A New Class of LRS Bianchi Type-I Cosmological Models in Lyra Geometry

Anirudh Pradhan*
Department of Mathematics, Hindu Degree College
Zamania-232 331, Ghazipur, U.P., India
and
Anil Kumar Vishwakarma
Department of Mathematics
A.I. College, Mahuwabag, Ghazipur-233 001, U.P., India.

Abstract

LRS Bianchi type-I models have been studied in the cosmological theory based on Lyra's geometry. A new class of exact solutions has been obtained by considering a time dependent and constant displacement field for constant deceleration parameter models of the universe. The physical behaviour of the models is examined in vacuum and in the presence of perfect fluids.

KEYWORDS : Cosmology, Lyra Geometry, LRS Bianchi type-I models.

*Corresponding Author

1 Introduction

In 1917 Einstein introduced the cosmological constant into his field equations in order to obtain a static cosmological model since, as is well known, without the cosmological term his field equations admit only nonstatic solutions. After the discovery of the redshift of galaxies and its explanation as being due to the expansion of the universe, Einstein regretted his introduction of the cosmological constant. Recently, there has been much interest in the cosmological term in context of quantum field theories, quantum gravity, supergravity theories, Kaluza-Klein theories and the inflationary-universe scenario. Lyra [1] proposed a modification of Riemannian geometry by introducing a gauge function into the structureless manifold, as a result of which the cosmological constant arises naturally from the geometry. In consecutive investigations Sen [2], Sen and Dunn [3] proposed a new scalar-tensor theory of gravitation and constructed an analog of the Einstein field equations based on Lyra's geometry.

Halford [4] has pointed out that the constant vector displacement field ϕ_i in Lyra's geometry plays the role of cosmological constant Λ in the normal general relativistic treatment. It is shown by Halford [5] that the scalar-tensor treatment based on Lyra's geometry predicts the same effects, within observational limits, as the Einstein's theory. Several authors Bhamra [6], Karade and Borikar [7], Kalyanshetti and Wagmode [8], Reddy and Innaiah [9], Beesham [10], Reddy and Venkateswarlu [11], Soleng [12], have studied cosmological models based on Lyra's manifold with a constant displacement field vector. However, this restriction of the displacement field to be constant is merely one of convenience and there is no a priori reason for it. Beesham [13] considered FRW models with time dependent displacement field. He has shown that by assuming the energy density of the universe to be equal to its critical value, the models have the $k = -1$ geometry. Singh and Singh [14-17], Singh and Desikan [18] have studied Bianchi-type I, III, Kantowaski-Sachs and a new class of cosmological models with time dependent displacement field and have made a comparative study of Robertson-Walker models with constant deceleration parameter in Einstein's theory with cosmological term and in the cosmological theory based on Lyra's geometry. Soleng [12] has pointed out that the cosmologies based on Lyra's manifold with constant gauge vector ϕ will either include a creation field and be equal to Hoyle's creation field cosmology [19-21] or contain a special vacuum field which together with the gauge vector term may be considered as a cosmological term. In the latter case the solutions are equal to the general relativistic cosmologies with a cosmological term.

The present investigation is concern with LRS Bianchi type-I cosmological model with both cases, viz., time-dependent and constant displacement vectors based on Lyra's geometry in normal gauge. It may be pointed out that similar results can be obtained in several other theories as well as Einstein's theory minimally coupled to a massless scalar field and in Hoyle's creation field theory [19], if the creation field is assumed to be time dependent. Such investigations have not been undertaken in Hoyle's theory so far.

2 Field Equations

We consider LRS Bianchi type I space-time

$$ds^2 = dt^2 - A^2 dx^2 - B^2(dy^2 + dz^2) \tag{1}$$

where $A = A(x,t), B = B(x,t)$.
We take a perfect fluid form for the energy momentum tensor

$$T_{ij} = (p+\rho)u_i u_j - p g_{ij} \tag{2}$$

together with comoving coordinates $u^i u_i = 1$.

