A stationary vacuum solution dual to the Kerr solution

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Abstract

We present a stationary axially symmetric two parameter vacuum solution which could be considered as "dual" to the Kerr solution. It is obtained by removing the mass parameter from the function of the radial coordinate and introducing a dimensionless parameter in the function of the angle coordinate in the metric functions. It turns out that it is in fact the massless limit of the Kerr - NUT solution.

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The axially symmetric stationary spacetime hosts the most interesting two parameter Kerr family of vacuum black hole solution of the Einstein equation. It is well known that the family is unique under the assumptions of asymptotic flatness and of existence of regular smooth horizon. The Kerr solution has turned out to be the most interesting solution of the Einstein equation for the astrophysical applications, particularly in the context of high energy sources like quasars, pulsars, gamma ray bursters and active galactic nuclei.

The two parameters in the solution represent mass and specific angular momentum of the hole. When rotation is switched off, the solution reduces to the static Schwarzschild black hole. In the limit of vanishing mass it reduces to flat space. That is, the rotation by itself cannot be a source of gravity. This is however quite understandable in the Newtonian framework. The question is, should that always be the case in general relativity as well? In this paper, we wish to address this question and would like to present a solution in which mass parameter is replaced by a dimensionless parameter appropriately. The solution so obtained would in a particular sense be "dual" to the Kerr solution.

The form of the axially symmetric stationary metric we choose is motivated by the physical considerations of separability of the Hamilton - Jacobi equation for particle motion and the Klein - Gordon equation. That is, we seek the solution of the Einstein vacuum equation in the metric form in which these equations characterizing the physical features of the solution are solvable. For this, we follow the method employed by one of

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us (ZT) [1]. This would determine the form and character of the certain metric functions, a priori, and the metric is written in the following form [1],

$$ds^2 = \Lambda(dt + a\frac{(F-U^2)V^2-(G-V^2)U^2}{U^2-a^2V^2}d\varphi)^2 - \Lambda^{-1}[(U^2-a^2V^2)(\frac{dr^2}{U^2}+\frac{dS^2}{V^2}) + U^2V^2d\varphi^2] \quad (1)$$

where

$$\Lambda = \frac{U^2-a^2V^2}{F-a^2G}, \quad (2)$$

$$F = r^2 + a^2, \;\; G = 1 - S^2. \quad (3)$$

Further we have $U = U(r), V = V(S)$ where S is an angle coordinate and a is a constant having the dimension of length.

Now the Kerr solution is specified by

$$F - U^2 = 2Mr, \;\; G - V^2 = 0 \quad (4)$$

with M and a having the usual meaning of mass and specific angular momentum of the rotating black hole. On the other hand, the specification

$$F - U^2 = 0, \;\; G - V^2 = -2NS \quad (5)$$

also gives a vacuum solution and where the new parameter N is dimensionless. This characterization is a kind of "dual". In the former the radial part $F - U^2$, which is dimensionful, is specified through the dimensionful mass parameter and the angle part $G-V^2$ is vacuous while in the latter the radial part is vacuous and the angle part, which is dimensionless, is specified through a dimensionless parameter N. In this sense the above two characterizing relations are "dual" to each-other, and so should be the spacetimes they specify.

Clearly, it is this form of the metric which has suggested us the duality relation leading to the dual solution. It is an example of pure gravomagnetic spacetime because the primary source of the field is introduced through the angle function in eqn. (4). Note that what is the radial coordinate to gravoelectric and so is the angle coordinate to gravomagnetic spacetime. The axially symmetric vacuum spacetime metric in the proper (separability of H-J and K-G equations) form (1) is characterized by the two functions $U(r)$ and $V(S)$. The mass parameter enters through the radial function U while the dimensionless parameter through the angular function V.

Further, the dual character of the parameters M and N will become clearly visible in the expressions for the curvature components. For instance, let us write one particular component, say $R^{rS}{}_{rS}$ for the metric (1) explicitly,

$$R^{rS}{}_{rS} = \frac{Mr(r^2-3a^2S^2)}{(r^2+a^2S^2)^3} \quad (6)$$

for the Kerr solution specified by eqn. (3). This component would go over to the one for the specification of the dual solution (4) by the transformation, $M \to -aN, \; r \leftrightarrow aS$. This is a true dual transformation which takes the Kerr solution to the dual solution;i.e.

Riemann(Kerr) $\rightarrow$ Riemann(Dual). Under this transformation, the functions of r and S interchange their roles, $(F - U^2) \leftrightarrow a^2(G - V^2)$. Then eqns (3) and (4) are clearly the dual of each-other and so are the spacetimes described by them.

For the dual solution, the metric would take the explicit form,

$$
\begin{aligned}
ds^2 &= \frac{r^2 + a^2\cos^2\theta - a^2N^2\sin^2\theta}{r^2 + a^2S^2}(dt + \frac{2aNS(r^2+a^2)}{r^2 + a^2\cos^2\theta - a^2N^2\sin^2\theta}d\varphi)^2 \\
&- \frac{(1+N^2)(r^2+a^2)(r^2+a^2S^2)\sin^2\theta}{r^2 + a^2\cos^2\theta - a^2N^2\sin^2\theta}d\varphi^2 - (r^2+a^2S^2)(\frac{dr^2}{r^2+a^2} + d\theta^2) \qquad (7)
\end{aligned}
$$

where we have written $S = N + (1+N^2)^{1/2}\cos\theta$.

It reduces to flat space when $N = 0$ or $a = 0$. The spacetime is asymptotically non flat. It turns out that it is in fact the Kerr - NUT solution [2] with $M = 0$. That is, it is the massless limit of the Kerr - NUT solution. By writing $l = aN, a^2 = b^2 - l^2$ (and then replacing b by a) and redefining the coordinates t and φ appropriately, the above metric can be transformed to the Kerr - NUT form,

$$
\begin{aligned}
ds^2 &= \frac{A}{B}[dt - (a\sin^2\theta - 2l\cos\theta)d\varphi]^2 - \frac{B}{A}dr^2 \\
&- \frac{\sin^2\theta}{B}[(r^2+a^2+l^2)d\varphi - adt]^2 - Bd\theta^2 \qquad (8)
\end{aligned}
$$

where

$$
A = r^2 + a^2 - l^2, \ B = r^2 + (l + a\cos\theta)^2. \qquad (9)
$$

This is the Kerr - NUT solution [2] with $M = 0$ and l being the NUT parameter. The massless limit of the Kerr - NUT solution and the Kerr solution are thus dual of each-other. The NUT solution [3-7] is the prime example of the gravomagnetic field, in particular it can be interpreted as the field of a grvomagnetic monopole charge [4]. In our solution, though $M = 0$, yet the Kerr and NUT parameters do contribute an effective gravitational mass $(l^2 - a^2)/r$. This follows from writing $A/r^2 = 1 + 2\phi$, where ϕ is the Newtonian potential. The effective mass would be positive or negative depending upon $l^2 \gtrless a^2$ and consequently the field would be attractive or repulsive. That is, the NUT parameter produces attractive field while the Kerr parameter, as is well - known, produces repulsive.

The maim aim of this exercise was to expose this interesting duality relation between the two exact vacuum solutions. This duality is however different from the elctrogravity duality considered in Refs. [8,9]. The latter referred to duality between active and passive electric parts of the Riemann curvature, which was also the symmetry of the vacuum equation. Here we are instead referring to duality in the prescription of the radial and angular functions in the metric.

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