu}$ AND $\Pi_{\mu\nu}^5$

It can be seen from Eq.(9) that, in general, the effective neutrino charge depends on $\Pi_{\mu\nu}(q)$ as well as on $\Pi_{\mu\nu}^5(q)$. Now, in vacuum we have,

$$\Pi_{\mu\nu}(q) = (g_{\mu\nu} q^2 - q_\mu q_\nu) \Pi(q^2) , \quad (37)$$

where $\Pi(k^2)$ vanishes for $q_0 = 0, \bar{q} \rightarrow 0$. The other contribution to the effective charge, coming from $\Pi_{\mu\nu}^5(k)$, also vanishes in vacuum. This can be understood from the general form factor analysis. We should be able to express $\Pi_{\mu\nu}^5(k)$, in vacuum, in terms of $g_{\mu\nu}, \epsilon_{\mu\nu\lambda\sigma}$ and q_λ . The parity structure of the theory forbids the appearance of $g_{\mu\nu}$. Therefore, the only possible combination, to obtain a second rank tensor is, $\epsilon_{\mu\nu\lambda\sigma} q_\lambda q_\sigma$. Since, this is identically zero, there can be no effective charge of a neutrino in vacuum.

On the other hand, in the presence of a medium the polarisation tensor can be expanded in terms of the form factors as follows [10]:

$$\Pi_{\mu\nu}(k) = \Pi_T T_{\mu\nu} + \Pi_L L_{\mu\nu} + \Pi_P P_{\mu\nu} , \quad (38)$$

where

$$T_{\mu\nu} = \tilde{g}_{\mu\nu} - L_{\mu\nu} \quad (39)$$

$$L_{\mu\nu} = \frac{\tilde{u}_\mu \tilde{u}_\nu}{\tilde{u}^2} \quad (40)$$

$$P_{\mu\nu} = \frac{i}{Q} \varepsilon_{\mu\nu\alpha\beta} q^\alpha u^\beta \quad (41)$$

and,

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} - \frac{q_\mu q_\nu}{k^2} \quad (42)$$

$$\tilde{u}_\mu = \tilde{g}_{\mu\rho} u^\rho \quad (43)$$

$$\mathcal{Q} = \sqrt{(q \cdot u)^2 - q^2} \quad (44)$$

in the rest frame of the medium where $u^\mu = (1, 0, 0, 0)$. It is easy to see that neither the longitudinal projection $L_{\mu\nu}$ or Π_L is non-zero in the limit $q_0 = 0, \bar{q} \rightarrow 0$. This then provides a non-zero contribution to the effective charge of the neutrino. And in [2] it has been shown that the non-zero contribution to the effective charge, in presence of a thermal medium, comes only from the $\Pi_{\mu\nu}(q)$ part. The tensor structure of $\Pi_{\mu\nu}^5$ in a medium is of the form $\varepsilon_{\mu\nu\alpha\beta} q^\alpha u^\beta$ and does not contribute to the zeroth component of Γ_ν . For a more physical understanding of the appearance of the effective charge of the neutrinos, in a medium, see [2].

Now, at the 1-loop level $\Pi_{\mu\nu}$ is invariant under charge conjugation, i.e., if we calculate the vacuum polarisation in a medium with a certain background field, it should be the same as that obtained in a charge-conjugated medium with an opposite background field. This means that, in the polarization tensor, the terms are either even in the background field or even in μ or odd in both. Therefore, in absence of a medium (which can be thought of as containing μ^0), the terms containing odd powers of the background field should vanish (see paper-I for a discussion). Therefore, the contribution to the effective charge of a neutrino, in presence of a background magnetic field in vacuum comes only from $\Pi_{\mu\nu}^5(q)$, to linear order in \mathcal{B} . Now, $\Pi_{\mu\nu}^5(q)$ in a magnetised vacuum is given by [9]:

$$\begin{aligned} \Pi_{\mu\nu}^5(q) &= \frac{i}{(4\pi)^2} \int_0^\infty ds \int_{-1}^1 dv / 2e^{-isx} \\ &\quad \left\{ (1-v^2) q_\parallel^2 e\tilde{F}q_\nu \right. \\ &\quad \left. + R \left(-q_\perp^2 e\tilde{F}^{\mu\nu} + q_\perp^\mu e\tilde{F}q_\nu + q_\perp^\nu e\tilde{F}q_\mu \right) \right\}, \quad (45) \end{aligned}$$

with,

$$\chi = m^2 + \frac{(1-v^2)}{4} q_\parallel^2 + \frac{\cos vz - \cos z}{2z \sin z} \quad (46)$$

$$R = \frac{1 - v \sin vz \sin z - \cos vz \cos z}{\sin^2 z}, \quad (47)$$

where, $z = eBs$ and $\tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\lambda\sigma} F_{\lambda\sigma}$ (note that the metric used in Eq. (45) is $g^{\mu\nu} = \text{diag}(-, +, +, +)$). It is evident that in the zero frequency and long wavelength limit $\Pi_{\mu\nu}^5$ vanishes resulting in zero effective charge of a neutrino in magnetised vacuum.

The tensorial form of $\Pi_{\mu\nu}$ has been discussed in detail in paper-I. It can be seen from that discussion that in a magnetised medium, the electromagnetic field always appears in the combination $u^\mu F_{\mu\nu}$ or $q^\mu F_{\mu\nu}$ in $\Pi_{\mu\nu}$, to linear order in the field strength. Since, we consider the case of a pure magnetic field and a stationary medium the

only terms that would survive would contain the combination $q^\mu F_{\mu\nu}$ (also borne out by the explicit calculations of paper-I). Hence, in the zero frequency, long wavelength limit it vanishes leaving only $\Pi_{\mu\nu}^5$ to contribute to the effective charge of the neutrino. Now, in a magnetized medium $\Pi_{\mu\nu}^5$ can be written, in terms of general basis vectors available, as follows:

$$\begin{aligned} \Pi_{\mu\nu}^5 &= \varepsilon_{\mu\nu 12} (q_\perp^2 f_1 + q_\parallel^2 f_2 + (q \cdot u) f_3) \\ &\quad + \varepsilon_{\mu 12 \alpha \parallel} (q_{\nu \parallel} u^{\alpha \parallel} f_3 + u_{\nu \parallel} u^{\alpha \parallel} f_4 + q^{\alpha \parallel} u_{\nu \parallel} f_5 \\ &\quad + q_{\nu \parallel} q^{\alpha \parallel} f_6 + u_{\nu \parallel} q^{\alpha \parallel} f_7 + q^{\alpha \parallel} q_{\nu \parallel} f_8) \\ &\quad + g_{\mu \alpha \parallel} q_{\nu \perp} (u^{\tilde{\alpha} \parallel} f_9 + q^{\tilde{\alpha} \parallel} f_{10}) \\ &\quad + g_{\mu \nu} q_{\alpha \parallel} q^{\tilde{\alpha} \parallel} f_{11} + g_{\mu \nu} u_{\alpha \parallel} q^{\tilde{\alpha} \parallel} f_{12} + g_{\nu \alpha \parallel} u^{\tilde{\alpha} \parallel} q_{\mu \perp} f_{13} \\ &\quad + g_{\nu \alpha \parallel} q^{\tilde{\alpha} \parallel} q_{\mu \perp} f_{14} + g_{\nu \alpha \parallel} q^{\tilde{\alpha} \parallel} q_{\mu \perp} f_{15}, \quad (48) \end{aligned}$$

where f_i s are the respective form factors. It can be easily seen that the terms proportional to the product of u 's are non-zero in the static long wavelength limit giving a finite contribution to the effective charge of a neutrino.

In this connection, it should be mentioned that the effective charge of the neutrinos bear a simple relation with the Debye screening length in the case of an unmagnetized plasma. As has been shown by [2] the contribution to the effective charge comes only from Π_L which corresponds to the Debye screening in the limit $q_0 = 0, \mathbf{q} \rightarrow 0$. In the case of a magnetised plasma $\Pi_{\mu\nu}^5$, in general, would have many more tensorial forms in it due to the presence of the electromagnetic tensor (paper-I0). Hence, there may not exist a simple correspondence between the Debye screening and the effective charge of a neutrino in a magnetised medium.