The field equations in normal gauge for Lyra's manifold, as obtained by Sen [2] are

$$R_{ij} - \frac{1}{2} g_{ij} R + \frac{3}{2}\phi_i \phi_j - \frac{3}{4} g_{ij}\phi_k \phi^k = -8\pi G T_{ij} \tag{3}$$

where ϕ is a displacement field vector defined by $\phi_i = (0,0,0,\beta(t))$ and other symbols have their usual meaning as in Riemannian geometry. We choose the coordinate system comoving with the matter $u^1 = u^2 = u^3 = 0, u^4 = 1$. Now the field equations can be set up and one obtains

$$\frac{2\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} - \frac{B'^2}{A^2 B^2} + \frac{3}{4}\beta^2 = -\chi p \tag{4}$$

$$\dot{B}' - \frac{B'\dot{A}}{A} = 0 \tag{5}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \frac{B''}{A^2B} + \frac{A'B'}{A^3B} + \frac{3}{4}\beta^2 = -\chi p \tag{6}$$

$$\frac{2B''}{A^2B} - \frac{2A'B'}{A^3B} + \frac{B'^2}{A^2B^2} - \frac{2\dot{A}\dot{B}}{AB} - \frac{\dot{B}^2}{B^2} + \frac{3}{4}\beta^2 = \chi\rho. \tag{7}$$

The energy conservation equation is

$$\chi\dot{\rho} + \frac{3}{2}\beta\dot{\beta} + [\chi(p+\rho) + \frac{3}{2}\beta^2](\frac{\dot{A}}{A} + \frac{2\dot{B}}{B}) = 0 \tag{8}$$

where $\chi = 8\pi G$. Here and in what follows a prime and a dot indicate partial differentiation with respect to x and t respectively. Equations (4) - (7) are four equations in five unknowns A, B, β, p and ρ. For complete determinacy of the system one extra condition is needed. One way is to use an equation of state. The other alternative is a mathematical assumption on the space-time and then to discuss the physical nature of the universe. In this paper we confine ourselves to assume an equation of state

$$p = \gamma\rho, \ 0 \leq \gamma \leq 1 \tag{9}$$

3 Solutions of the field equations

3.1 $\beta = \beta(t)$

Equation (5), after integration, yields

$$A = \frac{B'}{\sigma} \tag{10}$$

where σ is an arbitrary function of x

Equations (4) and (6), with the use of equation (10), reduces to

$$\frac{B}{B'}\left(\frac{\ddot{B}}{B}\right)' + \frac{\dot{B}}{B'}\frac{d}{dt}\left(\frac{B'}{B}\right) + \frac{\sigma^2}{B^2}\left(1 - \frac{B\sigma'}{B'\sigma}\right) = 0 \tag{11}$$

To get solution, let us assume

$$\frac{B'}{B} = \text{functions of } x \tag{12}$$

By this choice, equation (11) gives after integration

$$B = \sigma S(t), \tag{13}$$

where $S(t)$ is an arbitrary function of t.
With the help of equation (13), equation (10) becomes

$$A = \frac{\sigma'}{\sigma} S \tag{14}$$

Now the metric (1) takes the form

$$ds^2 = dt^2 - S^2(t)[dX^2 + e^{2X}(dy^2 + dz^2)], \tag{15}$$

where $X = \ln \sigma$.
Equations (4) and (7) give

$$\chi p = \frac{1}{S^2} - 2\frac{\ddot{S}}{S} - \frac{\dot{S}^2}{S^2} - \frac{3}{4}\beta^2 \tag{16}$$

$$\chi \rho = \frac{3\dot{S}^2}{S^2} - \frac{3}{S^2} - \frac{3}{4}\beta^2 \tag{17}$$

Using equation (9) and eliminating $\rho(t)$ from equations (16) and (17) we have

$$\frac{2\ddot{S}}{S} + (1 + 3\gamma)\frac{\dot{S}^2}{S^2} - (1 + 3\gamma)\frac{1}{S^2} + \frac{3}{4}(1 - \gamma)\beta^2 = 0 \tag{18}$$

Now the expressions for the energy density and the pressure are given by

$$\chi p = \chi \gamma \rho = \frac{4\gamma}{(1 - \gamma)} \left[\frac{\dot{S}^2}{S^2} - \frac{1}{S^2} + \frac{\ddot{S}}{2S}\right] \tag{19}$$

The function $S(t)$ remains undetermined. To obtain its explicit dependence on t, one may have to introduce additional assumptions. In the following we assume the deceleration parameter to be constant to achieve this objective i.e.

