

LETTER TO THE EDITOR

A comparison between semiclassical gravity and semiclassical electrodynamics

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Abstract. It is known how the equation of motion for a quantum field in a classical curved spacetime can be derived as an approximation to the Wheeler–DeWitt equation. In order to obtain a better understanding of this derivation, we develop an analogous approximation for quantum electrodynamics. We show that quantum field theory in an external, classical electromagnetic field can be obtained as a limiting case of quantum electrodynamics, by expanding the full wavefunctional in a power series in the coupling constant e^2 . The important difference in the two derivations is that unlike the metric, the electromagnetic potential has to be scaled with respect to the coupling constant before the semiclassical limit can be obtained.

There is by now a broad consensus as to how one recovers a quantum field theory in a classical curved spacetime, starting from a quantum theory of gravity expressed in the form of the Wheeler–DeWitt equation. The Wheeler–DeWitt wavefunctional is expanded in a power series in the gravitational constant G , and the limit $G \rightarrow 0$ is taken. In physical terms this is the long-wavelength limit of quantum gravity. An important aspect of this derivation is the emergence of time evolution in the semiclassical limit. (Moreover, it has been demonstrated [1] that this definition of time in semiclassical gravity can be generalized to obtain a definition of time in quantum gravity.) Objections concerning the special nature of WKB wavefunctions, and the complex (as contrasted to real) nature of semiclassical solutions can be satisfactorily countered (for details see, e.g., [2, 3]).

The vanishing of the Hamiltonian constraint in general relativity and the consequent apparent ‘timelessness’ of the Wheeler–DeWitt equation (WD equation for short) has led to a major segregation of this ‘canonical’ approach from conventional studies of other interactions. One of the reasons for this segregation is that the quantum theory based on the WD equation uses the Schrödinger picture, which is not convenient for the usual perturbative approach to quantum field theories. (It is, however, convenient for certain non-perturbative features [4].) As a small step towards bridging this gap, we recast quantum electrodynamics (QED) in the Schrödinger picture, and derive the limit in which the electromagnetic field is classical, while the matter field coupled to it is quantum mechanical. The derivation is focused on bringing out the similarities to and differences from the corresponding derivation for the WD equation.

For quantum electrodynamics two different semiclassical limits can be meaningfully discussed: quantized radiation interacting with classical charges, and quantized matter interacting with an external, classical field. In contrast, we do not expect circumstances where classical matter could be interacting with quantum gravity. The difference can be traced to the equivalence principle and the vast gap between Planck length and atomic length scales. Because of the equivalence principle, the gravitational coupling constant G appears—relative to the gravitational action—as a multiplicative factor in front of the matter action. This provides G as a natural expansion parameter for obtaining the semiclassical limit, and since Compton wavelengths $\lambda = \hbar/m$ for known matter are much larger than the Planck length $L_P = \sqrt{G\hbar}$, a semiclassical limit with quantum matter and classical gravity is obtained. This limit is valid over length scales l such that $\lambda > l > L_P$. (We use units in which $c = 1$ throughout the letter).

On the other hand, QED does not obey an equivalence principle, does not have a fundamental length scale and the coupling constant e^2 does not appear as a multiplicative factor in front of the matter action. Thus e^2 as such is not a natural parameter for the semiclassical expansion (where the term 'semiclassical' is used in the sense defined above), neither is there a natural domain in length (or energy) separating some fundamental length from Compton wavelengths for matter. However, we show below that a field redefinition brings the QED action into the same universal form as the gravitational action, and then e^2 , like G , becomes the appropriate expansion parameter for obtaining the semiclassical limit.