V. EFFECTIVE CHARGE OF A NEUTRINO

It is evident from Eq.(11) that to find the effective charge of the neutrino we need only to calculate $\Pi_{\mu 0}^5$ in the limit ($q_0 = 0, \mathbf{q} \rightarrow 0$). From Eq.(36) it can be seen that in this limit the only surviving terms in Π_{00}^5 are the ones containing odd powers of p_0 . Now, $\eta_+(p_0)$ is even in p_0 and so is the exponent in the zero frequency limit. Hence, p_0 integration makes Π_{00}^5 vanish. We also have,

$$\begin{aligned} R_{10} &= p_3 q_1 (\tan e\mathcal{B}s - \tan e\mathcal{B}s') \\ &\quad + q_3 p_1 \sec^2 e\mathcal{B}s \tan e\mathcal{B}s', \quad (49) \end{aligned}$$

$$\begin{aligned} R_{20} &= p_3 q_2 (\tan e\mathcal{B}s - \tan e\mathcal{B}s') \\ &\quad + q_3 p_2 \sec^2 e\mathcal{B}s \tan e\mathcal{B}s'. \quad (50) \end{aligned}$$

It can be seen that after p integration we shall have terms proportional to $q_3 q_1$ and $q_3 q_2$ in Π_{10}^5 and Π_{20}^5 respectively, as the integrals in p_1 and p_2 are Gaussian. In the limit $\mathbf{q} \rightarrow 0$ these terms would vanish. Therefore, in the relevant limit of vanishing external photon momenta only Π_{30}^5 , given by,

$$\begin{aligned}
& \Pi_{30}^5(q_0 = 0, \vec{q} \rightarrow 0) \\
&= \lim_{q_0=0, \vec{q} \rightarrow 0} 4e^2 \int \frac{d^4 p}{(2\pi)^4} \int_{-\infty}^{\infty} ds e^{\Phi(p,s)} \int_0^{\infty} ds' e^{\Phi(p',s')} \\
&\times \eta_+(p_0) ((q \cdot p)_{\parallel} - 2p_0^2) (\tan e\mathcal{B}s + \tan e\mathcal{B}s') \quad (51)
\end{aligned}$$

has a non-zero contribution to the effective charge. It can be seen that in the above expression, except for the exponents, the integrand is free of the perpendicular components of the momentum. Therefore, the perpendicular components of the loop momentum can be integrated out. Now, the exponential factors can be written as,

$$\Phi(p, s) + \Phi(p', s') = \Phi_{\parallel} + \Phi_{\perp}, \quad (52)$$

where,

$$\Phi_{\parallel} = is(p_{\parallel}^2 - m^2) + is'(p'_{\parallel}{}^2 - m^2) - \varepsilon|s| - \varepsilon|s'|, \quad (53)$$

$$\Phi_{\perp} = -\frac{i \tan e\mathcal{B}s}{e\mathcal{B}} p_{\perp}^2 - \frac{i \tan e\mathcal{B}s'}{e\mathcal{B}} p'_{\perp}{}^2. \quad (54)$$

Therefore, integration of the perpendicular part of the momentum gives us,

$$\begin{aligned}
& \int \frac{d^2 p_{\perp}}{(2\pi)^2} e^{\Phi_{\perp}} \\
&= \exp\left(-\frac{i}{e\mathcal{B}} \frac{\tan e\mathcal{B}s \tan e\mathcal{B}s'}{\tan e\mathcal{B}s + \tan e\mathcal{B}s'} k_{\perp}^2\right) \int \frac{d^2 p_{\perp}}{(2\pi)^2} \\
&\times \exp\left(-i \frac{\tan e\mathcal{B}s \tan e\mathcal{B}s'}{e\mathcal{B}} \left(p_{\perp} + \frac{\tan e\mathcal{B}s'}{\tan e\mathcal{B}s + \tan e\mathcal{B}s'} q_{\perp}\right)^2\right) \\
&= -\frac{1}{4\pi} \frac{i e\mathcal{B}}{\tan e\mathcal{B}s + \tan e\mathcal{B}s'}; \quad (55)
\end{aligned}$$