γ	β^2	ρ
0	$\frac{4}{3}[1/S^2-(1-2b)H^2]$	$\frac{2}{\chi}[(2-b)H^2-2/S^2]$
$\frac{1}{3}$	$4[1/S^2-(1-b)H^2]$	$\frac{3}{\chi}[(2-b)H^2-2/S^2]$

Table 1: Value of β^2 and ρ for dust and radiation power-low models.

$$q=-\frac{S\ddot{S}}{\dot{S}^2}=-\left(\frac{\dot{H}+H^2}{H^2}\right)=b(\text{constant}), \tag{20}$$

where $H=\dot{S}/S$ is the Hubble parameter.

This equation (20) is integrated to obtain

$$S(t)=\left[\begin{array}{ll}(D+Ct)^{\frac{1}{(1+b)}} & \text{when } b\neq -1\\ m_1e^{m_2t} & \text{when } b=-1\end{array}\right. \tag{21}$$

where C, D, m_1 and m_2 are constants of integration.
Using eq. (20) in eqs. (18) and (19) lead to

$$\beta^2=\frac{4}{3(1-\gamma)}\left[(1+3\gamma)\frac{1}{S^2}-(1+3\gamma-2b)H^2\right] \tag{22}$$

$$\chi\rho=\frac{1}{(1-\gamma)}\left[2(2-b)H^2-\frac{4}{S^2}\right] \tag{23}$$

The expressions for β^2 and ρ corresponding to $\gamma=0,\frac{1}{3}$ are summarized in the **Table 1**.

3.1.1 Flat Models

The condition for flat model is obtained as

$$\frac{1}{S^2}=(1-b)H^2 \tag{24}$$

Using equation (24), equations (22) and (23) reduce to

$$\beta^2=\frac{4(3\gamma-1)}{3(\gamma-1)}bH^2 \tag{25}$$

and

γ	β^2	ρ
0	$\frac{4}{3}bH^2$	$2bH^2$
$\frac{1}{3}$	0	$3bH^2$

Table 2: Values of β^2 and ρ for dust and radiation power-law models.

$$\chi\rho = -\frac{2bH^2}{(\gamma-1)} \tag{26}$$

From equation (26) we see that $\rho \geq 0$ if $1 > b > 0$ since $(\gamma - 1) < 0$. From eq. (25) we see that since $(\gamma - 1) < 0$, $\beta^2 > 0$ if $\gamma < 1/3$ and $\beta^2 < 0$ if $\gamma > 1/3$.

The expressions for β^2 and ρ corresponding to $\gamma = 0, 1/3$ are given in **Table 2**.

Here we observe that when $\gamma = 1/3, \beta^2 = 0$ and the equations reduce to those of LRS Bianchi type-I flat universe. For $b = -1$, the energy density always comes negative. So we only consider the case $b \neq -1$.

Case (i) : $b \neq -1$. For singular models, equation (21) leads to

$$S = mt^{\frac{1}{(1+b)}} \tag{27}$$

Using eq. (27) in eqs. (25) and (26) yield

$$\beta^2 = \frac{4(3\gamma-1)}{3(\gamma-1)}\frac{b}{(1+b)^2t^2} \tag{28}$$

and

$$\chi\rho = -\frac{2b}{(\gamma-1)(1+b)^2t^2} \tag{29}$$

The above expressions for β^2 and energy density $\rho(t)$ are similar to those obtained by Beesham [22] for a variable cosmological term $\wedge(t)$ and energy density $\rho(t)$. Here β^2 plays the role of a variable cosmological term $\wedge$.

