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Quantum Stationary States in the Bianchi Universes

T. PADMANABHAN

Tata Institute of Fundamental Research,
Bombay 400 005, India

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Abstract

A path integral formulation is used in the superspace of geometries leading to a model of quantum gravity. It is shown that this method agrees with the standard path integral technique in the special case of Friedmann universes. A Schrödinger-like equation is set up, leading to the stationary states. Some solutions to these equations are presented.

§(1): *Introduction*

Path integrals provide a powerful way of attacking some of the problems in quantum gravity. One can make considerable progress by using this technique with selected degrees of freedom (for a detailed discussion on the logistics and methodology of this approach, which we shall not present here, see references 1-3).

Two major features emerge from this type of analysis: (i) It is seen that the quantum fluctuations of the conformal part of the geometry diverge at the classical singularity [2, 4], leading to transitions to other nonsingular geometries. (ii) Quantum geometries admit stationary state solutions which (under reasonably general conditions) lead to a lower bound on the physical length scale [5, 6]. In simple cases one can show that these stationary states provide stable configurations for the geometry and thus avoid the singularities [7].

The picture that arises from all these is reminiscent of early days of quantum theory of the hydrogen atom—an analogy stressed by Wheeler again and again [8, 9]. Classically, Einstein's equations predict a deterministic evolution for

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3-geometry leading, in certain cases, to singularities, just as classically an electron would spiral down in a hydrogen atom. Quantum mechanically, however, this cannot happen since it violates the uncertainty principle. This leads to the emergence of stationary states at which the electron (or the geometry of the space-time) will settle down.

The above interpretation, certainly, is very tentative. Considerably more work has to be done before any definite picture will emerge. We shall investigate the situation from a somewhat different angle in this paper.

The motivation for the present work arises from the following: When one uses path integrals one has to make sure that the degrees of freedom are the true independent degrees of freedom. It is also necessary to choose a physically meaningful time coordinate which can be used in the formulation. One would also like to treat all the degrees of freedom at equal footing, rather than to concentrate on few selected degrees of freedom. It is also necessary to show that the results are covariant and do not depend on the coordinates chosen.

One way of attacking these questions is to describe physics in a synchronous frame [5], where the metric has the form

$$ds^2 = -dt^2 + g_{ij}\, dx^i\, dx^j \tag{1}$$

This provides a physically meaningful time coordinate. Also the conformal part of the 3-geometry $g_{ij}(\mathrm{x}, t)$ is a true degree of freedom and is better suited for path integrals than the total conformal part [10]. This scale factor can be quantized in a variety of static background metrics (including the Friedmann model) leading to results quoted above. However, this requires suppression of other degrees of freedom and thus cannot be considered to be a complete quantization of the system. Also, the coordinate independence is not evident.

In this paper, we shall present an approach based on the superspace version of gravity (for an excellent discussion see [11]). Since a point in the superspace represents the geometry of the space-time, the quantum evolution of geometry is given by a path in the superspace. Using the metric in the superspace, one can write down an action integral in superspace, whose variation will lead to Einstein's equations in the space-time. The method now consists of using this superspace action in the Feynman path integral [rather than the normal space-time action $\int R(-g)^{1/2} d^4x$]. The classical limits coincide trivially, since the variational principle leads to Einstein's equations in both cases. We shall show that the quantum version based on the superspace leads to our previous results for the special cases we have considered earlier.

Since the superspace has a built-in coordinate-independent structure, the covariance of the theory is assured. Superspace also provides one with a natural choice of supertime parameter.

However, one problem remains. It is not easy to evaluate the superspace action for an arbitrarily complicated geometry. Since we do not want to suppress any degrees of freedom, it is necessary to impose additional symmetries on the system. The Bianchi-type universes provide exactly such a system. They admit a three-parameter group of motions and thus have only a manageable number of degrees of freedom.

The plan of the paper is as follows: In the next section we present the formalism based on the superspace. Section 3 shows the equivalence of this formalism with the conventional action integral used in earlier works. The stationary state equations are presented for various Bianchi universes in the next section with solutions in some cases. No detailed discussion of the solutions is attempted at this stage.