where we have neglected terms up-to $\mathcal{O}(q_{\perp}^2)$ since, to calculate the effective charge of the neutrinos we ultimately have to take the limit $\mathbf{k} \rightarrow 0$. Hence, eq.(51) can be written as

$$\begin{aligned}
& \Pi_{30}^5(q_0 = 0, \vec{q} \rightarrow 0) \\
&= -\lim_{q_0=0, \vec{q} \rightarrow 0} \frac{i e^3 \mathcal{B}}{\pi} \int \frac{d^2 p_{\parallel}}{(2\pi)^2} \eta_+(p_0) \int_{-\infty}^{\infty} ds e^{is(p_{\parallel}^2 - m^2) - \varepsilon|s|} \\
&\times \int_0^{\infty} ds' e^{is'(p'_{\parallel}{}^2 - m^2) - \varepsilon|s'|} ((q \cdot p)_{\parallel} - 2p_0^2) \\
&= \lim_{k_0=0, \vec{q} \rightarrow 0} 2e^3 \mathcal{B} \int \frac{d^2 p_{\parallel}}{(2\pi)^2} \eta_+(p_0) \frac{(p_{\parallel}^2 - m^2)}{(p'_{\parallel}{}^2 - m^2) + i\varepsilon} \\
&\times ((q \cdot p)_{\parallel} - 2p_0^2), \quad (56)
\end{aligned}$$

where we have used the following relation,

$$\begin{aligned}
& \int_{-\infty}^{\infty} ds e^{is(p_{\parallel}^2 - m^2) - \varepsilon|s|} \int_0^{\infty} ds' e^{is'(p'_{\parallel}{}^2 - m^2) - \varepsilon|s'|} \\
&= 2\pi i \frac{\delta(p_{\parallel}^2 - m^2)}{(p'_{\parallel}{}^2 - m^2) + i\varepsilon}. \quad (57)
\end{aligned}$$

The expression for Π_{30}^5 given by Eq.(56) contains two parts which can be integrated separately to obtain,

$$\begin{aligned}
& \lim_{q_0=0, \vec{q} \rightarrow 0} \int \frac{d^2 p_{\parallel}}{(2\pi)^2} (2\pi) \frac{(p_{\parallel}^2 - m^2)}{(p'_{\parallel}{}^2 - m^2) + i\varepsilon} \eta_+(p_0) (q \cdot p)_{\parallel} \\
&= \frac{1}{2} \int \frac{dp}{2\pi} \frac{\eta_+(E_p)}{E_p}, \quad (58)
\end{aligned}$$

and,

$$\begin{aligned}
& \lim_{q_0=0, \vec{q} \rightarrow 0} \int \frac{d^2 p_{\parallel}}{(2\pi)^2} (2\pi) p_0^2 \frac{(p_{\parallel}^2 - m^2)}{(p'_{\parallel}{}^2 - m^2) + i\varepsilon} \eta_+(p_0) \\
&= \frac{1}{4} \int \frac{dp}{2\pi} \left[\frac{\eta_+(E_p)}{E_p} - \beta \eta_+(E_p) \right], \quad (59)
\end{aligned}$$

where $E_p^2 = p^2 + m^2$. Therefore, we have,

$$\Pi_{30}^5(q_0 = 0, \vec{q} \rightarrow 0) = \frac{(e^3 \mathcal{B})}{2\pi} \beta \int \frac{dp}{2\pi} \eta_+(E_p). \quad (60)$$