Equations (9) and (29) give

$$\rho + 3p = \frac{-2(1+3\gamma)}{\chi(\gamma-1)}\frac{b}{(1+b)^2t^2} \tag{30}$$

γ	β^2	ρ
0	$\frac{4b}{3(1+b)^2t^2}$	$\frac{2b}{(1+b)^2t^2}$
$\frac{1}{3}$	0	$\frac{3b}{(1+b)^2t^2}$

Table 3: Values of β^2 and ρ for dust and radiation power-law models.

It can be seen from the above expression that the condition $\rho + 3p \geq 0$ would hold only for $1 + 3\gamma \geq 0$ i.e. $\gamma \geq -1/3$. So for values of $\gamma < -1/3$, we cannot have viable models.

We observe from (28) and (29) that β^2 and ρ fall off as $\frac{1}{t^2}$ irrespective of the equation of state. The expressions for β^2 and ρ corresponding to $\gamma = 0, \frac{1}{3}$ are given in **Table 3**.

It can be easily seen from equations (28) and (29) or otherwise also that the expressions for β^2 and ρ will not be valid for the empty universe (i.e. $p = \rho = 0$) and the stiff matter (i.e.$p = \rho$) models. So we shall not discuss these models.

3.1.2 Non-flat Models

Case (i) : $b \neq -1$. Using eq. (27) in eqs. (22) and (23) lead to

$$\beta^2 = \frac{4}{3(\gamma - 1)}\left[\frac{(1 + 3\gamma - 2b)}{(1 + b)^2 t^2} + \frac{(1 + 3\gamma)}{m^2 t^{2/(1+b)}}\right] \tag{31}$$

and

$$\chi\rho = \frac{2}{(\gamma - 1)}\left[\frac{(b - 2)}{(1 + b)^2 t^2} + \frac{2}{m^2 t^{2/(1+b)}}\right] \tag{32}$$

From eq. (32), we see that $\rho \geq 0$ when $-1 < b < 2$ as $(\gamma - 1) < 0$. From equation (31), we see that for $b < (1 + 3\gamma)/2, \beta^2 < 0$ for all times $t > 0$ as $(\gamma - 1) < 0$. It is also observed that for $(1 + 3\gamma)/2 < b \leq 2, \beta^2 > 0$ for

$$0 < t^{2b/(1+b)} < \frac{(2b - 1 - 3\gamma)m^2}{(1 + 3\gamma)(1 + b)^2} \tag{33}$$

and $\beta^2 < 0$ for

$$t^{2b/(1+b)} > \frac{(2b-1-3\gamma)m^2}{(1+3\gamma)(1+b)^2}. \tag{34}$$

At

$$t^{2b/(1+b)} = \frac{(2b-1-3\gamma)m^2}{(1+3\gamma)(1+b)^2}, \beta^2 = 0. \tag{35}$$

When $b = (1+3\gamma)/2$, equations (31) and (32) reduce to

$$\beta^2 = \frac{4(1+3\gamma)}{3(\gamma-1)m^2 t^{4/3(1+\gamma)}} \tag{36}$$

and

$$\chi\rho = \frac{1}{(1-\gamma)}\left[\frac{3(1-\gamma)}{(1+b)^2t^2} - \frac{4}{m^2 t^{4/3(1+\gamma)}}\right] \tag{37}$$

From equation (36), it is obvious that $\beta^2 < 0$ for all times as $(\gamma - 1) < 0$. The expressions for β^2 and ρ cannot be determined for the empty universe (i.e. $p = \rho = 0$) and stiff matter ($p = \rho$) models. The above expressions for ρ and β^2 are similar to those of ρ and β^2 given by equations (35)-(38) of reference Singh and Desikan [18].
Physical Behaviour of the Model: In the case of a non flat model when $b \neq -1$, the Ricci scalar becomes

$$R = \frac{1}{m^2 t^{2/(1+b)}} - \frac{(1-b)}{(1+b)}t \tag{38}$$

It is observed from equation (38) that when $t \to 0$ (i) $R \to \infty$ if $b = 0$, (ii) $R \to \infty$ if $b \geq 1$ and (iii) $R \to \infty$ if $b \leq -2$. The equation (38) also suggests that when $t \to \infty$ (i) $R \to 0$ if $b \geq 0$ and (ii) $R \to \infty$ if $b \leq -2$.