Before going over to QED, we briefly recall the semiclassical expansion procedure for the WD equation. This equation reads

$$\left(-\frac{\hbar^2}{2M} G_{AB} \frac{\delta^2}{\delta h_A \delta h_B} + MV(h_A) + H_m(h_A, \phi) \right) \Psi(h_A, \phi) = 0. \quad (1)$$

Here, $M \equiv 1/32\pi G$. We have used a condensed notation and labelled the components h_{ij} of the 3-metric as h_A . The DeWitt metric G_{ijkl} on superspace is written as G_{AB} . The term $V(h_A)$ stands for $-2c^2\sqrt{h} {}^3R$, where h is the determinant of the 3-metric, and 3R is the Ricci-scalar on the 3-space. We have ignored the factor ordering ambiguity in the gravitational kinetic term as it does not affect our discussion. H_m is the Hamiltonian density of all matter fields, but for simplicity we assume that only a scalar matter field is present, and that H_m has the form

$$H_m = -\frac{\hbar^2}{2\sqrt{h}} \frac{\delta^2}{\delta \phi^2} + U(h_{AB}, \phi, \phi_{,a}). \quad (2)$$

We write the wavefunctional $\Psi[h_A(x), \phi(x)]$ as

$$\Psi = e^{iS/\hbar} \quad (3)$$

and expand S in the form

$$S = MS_0 + S_1 + M^{-1}S_2 + \dots \quad (4)$$

We now insert the expansion defined by (3) and (4) into (1) and compare terms at the same order in M . The highest order (M^2) yields

$$\left(\frac{\delta S_0}{\delta \phi} \right)^2 = 0. \quad (5)$$

Thus S_0 depends *only* on the 3-metric. The next order (M^1) yields the Hamilton–Jacobi equation for gravity alone

$$\frac{1}{2}G_{AB} \frac{\delta S_0}{\delta h_A} \frac{\delta S_0}{\delta h_B} + V(h_A) = 0. \quad (6)$$

It is well known that (6) is—together with the principle of constructive interference—equivalent to all ten (vacuum) Einstein field equations [5]. Once S_0 is given, every 3-geometry can be integrated to give a full four-dimensional solution of the field equations.

The next order (M^0) yields

$$G_{AB} \frac{\delta S_0}{\delta h_A} \frac{\delta S_1}{\delta h_B} - \frac{i\hbar}{2} G_{AB} \frac{\delta^2 S_0}{\delta h_A \delta h_B} + \frac{1}{2\sqrt{h}} \left(\frac{\delta S_1}{\delta \phi} \right)^2 - \frac{i\hbar}{2\sqrt{h}} \frac{\delta^2 S_1}{\delta \phi^2} + U(h_A, \phi, \phi_{,a}) = 0. \quad (7)$$

We now define a functional f according to

$$f \equiv D(h) e^{iS_1/\hbar} \quad (8)$$

and, using a condition on D , derive an equation for f . We choose $D(h_A)$ to satisfy the equation

$$G_{AB} \frac{\delta S_0}{\delta h_A} \frac{\delta D}{\delta h_B} - \frac{1}{2} G_{AB} \frac{\delta^2 S_0}{\delta h_A \delta h_B} D = 0. \quad (9)$$

Thus, D plays the role of a Van Vleck determinant. From (7), (8) and (9) it is then easy to show that f satisfies the functional equation

$$i\hbar G_{AB} \frac{\delta S_0}{\delta h_A} \frac{\delta f}{\delta h_B} \equiv i\hbar \frac{\delta f}{\delta \tau} = H_m f. \quad (10)$$

This is the functional Schrödinger equation for matter fields (the Tomonaga–Schwinger equation) propagating on a *fixed* curved background given by (6). The time τ in (10) (which is a ‘many-fingered time’) labels the ‘trajectories’ in superspace which run orthogonal to hypersurfaces $S_0 = \text{constant}$. This is a natural definition of time because the operator on the left-hand side of (9) is gravitational momentum times derivative with respect to the metric. It is usually called the WKB time [7].

To this order of approximation, the wavefunctional of the system is

$$\Psi = \frac{1}{D} e^{iMS_0/\hbar} f. \quad (11)$$

We will now demonstrate the analogous procedure for semiclassical electrodynamics.