Now it is evident that in the limit ($q_0 = 0, \mathbf{q} \rightarrow 0$) the dominant component of $\Pi_{\mu\nu}^5$ is Π_{30}^5 . Therefore, the index μ in Eq.(11) can only take the value 3. Incorporating this fact and after taking the trace, Eq.(11) takes the following form:

$$e_{\text{eff}} = -\frac{G_F}{\sqrt{2}e} g_A (1 - \lambda) \Pi_{30}^5(q_0 = 0, \mathbf{q} \rightarrow 0) \cos \theta, \quad (61)$$

where, θ is the angle between the magnetic field and the direction of the neutrino propagation. Therefore, in the limit of $m \geq \mu$ we obtain :

$$\begin{aligned}
e_{\text{eff}}^{\nu} &= \frac{1}{8\sqrt{2}\pi^3} e^2 G_F g_A (1 - \lambda) \mathcal{B} \cos \theta \\
&\times \sum_{n=0}^{\infty} (-1)^n (1 + n) \beta m K_1[(n + 1)\beta m] \\
&\times \cosh\{(n + 1)\beta \mu\}, \quad (62)
\end{aligned}$$

where K_n is the n-th order modified Bessel function of the second kind. It should be noted here that even though this result is linear in the field strength \mathcal{B} , this result is exact to all odd powers of \mathcal{B} . In the zero frequency, long wavelength limit it is only the term linear in \mathcal{B} that survives. To get a feeling of the magnitude of the effective charge of a neutrino in presence of a background magnetic field, we compare this with that in absence of a magnetic field. In the limit of vanishing chemical potential, the ratio between the effective charge in a magnetised medium and in a simple thermal medium turns out to be :

$$\frac{e_{\text{eff}}^{\nu}(\mathcal{B})}{e_{\text{eff}}^{\nu}(\mathcal{B} = 0)} = \frac{c_A}{4\pi^3} \left(\frac{\mathcal{B}}{\mathcal{B}_c}\right) (m\beta)^3 K_1(m\beta) \cos \theta \quad (63)$$

It can be seen from the equation above that the ratio is proportional to $\frac{\mathcal{B}}{\mathcal{B}_c}$. Since, in almost all astrophysical situations encountered so far, this ratio is less than one, in the weak field limit the charge due to the unmagnetized plasma is larger than in the case of a magnetised plasma.

VI. CONCLUSION

To conclude, we note that only left-handed neutrinos acquire an effective charge. Since we have performed our calculation for massless, standard-model neutrinos that should automatically come out of the theory. But recent observations indicate that the neutrinos have mass (which allows for both left-handed and right-handed neutrinos). Our treatment can be modified for massive neutrinos following the method adopted in [2] and we expect that the qualitative aspects of our result should remain the same.

More importantly, we notice that the presence of a magnetic field breaks the isotropy of space and introduces a preferential direction. As a consequence neutrinos propagating along the direction of the magnetic field acquire a positive charge whereas those propagating in the opposite direction acquire a negative charge. The net effect of this would then be the creation of a charge current along the direction of the field. Interestingly, neutrinos propagating in a direction perpendicular to the field would acquire zero effective charge. Whereas in an unmagnetized thermal medium neutrinos acquire an effective charge irrespective of their direction of propagation. Therefore for isotropic neutrino propagation no net current is generated in that case unlike in the presence of a magnetic field.

It is worth mentioning here that the generation of this charge current may play a significant role in the magnetic field generation of neutron stars being produced in supernova explosion. Another possible application could be to the neutrino wind driven instability responsible for blowing up the outer mantle of supernova as proposed by [6]. The basic mechanism of this instability can be understood as follows. Consider the collision of two plasma fluids. If we consider the motion of one plasma with respect to the centre of mass of the other then the dispersion relation of the particles of the first would depend on the relative velocity between the two plasma. Now this velocity dependent dispersion relation can give rise to damping or instability of the plasma modes depending on the relative velocity between the two media. The growth rate of the plasmons, in such systems, has been estimated using formalism of finite temperature field theory [22,23] as well as using plasma physics techniques [6]. It is important to note that the finite temperature field theory techniques show the damping/growth to be proportional to G_F^2 whereas that calculated using the plasma physics techniques show a scaling as G_F [6].

In conclusion, we have calculated the effective charge of neutrinos in a weakly magnetized plasma, in the limit $m > \mu$. It is observed that the neutrino charge acquires a direction dependence as a result of the presence of an external magnetic field and it is also proportional to the magnitude of the field strength present in the system.

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