§(2): *Path Integral in Superspace*

The space-time metric in the synchronous frame is (essentially) determined by the six components of $g_{ij}(x)$ (with $i, j = 1, 2, 3$) (except for an arbitrary function determining the time coordinate). The superspace is essentially the space of all such 3-geometries, where two metrics are identified as one if they are related by a coordinate transformation. Thus a point in the superspace corresponds to a metric $g_{ij}(x)$ and all other metric functions derivable from this by coordinate transformations. One can in turn put a metric in the superspace and write the "distance" between two points in the superspace as

$$dS^2 = G_{ijkl}\, dg^{ij}\, dg^{kl} = G_{AB}\, dg^A\, dg^B \tag{1}$$

We shall use A, B, etc. to denote a pair of indices. One expects the coordinate transformation to be isometries of the superspace metric. Also one wants G_{AB} to be local in the sense that it should not pick up contributions from the space-time distance. This still does not fix G_{AB} uniquely. We shall use the metric suggested by Misner [see Reference 11, p. 451, equation (38)]:

$$G_{AB} = G_{(ij)(lm)} = \tfrac{1}{2}\left(g_{il}g_{jm} + g_{im}g_{jl} - 2g_{ij}g_{lm}\right) \tag{3}$$

With this metric, one can show that Einstein's equation can be written as a driven geodesic equation,

$$\frac{d^2 g^A}{d\lambda^2} + \Gamma^A{}_{BC}\,\frac{dg^B}{d\lambda}\,\frac{dg^C}{d\lambda} = \frac{1}{2}\,\mathcal{R}^{;A} \tag{4}$$

where $\mathcal{R} = g\,{}^3R$, in which g is the metric determinant and ${}^3R$ is the scalar curvature. [$\mathcal{R}$ has to be treated as a functional of the metric tensor and $\mathcal{R}^{;A} = G^{AB}(\partial\mathcal{R}/\partial g^B)$, etc.] Thus one can write an action integral in this superspace in the form

$$J_{\text{sup}} = \int G_{AB}\,\frac{dg^A}{d\lambda}\,\frac{dg^B}{d\lambda}\,d\lambda + \int \mathcal{R}\,d\lambda = \int \left(\frac{dS}{d\lambda}\right)^2 d\lambda + \int \mathcal{R}\,d\lambda \tag{5}$$

such that

$$\delta J_{\text{sup}} = 0 \Longrightarrow \text{equation (4)} \Longrightarrow \text{Einstein equations} \tag{6}$$

We shall now quantize this system, using path integrals. That is to say, we postulate that the probability amplitude for the system to make a transition from g_1^A to g_2^A is given by the functional integral,

$$K[g_1^A \lambda_1 ; g_2^A \lambda_2] = \int \mathcal{D} g^A(\lambda) \exp \frac{i}{h} J_{\text{sup}}[g^A(\lambda)] \tag{7}$$

Our problem, in principle, can be solved by evaluating K by equation (7). However, there are several operational difficulties in this procedure. It is extremely difficult to evaluate dS^2 (in terms of the true degrees of freedom of the geometry) for an arbitrary metric. Even if this is done, there is no guarantee that equation (7) can be exactly evaluated. Moreover, what we are interested in are the stationary states, rather than the propagation kernal.

All these suggest that one should write down the "Schrödinger equation" corresponding to equation (7). But before this can be done, one has to identify the true degrees of freedom and express J_{sup} in terms of those coordinates. We shall do this for a class of homogeneous space-times in the next section.

§(3): *Bianchi Universes and the Superspace Metric*

Consider the class of metrics which allow a 3-parameter group of isometries with the structure constants c^i_{jk}. We parametrize the metric to read

$$ds^2 = -dt^2 + R_0^2 e^{-2\Omega} (e^{2\beta})_{ij} \omega^i \omega^j \tag{8}$$

where the one-form ω^k satisfies the relation

$$d\omega^i = \tfrac{1}{2} c^i_{jk} \omega^j \Lambda \omega^k \tag{9}$$

Using the rotation matrix techniques one can write an arbitrary β matrix as

$$\beta = e^{-\psi k_3} e^{-\theta k_1} e^{-\theta k_3} \beta_d e^{\theta k_3} e^{\theta k_1} e^{\psi k_3} \tag{10}$$

where

$$k_3 = \begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}, \quad k_1 = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix} \tag{11}$$

$$\beta_d = \text{diag}\,(\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+) \tag{12}$$

One can show that this is the most general space-time which admits a 3-parameter group of motions. (For a detailed discussion see Reference 12.)

The line interval for this case (in the superspace) has the form

$$\begin{aligned} dS^2 = 24[-d\Omega^2 + d\beta_+^2 + d\beta_-^2 + \tfrac{1}{3}\sinh^2(2\sqrt{3}\beta_-)(\sigma_3)^2 \\ + \tfrac{1}{3}\sinh^2(3\beta_+ + \sqrt{3}\beta_-)(\sigma_2)^2 + \tfrac{1}{3}\sinh^2(3\beta_+ - \sqrt{3}\beta_-)(\sigma_1)^2] \end{aligned} \tag{13}$$

where $\{\sigma_i\}$ satisfies $d\sigma^i = c^i_{jk} \sigma^j \Lambda \sigma^k$. Even though one can tackle this general case, the physics is badly obscured by complicated mathematics. It is therefore preferable to consider the diagonal Bianchi models where the metric has the form