For simplicity we consider the case of scalar electrodynamics, instead of fermionic electrodynamics, since additional care is required in constructing a Schrödinger picture for the latter. Thus the Lagrangian is

$$L = L_S(\phi, \phi^+) + L_{EM}(A^\mu) + L_{\text{int}}(\phi, \phi^+, A^\mu) \quad (12)$$

with

$$L_S(\phi, \phi^+) = (\partial_\mu \phi)(\partial^\mu \phi^+) - m^2 \phi^+ \phi \quad (13)$$

$$L_{EM}(A^\mu) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} (\mathbf{E}^2 - \mathbf{B}^2) \quad (14)$$

$$L_{int}(\phi, \phi^+, A^\mu) = -ieA^\mu(\phi^+ \partial_\mu \phi - \phi \partial_\mu \phi^+) + e^2 A^2 \phi^+ \phi. \quad (15)$$

The gauge $A^0 = 0$, $\nabla \cdot \mathbf{A} = 0$ is convenient while working in the Schrodinger picture, and we will make this choice. While one knows that this gauge is inconsistent in the presence of matter, it is allowed in the present case because our semiclassical expansion leads to source-free Maxwell equations (see equation (22) below). Since the full theory is gauge invariant, its semiclassical and classical limits will also respect this invariance.

The various canonical momenta are then given by

$$\Pi_\phi^+ = \partial^0 \phi \quad \Pi_\phi = \partial^0 \phi^+ \quad \Pi^A = \dot{A} = E \quad (16)$$

and the Hamiltonian is

$$H = H_{EM} + H_\phi \quad (17)$$

where

$$H_{EM} = \frac{1}{2} (\Pi^A)^2 + \frac{1}{2} (\nabla \times \mathbf{A})^2 \quad (18)$$

$$H_\phi = \Pi_\phi \Pi_\phi^+ + U(\phi, \mathbf{A}) \quad (19)$$

$$U(\phi, \mathbf{A}) = \partial_k \phi^+ \partial^k \phi + m^2 \phi^+ \phi + ieA^k (\phi^+ \partial_k \phi - \phi \partial_k \phi^+) - e^2 A^k A_k \phi^+ \phi. \quad (20)$$

Let Ψ be the wavefunctional of the system in the quantum field theory for this Hamiltonian. The evolution is then described by the functional Schrödinger equation

$$i\hbar \frac{\delta \Psi}{\delta t} = \left(-\frac{\hbar^2}{2} \frac{\delta^2}{\delta \mathbf{A}^2} + \frac{1}{2} (\nabla \times \mathbf{A})^2 - \hbar^2 \frac{\delta^2}{\delta \phi \delta \phi} + U(\phi, \mathbf{A}) \right) \Psi = \epsilon \Psi. \quad (21)$$

In (21) we first consider the case where $\Psi(\phi, \mathbf{A})$ describes a stationary state of energy ϵ , and later we will comment on the time-dependent case. In the WD equation, $\epsilon = 0$, so the appropriate analogue to consider in QED is a stationary state. Moreover, we may also set $\epsilon = 0$ by a shift in the phase of Ψ . The wavefunctional Ψ also obeys the quantized version of Gauss' law:

$$\partial_k \frac{\delta \Psi}{\delta A^k} = ie\phi \frac{\delta \Psi}{\delta \phi} - ie\phi^+ \frac{\delta \Psi}{\delta \phi^+}$$

which expresses the invariance of Ψ under simultaneous gauge transformations of \mathbf{A} , ϕ and ϕ^+ .

We are interested in developing a systematic approximation to (21), so as to arrive at the limiting case where \mathbf{A} is classical, whereas ϕ is a quantum field. Thus, starting from (21) we should be able to arrive at a classical equation of motion for \mathbf{A}

$$\ddot{\mathbf{A}} - \nabla^2 \mathbf{A} = 0 \quad (22)$$

and a quantum mechanical equation for ϕ :

$$i\hbar \frac{\partial f(\phi; \mathbf{A}(t))}{\partial t} = H_\phi f(\phi, \mathbf{A}) \quad (23)$$

where H_ϕ is the Hamiltonian of (19). The functional $f(\phi, \mathbf{A})$ will describe the evolution of the quantum field ϕ in the classical background \mathbf{A} obtained by solving (22). It is assumed that to the leading order of approximation, ϕ does not 'back-react' on \mathbf{A} , though \mathbf{A} could be driven by a classical source in (22).