The scalars of expansion and shear are given by

$$\theta = \frac{3}{(1+b)t}, \sigma = 0 \tag{39}$$

The model has singularity at $t = 0$. At $t \to \infty$, the expansion ceases. The gauge function β was large in the beginning but decreases fast with the evolution of the model. Similar results can be obtained for Hoyle's creation field theory if the creation field is time dependent. Here $\sigma/\theta = 0$, which confirms

γ	β^2	ρ
0	$-\frac{4}{3}[3H_0^2 - e^{-2m_2t}/m_1^2]$	$2[3H_0^2 - 2e^{-2m_2t}/m_1^2$
$\frac{1}{3}$	$-4[2H_0^2 - e^{-2m_2t}/m_1^2]$	$3[3H_0^2 - 2e^{-2m_2t}/m_1^2]$

Table 4: Values of β^2 and ρ for dust and radiation exponential.

the isotropic nature of the space-time which we have already obtained in equation (15).

Case (ii) : $b = -1$. In this case, equation (20) becomes

$$\dot{H} = 0 \quad \text{and} \quad H = H_0 = \text{constant} \tag{40}$$

Using eq. (4o) in eqs. (22) and (23) we have

$$\beta^2 = \frac{4}{3(\gamma - 1)}\left[3(1+\gamma)H_0^2 - \frac{(1+3\gamma)}{m_1^2}e^{-2m_2t}\right] \tag{41}$$

$$\chi\rho = \frac{2}{(1-\gamma)}\left[3H_0^2 - \frac{2}{m_1^2}e^{-2m_2t}\right] \tag{42}$$

From equation (42), we observe that $\rho > 0$ only for $H_0^2 > 2/3m_1^2$ as $(1-\gamma) > 0$. $\beta^2 < 0$ for all times as can be seen from the equation (41) as $(\gamma - 1) < 0$. For large times i.e. $t \to \infty$ we see that β^2 and ρ would reach steady state i.e.

$$\beta^2 \to \frac{4}{(\gamma-1)}(1+\gamma)H_0^2 \quad \text{and} \quad \chi\rho \to \frac{6}{(1-\gamma)}H_0^2. \tag{43}$$

The strong energy condition, as mentioned by Ellis [23], $\rho + 3p > 0$ is also satisfied for $H_0^2 > 2/3m_1^2$.
The expressions for β^2 and ρ corresponding to $\gamma = 0, 1/3$ are given in **Table 4**.

3.1.3 Empty Universe

In the case of empty universe ($p = \rho = 0$) equation(42) reduces to

$$3H_0^2 - \frac{2}{m_1^2}e^{-2m_2t} = 0 \tag{44}$$

Using eq. (44) in eq. (41) leads to

$$\beta^2 = -2H_0^2 \tag{45}$$

For H_0 to be real, β must be imaginary.

Physical Behaviour of the Model. The Ricci scalar R is

$$R = 2H_0^2 - \frac{1}{m_1^2}e^{-2m_2 t} \tag{46}$$

It is easy to see that (i) when $t \to 0, R \to 2H_0^2 - 1/m_1^2$, and (ii) when $t \to \infty, R \to 2H_0^2$ when $m_2 > 0$ and $R \to \infty$ when $m_2 < 0$. The expansion and shear scalars are

$$\theta = 3H_0, \sigma = 0 \tag{47}$$

The model represents an uniform expansion as can be seen from eq.(47). The flow of the fluid is geodetic as the acceleration vector $f_i = (0,0,0,0)$.