$$dS^2 = 24[-d\Omega^2 + d\beta_+^2 + d\beta_-^2] \tag{14}$$

The superspace action has the form

$$J_{\text{sup}} = \int \left(\frac{dS}{d\lambda}\right)^2 d\lambda + \int \mathcal{R}\, d\lambda = \int d\lambda [24(\dot{\beta}_4^2 + \dot{\beta}_-^2 - \dot{\Omega}^2) - R_0^4 e^{-4\Omega}(V - 1)] \tag{15}$$

where a dot denotes differentiation with respect to λ. We have used the fact that, for a diagonal Bianchi universe, $\mathcal{R}$ has the form

$$\mathcal{R} = gR = -R_0^4 e^{-4\Omega} [V(\beta_+, \beta_-) - 1] \tag{16}$$

where the form of the potential $V(\beta_+, \beta_-)$ depends on the Bianchi type chosen. Equation (15) is in a form suitable for quantization. One can uniquely write a Schrödinger equation corresponding to this form of action. [Since it has a $(T - V)$ structure, no ambiguities will arise.] The eigenvalue equation for the stationary states will read

$$\left(-\frac{\partial^2}{\partial \beta_+^2} - \frac{\partial^2}{\partial \beta_-^2} + \frac{\partial^2}{\partial \Omega^2}\right) \psi + 96 R_0^4 e^{-4\Omega} (V - 1)\, \psi = E\psi \tag{17}$$

Given the Bianchi type, V is fixed and this equation can be analyzed for stationary states.

At this stage one should make a distinction between the present approach and the conventional approaches to quantum gravity based on superspace. In the conventional approaches one identifies the Hamiltonian (either in the ADM form or in the superspace form) with the differential operator $i(\partial/\partial\Omega)$ and tries to solve the equations, respecting the constraints. We follow a different line of thought. The Bianchi diagonal model is described by Ω, β_+, and β_-. Thus the state of the geometry is described by a wave function of the form $\psi = \psi(\Omega, \beta_+, \beta_-)$ which satisfies the eigenvalue equation (17). In other words we simply use the superspace action as a tool to arrive at a manageable Lagrangian which can be quantized. [Incidentally, the standard approaches would lead to equation (17) with $E = 0$; this is certainly not true in our approach since we need a set of stationary states to construct the kernel $K[g_1^A ; g_2^A]$ of equation (7)].

However, it is essential that this method agrees with the previous results regarding the stationary states in the Robertson-Walker (RW) metric. This approach to quantization is different from the conventional superspace and ADM approaches in the sense that it predicts definite stationary states. But one expects it to lead to the same results as derived without using the superspace.

We shall now show that the superspace leads to the same equations as derived previously [5] when specialized to the RW metrics. This example will also illustrate some features of the affine parameter λ.

The Robertson-Walker metrics are special cases of Bianchi type I, V, and IX with the additional assumption of isotropy. This makes the superspace one-dimensional with the metric,

$$dS^2 = -24\, d\Omega^2 \tag{18}$$

(This is understandable since FRW metrics are all conformally flat and thus have only one degree of freedom.) The potential term has the form

$$\mathcal{R} = \tfrac{3}{2} R_0^4 k e^{-4\Omega}, \qquad \text{where} \quad k = 0, \pm 1 \tag{19}$$

The action in superspace thus has the form

$$J_{\text{sup}} = \int d\lambda [-24\dot{\Omega}^2 + \tfrac{3}{2} R_0^4 k e^{-4\Omega}] \tag{20}$$

In order to compare this with our previous result, notice the following: The *space-time* metric, using λ as the parameter, has the form

$$ds^2 = -e^{-6\Omega} d\lambda^2 + g_{ij} e^{-2\Omega} \sigma^i \sigma^i \tag{21}$$

[see Reference 11, p. 453, equation (45)], while in our previous paper [see Reference 5, equation (4)] we have used a parametrization

$$ds^2 = -dt^2 + a^2(t) g_{ij} \sigma^i \sigma^j \tag{22}$$

This suggests a substitution, $d\lambda = e^{3\Omega}\, dt$ in the independent variable, and $a = e^{-\Omega}$ in the dependent variable, leading to

$$J_{\text{sup}} = \int dt [-24 a\dot{a}^2 + \tfrac{3}{2} k R_0^4 a] \tag{23}$$

which is the form used in Reference 5 [see equation (6)] except for a trivial scale change. Thus the present approach differs from the previous one, only through the choice of the form for space-time metric [equations (22) or (21): equation (22) is more suited to space-time based approach, while equation (21) is useful in the superspace approach.]

The problem is now one of mathematics, where one has to solve the equation (17) for various Bianchi types. We shall present some special cases in the next section though no detailed or general discussion is attempted.