If equations (22) and (23) are to be obtained from (21) by a power series expansion in the coupling constant e^2 , this cannot be done without scaling at least one of the configuration variables \mathbf{A}, ϕ with respect to e . This follows by noting that H_ϕ in (23) explicitly depends on e , and hence (23) cannot be obtained by equating the coefficients of a certain power of e to zero. In this regard, (23) should be contrasted with the corresponding equation (9) in the gravitational case: *The coupling constant G does not appear in the equation of motion for a quantum field in curved spacetime.* This in turn is because of the equivalence principle, which allows gravity to be interpreted as spacetime curvature.

Inspection shows that the scaled variable

$$\mathbf{Z} \equiv e\mathbf{A} \quad (24)$$

is the right one to work with, and ϕ need not be scaled. Equation (21) then reads

$$\epsilon \Psi(\phi, \mathbf{Z}) = \left(-\frac{\hbar^2}{2} e^2 \frac{\delta^2}{\delta \mathbf{Z}^2} + \frac{1}{2} \frac{(\nabla \times \mathbf{Z})^2}{e^2} - \hbar^2 \frac{\delta^2}{\delta \phi \delta \phi} + U(\phi, \mathbf{Z}) \right) \Psi. \quad (25)$$

Equation (25) is now of the same form as the WD equation (1), with \mathbf{Z} replacing the metric, and e^2 replacing G .

We can also do the scaling in the Lagrangian (12), which then is of the form

$$L = L_{\text{EM}}(\mathbf{Z}) + L_{\text{matter}}(\phi, \mathbf{Z}) \quad (26)$$

where now

$$L_{\text{EM}}(\mathbf{Z}) = \frac{1}{2e^2} (\dot{\mathbf{Z}}^2 - (\nabla \times \mathbf{Z})^2) \quad (27)$$

and

$$L_{\text{matter}}(\phi, \mathbf{Z}) = L_S(\phi) + L_{\text{int}}(\phi, \mathbf{Z}). \quad (28)$$

Clearly the scaled matter Lagrangian is now independent of e , and the coupling constant e^2 appears as an overall multiplicative factor relating L_{EM} and L_{matter} . The total Lagrangian now has the same form as the Lagrangian of general relativity. If we proceed with the quantization of the scaled Lagrangian (26), we will arrive at the scaled equation (25). Because of the similarity in the forms of equations (1) and (25), we can apply to the latter equation the semiclassical expansion procedure used earlier for the WD case [6], and briefly outlined above. We write the Ψ in (25) as $\Psi = \exp(iS/\hbar)$ and expand $S(\phi, \mathbf{Z})$ as a power series in e^2

$$S(\phi, \mathbf{Z}) = \frac{S_0}{e^2} + S_1 + e^2 S_2 + \dots \quad (29)$$

Substituting this expansion in (25), we consider the limit $e^2 \rightarrow 0$. This turns out to be equivalent to taking the classical limit $\hbar \rightarrow 0$ for the electromagnetic part of the system, while leaving the scalar field part quantum mechanical. At various orders, we get the following equations:

$$O(1/e^4): \quad \left(\frac{\delta S_0}{\delta \phi} \right) \left(\frac{\delta S_0}{\delta \phi} \right) = 0 \quad (30)$$

$$O(1/e^2): \quad \frac{1}{2} \frac{\delta S_0}{\delta Z_i} \frac{\delta S_0}{\delta Z^i} + \frac{1}{2} (\nabla \times \mathbf{Z})^2 = \epsilon (\equiv \epsilon_0/e^2) \quad (31)$$

$$O(e^0): \quad -\frac{i\hbar}{2} \frac{\delta^2 S_0}{\delta Z_i \delta Z^i} + \frac{\delta S_0}{\delta Z_i} \frac{\delta S_1}{\delta Z^i} - i\hbar \frac{\delta^2 S_1}{\delta \phi \delta \phi} + \frac{\delta S_1}{\delta \phi} \frac{\delta S_1}{\delta \phi} + U(\phi) = 0. \quad (32)$$