3.2 β = constant

Let us suppose that displacement vector ϕ_i is a constant vector, i.e., gauge function β = const. In the normal gauge the solutions of the field equations for vacuum, dust and radiation can be written down in a straight-forward manner like general relativistic models with a cosmological constant.

3.2.1 Vaccum Model

Taking β as a constant in an empty universe ($p = \rho = 0$), we find that equation (17) reduces to

$$4\frac{\dot{S}^2}{S^2} - \frac{4}{S^2} - \beta^2 = 0 \tag{48}$$

The solutions to equation (48) are given as below:
(i) when $\beta^2 > 0$ i.e. β is real,

$$S = \frac{2}{\beta}\sinh\left(\frac{\beta t}{2}\right) \tag{49}$$

(ii) When $\beta^2 < 0$ i.e. β is pure imaginary.

$$S = \frac{2}{\alpha} \sin\left(\frac{\alpha t}{2}\right) \tag{50}$$

when we have set $\beta^2 = -\alpha^2$ and where, once again, we choose the constant of integration to be zero so that $S(0) = 0$.

(iii) When $\beta^2 = 0$, we obtain the Milne Universe.

These solutions are of the same type as obtained by Beesham [10].

4 Conclusion

In this paper we have obtained exact solutions of Sen equations in Lyra geometry for constant deceleration parameter. The nature of the displacement field β and the energy density ρ have been examined for both the (i) power-law and (ii) exponential expansion of both the flat universe and the non-flat universe.

Acknowledgements

The authers would like to thank Inter-University Centre for Astronomy and Astrophysics, Pune for providing a facility where this work was carried out.

References

[1] G. Lyra, Math. Z. **54**, 52 (1951).

[2] D. K. Sen, Z. Phys. **149**, 311 (1957).

[3] D. K. Sen and K. A. Dunn, J. Math. Phys. **12**, 578 (1971).

[4] W. D. Halford, Austr. J. Phys. **23**, 863 (1970).

[5] W. D. Halford, J.Math.Phys. **13**, 1399 (1972).

[6] K. S. Bhamra, Austr. J. Phys. **27**, 541 (1974).

[7] T. M. Karade and S. M. Borikar, Gen.Rel. Gravit. **9**, 431 (1978).

[8] S. B. Kalyanshetti and B. B. Waghmode, Gen.Rel. Gravit. **14**, 823 (1982).

[9] D. R. K. Reddy and P. Innaiah, Astrophys.Space Sci. **123**, 49 (1986).

[10] A.Beesham, Astrophys. Space Sci. **127**, 189 (1986).

[11] D. R. K.Reddy and R. Venkateswarlu, Astrophys. Space Sci. **136**, 191 (1987).

[12] H. H. Soleng, Gen.Rel.Gravit. **19**, 1213 (1987).

[13] A. Beesham, Austr. J. Phys. **41**, 833 (1988).

[14] T. Singh and G. P. Singh, J.Math.Phys. **32**, 2456 (1991a).

[15] T. Singh and G. P. Singh, Il. Nuovo Cimento **B106**, 617 (1991b).

[16] T. Singh and G. P. Singh, Int.J.Theor.Phys. **31**, 1433 (1992).

[17] T. Singh, and G. P. Singh, Fortschr. Phys. **41**, 737 (1993).

[18] G. P. Singh and K. Desikan, Pramana, **49**, 205 (1997).

[19] F. Hoyle, Monthly Notices Roy. Astron. Soc. **108**, 252 (1948).

[20] F. Hoyle and J. V. Narlikar, Proc. Roy. Soc. London Ser. A, **273**, 1 (1963).

[21] F. Hoyle and J. V. Narlikar, Proc.Roy.Soc. London Ser.A, **282**, 1 (1964).

[22] A. Beesham, Phys. Rev. **D48**, 3539 (1993).

[23] G. F. R. Ellis," in General Relativity and Cosmology", R.K.Sach (ed.), Academic Press, New York, p117 (1971).