§(4): *Stationary States*

The simplest case, of course, is that of Bianchi type I. For this $V - 1 = 0$, and hence we are left with a free equation,

$$\left(-\frac{\partial^2}{\partial_+^2} - \frac{\partial^2}{\partial\beta_-^2} + \frac{\partial^2}{\partial\Omega^2}\right)\psi = E\psi \tag{24}$$

The general solution is of the form

$$\psi = A \exp(\bar{p}_+\beta_+ + \bar{p}_-\beta_- + \bar{q}\Omega), \qquad \text{with} \quad \bar{q}^2 - \bar{p}_+^2 - \bar{p}_-^2 = E \tag{25}$$

The metric contains these parameters in the form $\exp[-2(\Omega + \beta_+ + \sqrt{3}\beta_-)]$, $\exp[-2(\Omega + \beta_+ - \sqrt{3}\beta_-)]$, and $\exp[-2(\Omega - 2\beta_+)]$. Since we expect quantum fluctuation not to be large for large departures from identity, one expects $\bar{p}_+$ and $\bar{p}_-$ to be imaginary and $q^- < 0$ (real). We shall write it as (for $\Omega > 0$)

$$\psi_E = A \exp i(P_+\beta_+ + P_-\beta_-)\, e^{-q\Omega}, \qquad E = q^2 + P_+^2 + P_-^2 \tag{26}$$

This type of stationary states occur also in scale fluctuations of static metrics and is discussed in detail elsewhere.

The Bianchi type-II universe has a potential given by

$$V - 1 = Ae^{-8\beta_+} \tag{27}$$

leading to the equation

$$\left(-\frac{\partial^2}{\partial\beta_+^2} - \frac{\partial^2}{\partial\beta_-^2} + \frac{\partial^2}{\partial\Omega^2}\right)\psi + Ae^{-(4\Omega+8\beta_+)}\psi = E\psi \tag{28}$$

One can separate out the β_- dependence at once. Then a change of variables to

$$x = -4\Omega - 8\beta_+, \qquad y = -8\Omega - 4\beta_+ \tag{29}$$

gives an equation of the form

$$\frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial x^2} + A'e^x\psi = E'\psi \tag{30}$$

which has the solution

$$\psi(x, y) = e^{\alpha y} I_\nu(2\sqrt{A'}e^{x/2}), \qquad \nu^2 = \alpha^2 - \bar{E} \tag{31}$$

where I_ν is the modified Bessel function.

The Bianchi types V and VI have the potentials of the form

$$V - 1 = Be^{4\beta_+} \tag{32}$$

leading to the Schrödinger equation (after separating out the β_- dependence)

$$\left(\frac{\partial^2}{\partial\Omega^2} - \frac{\partial^2}{\partial\beta_+^2}\right)\phi + ABe^{-4\Omega+4\beta_+}\phi = E\phi \tag{33}$$

changing to the null coordinates,

$$x = 4(\beta_+ - \Omega), \quad y = 4(\beta_+ + \Omega) \tag{34}$$

the equation can be solved to give

$$\phi(x,y) = ce^{ip_+ y} \exp(x - \alpha e^x), \quad (c, \alpha, p_+ \text{ constants}) \tag{35}$$

Are entirely different type of metric, which lends itself to this kind of analysis, is the Kanatowski-Sachs universe (for a detailed discussion see [12]). The superspace line element has the form

$$dS^2 = [-d\Omega^2 + d\beta^2] \tag{36}$$

with

$$\mathcal{R} = Ae^{-4\Omega+2\beta} \tag{37}$$

As one can easily see, this leads to the equation

$$\left(\frac{\partial^2}{\partial\Omega^2} - \frac{\partial^2}{\partial\beta^2}\right)\psi + Ae^{(-4\Omega+2\beta)}\psi = E\psi \tag{38}$$

The transformation

$$x = -4\Omega + 2\beta, \quad y = -2\Omega + 4\beta \tag{39}$$

leads to

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\psi + Ae^x\psi = E\psi \tag{40}$$

with the solution

$$\psi(x,y) = e^{ip_- y} J_\nu(2\sqrt{A}e^{x/2}), \quad \nu^2 = E + p^2 \tag{41}$$

where J_ν is the Bessel function.

Other Bianchi types have potentials which are not separated, by these simple techniques. Special solutions, however, can be found for Bianchi types VII and IV in the $\beta_- = 0$ plane (as function of β_+ and Ω).

The detailed features of these solutions are under investigation.

§(5): *Conclusion*

To summarize we have demonstrated the following.

i. The variation principle in the superspace can be used in the path integral formalism. This leads to results which agree with the previous approach.

ii. This method leads to stationary states in all Bianchi universes which can be analyzed by this method. Thus stationary states are a general phenomenon.

iii. Explicit solutions are presented for some of the above cases.

Other features of this approach including the physical meaning of the constant E, are under investigation.

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