Equation (30) implies that S_0 does not depend on ϕ or ϕ . Because S_0 has to be gauge invariant (this follows from an expansion of the Gauss constraint), it can, however, not depend on ϕ or ϕ alone. Equation (31) is the Hamilton-Jacobi equation for \mathbf{Z} , and we have redefined ϵ , since we know that the leading order of the total energy is $(1/e^2)$. The momentum conjugate to \mathbf{A} is seen to be $(\delta S_0/\delta A_i)/e^2$, and is equal to the electric field \mathbf{E} , (from (16)). So (31) is also the expression for the energy of the electromagnetic field. Moreover, differentiation of (31) leads to (22) after the gauge condition $\nabla \cdot \mathbf{A} = 0$ is used.

Equation (32) can be simplified, as in the gravitational case, by defining

$$f(\phi, \mathbf{Z}) = D(\mathbf{Z}) e^{iS_1/\hbar} \quad (33)$$

and choosing D to satisfy the equation

$$\frac{\delta S_0}{\delta Z_i} \frac{\delta D}{\delta Z^i} - \frac{1}{2} \frac{\delta^2 S_0}{\delta Z_i \delta Z^i} D = 0. \quad (34)$$

This, like before, is the equation for the WKB prefactor. Equation (32) then becomes

$$i\hbar \frac{\delta S_0}{\delta Z_i} \frac{\delta f}{\delta Z^i} = H_\phi f. \quad (35)$$

From the defining equation (31) we know that $(\delta S_0/\delta Z_i)$ is equal to \dot{Z}^i , so that the operator $(\dot{Z}^i \delta/\delta Z^i)$ acting on f in (35) is same as $(\partial/\partial t)$. Equation (35) is thus of the same form as (10) and (23).

The wavefunctional at $O(1/e^2)$ is $\Psi = \exp(iS_0/e^2\hbar)$, and hence at this order \mathbf{A} can be interpreted as being classical. At the next order the wavefunctional is

$$\Psi(\mathbf{Z}, \phi) = \frac{1}{D} e^{iS_0/e^2\hbar} f(\phi, \mathbf{Z}) \quad (36)$$

and can be interpreted as describing the evolution of ϕ in the classical background \mathbf{A} .

The time evolution in (35) needs to be interpreted carefully. Even though we began with the time-independent equation (25), the dependence of f on \mathbf{A} in (35) implies that f is in general time dependent. If we begin with the time-dependent version of (25), then in (35) we will also have the term $(i\hbar \partial f/\partial t)$ on the left-hand side, in addition to the term which induces time evolution through \mathbf{A} . This may

appear strange but it simply shows that the net time dependence of f comes from two different components—one explicit (through the dependence of Ψ on time), and the other implicit (through the dependence of f on Z). Also, in this case the energy term on the right-hand side of the Hamilton–Jacobi equation (31) gets replaced by $(\delta S_0/\delta t)$. This implies that the classical field Z is coupled to an external system which is changing its energy. From the point of view of studying semiclassical electrodynamics, we are interested in a closed system consisting only of the fields ϕ and Z , which thus restricts us to start from a stationary state Ψ . The situation is different in the WD case, where the term with explicit time dependence is absent. This is of course because the WD equation is a constraint equation expressing the time-reparametrization invariance of quantum general relativity. Thus semiclassical gravity recovers the true time evolution of the world in a fundamental way, starting from a theory in which there is no apparent time evolution.

The above expansion may also allow us to discuss effects which are—from the point of view of the conventional perturbation theory—of a non-perturbative nature like the particle generation in strong electric fields (for which Z can be taken proportional to t). The reason for this lies in the scaling of the vector potential above. Equation (35) then leads to an explicitly time-dependent Hamiltonian for the matter field which is very similar, e.g., to the case of scalar fields on a de Sitter background spacetime [8].

The above scaling and derivation will also work for other forms of quantized matter coupled to A^μ , e.g., the Dirac field, or a charged particle. One may think that a different, suitable scaling and an expansion of the form (29) will recover also the limit where ϕ is classical, and A^μ is quantized. However, an examination of the equations shows that this cannot be done. Let us first write down the equations we expect in this semiclassical limit, which will be complimentary to the semiclassical equations (22) and (23). There will be the classical equation for the free ϕ field (no back-reaction from A^μ to leading order)

$$\partial^\mu \partial_\mu \phi + m^2 \phi = 0. \quad (37)$$

Then there will be the Schrödinger equation for A^μ coupled to the classical ϕ -current

$$i\hbar \frac{\delta\psi}{\delta t} = \left(-\frac{\hbar^2}{2m} \frac{\delta^2}{\delta A^2} + (\nabla \times \mathbf{A})^2 + ieA^k(\phi^+ \partial_k \phi - \phi \partial_k \phi^+) - e^2 A^2 \phi^+ \phi \right) \psi. \quad (38)$$

It is natural that the form of the expansion (29) be retained as such, without changing the order of the leading term, or the expansion parameter. If we have to recover (37) from (21) using this expansion, we are compelled to *scale* ϕ by defining a new variable $\eta = e\phi$. While this recovers (37) at $O(e^{-2})$ it creates a problem at the next order, $O(e^0)$. Equation (38), when written in terms of the scaled variable η , explicitly depends on e , because the current term is quadratic in ϕ , though linear in A^μ . So (38) will not be recovered at $O(e^0)$. A scaling of \mathbf{A} , in addition to that of ϕ , does not help.

Thus the problem lies with the current term, which while linear in e , is quadratic in ϕ . The situation is similar for the Dirac field. On the other hand, if A^μ is coupled to a charged particle, the current term is of the form $eA_\mu \dot{x}^\mu$, which is linear in the matter configuration variable x . Hence an equation of the form (38) is recovered by our expansion procedure, when A^μ is coupled to a charge. These

results are consistent with the fact that while there are physically relevant interactions of photons with classical charges [9] there are none which involve photons interacting with classical charged scalar fields. However, we do not know of a physical reason as to why the above derivation has this limitation. This issue is under investigation.

Thus the derivation of the semiclassical limit is analogous to that for the WD case, the difference being that, unlike gravitation, the variables in electrodynamics have first to be scaled. This is because all matter couples to gravity (equivalence principle) but not to electromagnetism. The scaling brings the Lagrangian (12) and (21) to a form which *pretends* that all matter couples to the field Z . Of course it is easy to see through the pretence. Consider the total low-energy Lagrangian of the world, written in the form

$$L = L_{\text{grav}}[g] + L_{\text{neu}}[\phi; g] + L_{\text{cha}}[\psi; A^\mu; g] + L_{\text{EM}}[A^\mu; g] + L_{\text{rest}} \quad (39)$$

where L_{grav} and L_{neu} are respectively the Lagrangians for the gravitational field and neutral matter, and L_{cha} and L_{EM} are the Lagrangians for the electromagnetic field and charged matter. While everything couples to gravity but not to electromagnetism, scaling of A^μ would wrongly suggest that uncharged matter is coupled to electromagnetism. Further, by measuring the strength of the gravitational field *produced* by the electromagnetic field, we can conclude that the true field is A^μ and not Z^μ . Thus the scaling of A^μ is a correct procedure only so long as we restrict ourselves to the Lagrangian for charged matter coupled to electrodynamics.

We end with a brief comment on related work currently in progress. It may be possible to derive the equation for the wavefunctional at the next order, $O(e^2)$, by first computing the equation for the function S_2 in (29). This gives rise to e^2 dependent corrections in the Schrödinger equation (35). The interpretation of these corrections, their validity, and their connection with conventional perturbation theory (if any), is presently under investigation. A preliminary analysis of the analogous gravitational corrections to quantum field theory in curved spacetime has been carried out in [3].

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