Spherical and non-spherical gravitational collapse in Husain spacetime

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Abstract

We investigate the nature of singularities arising in Husain solution. We analyze both spherical and non-spherical gravitational collapse in Husain spacetime. An interesting feature that emerges is that gravitational collapse of spherical cosmological Husain solution lead to the formation of naked singularities, while non-spherical cosmological collapse proceeds to form a black hole. Further strength of naked singularities arising in these spacetimes has been analyzed. It is found that these naked singularities are strong in Tipler's sense.

Key words: cosmic censorship, naked singularity, gravitational collapse.

PACS number(s): 04.20.Dw; 97.60.Lf.

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1. Introduction

It is now well established fact that general relativity admits solution with singularities, and such solutions can be produced by the gravitational collapse of non-singular initial data. Various models of gravitational collapse have been studied over the last few years. These models include collapse of dust [1] radiation [2], perfect fluid [3] etc. All these models have shown the existence of a black hole or a naked singularity. Penrose has proposed a conjecture called cosmic censorship conjecture [4], which states that singularities arising from physically reasonable initial data are not visible to any observer. However this conjecture has not been proven yet. Rather counter examples to this conjecture have been found. Vaidya solution [5] is one of the most important solution among these. Papapetrou [6] first showed that this solution can yield a naked singularity, since then this solution is being used to analyze the scenario of gravitational collapse in general relativity.

The focus of our investigation is the singularity that may possibly form during the collapse, at the center. According to the singularity theorem by Hawking and Penrose, the collapsing massive body which develops a singularity, need not necessarily lead to a black hole. The other possibility may be of a naked singularity. Naked singularities could be formed when the center of the collapsing star gets trapped before its boundary has entered the Schwarzschild radius [c.f. 7].

Anzhong Wang [8] has generalized the Vaidya solution which include most of the known solutions to the Einstein equation such as anti-de Sitter-Charged Vaidya solution, Husain solution. Husain solution of null fluid with $P = k\,\rho$ have been lately used as the formation of black hole with short hair [9]. Our aim in this paper is to discuss the nature of the singularities forming in Husain solution. We show that under certain conditions on mass function, strong curvature naked singularities exist in spherically symmetric Husain solution. Even though in the non-spherical cases, there are many examples of naked singularities, it would be interesting to see whether non-spherical Husain solution also admits a naked singularity.

The paper is organized as follows. In sec 2, we define the Husain solution in spherical symmetric spacetime and discuss about the nature of singularities arising in this spacetime. In Sec.3 we analyze the structure of singularities arising in non-spherical collapse (toroidal, cylindrical or planar symmetry). This is followed by concluding section 4.

2. Husain solution in spherically symmetric spacetime

Let us consider the spherically symmetric spacetime [8].

$$ds^2 = -\left[1 - \frac{2m(v,r)}{r}\right] dv^2 + 2\,dv\,dr + r^2 (d\theta^2 + \sin^2\theta\; d\phi^2)\,, \tag{1}$$

where v is a advanced Eddington time coordinate, r is the radial coordinate with $0 < r < \infty$, and $m(v, r)$ is the mass function giving gravitational mass inside the sphere of radial coordinate r.

Non-vanishing components of the Einstein tensor are:

$$G_0^0 = G_1^1 = -\frac{2m'}{r^2}\,, \qquad G_0^1 = \frac{2\dot{m}}{r^2}\,, \qquad G_2^2 = G_3^3 = \frac{-m''}{r}\,, \tag{2}$$

where $\{x^\mu\} = \{v, r, \theta, \phi\}$, ($\mu = 0, 1, 2, 3$), and

$$\dot{m} = \frac{\partial m}{\partial v}\,, \qquad m' = \frac{\partial m}{\partial r}\,.$$

Einstein field equations are

$$G_{\mu\nu} = \kappa T_{\mu\nu}\,, \tag{3}$$

where $G_{\mu\nu}$ is Einstein tensor, κ is a gravitational constant and $T_{\mu\nu}$ is energy momentum tensor given by

$$T_{\mu\nu} = T_{\mu\nu}^{(n)} + T_{\mu\nu}^{(m)}\,, \tag{4}$$

where

$$T_{\mu\nu}^{(n)} = \sigma\, l_\mu l_\nu\,,$$

$$T_{\mu\nu}^{(m)} = (\rho + P)(l_\mu \eta_\nu + l_\nu \eta_\mu) + P g_{\mu\nu}\,. \tag{5}$$

Using equations (3), (4), (5) we can write the expressions for σ, ρ and P as

$$\sigma = \frac{2\dot{m}}{\kappa r^2}\,, \qquad \rho = \frac{2m'}{\kappa r^2}\,, \qquad P = \frac{-m''}{\kappa r}\,. \tag{6}$$

Here ρ, P are energy density and pressure, while σ is the energy density of the Vaidya null radiation.

We have considered null vectors l_μ, η_μ such that

$$l_\mu = \delta^0_\mu, \qquad \eta_\mu = \frac{1}{2}\left[1-\frac{2m}{r}\right]\delta^0_\mu - \delta^1_\mu,$$

$$l_\lambda l^\lambda = \eta_\lambda \eta^\lambda = 0, \qquad l_\lambda \eta^\lambda = -1. \tag{7}$$

Energy conditions for such type of fluids are given by [8, 10]

I) The weak and strong energy conditions:

$$\sigma > 0,\ \rho \geq 0,\ P \geq 0,\ (\sigma \neq 0). \tag{8}$$

II) The dominant energy condition:

$$\sigma > 0,\ \rho \geq P \geq 0,\ (\sigma \neq 0). \tag{9}$$

With the proper choice of the mass function *m(v, r)*, above conditions can be satisfied.

By imposing the condition $P = k\rho$, Husain considered the mass function *m(v, r)* as [11]

$$m(v,r) = f(v) - \frac{g(v)}{(2k-1)r^{2k-1}}, \qquad k \neq ½$$

$$= f(v) + g(v)\ ln\ r, \qquad k = ½, \tag{10}$$

where *f(v)* and *g(v)* are arbitrary functions restricted by the energy conditions.

It can be observed that for $k = ½$, energy conditions are not always satisfied for all r [11], hence in the present work, we consider the first case only.

Using above mass function, we can find that

$$P = k\rho = \frac{2kg(v)}{kr^{2k+2}}, \tag{11}$$

and

$$\sigma = \frac{2}{kr^2}\left[f(v) - \frac{\dot{g}(v)}{(2k-1)r^{2k-1}}\right]. \tag{12}$$

$T_{\mu\nu}^{(n)}$ and $T_{\mu\nu}^{(m)}$ from equation (4) are related to the EMT of Vaidya null fluid and to the electromagnetic tensor respectively.

Following [8, 9] we can write $T_{\mu\nu}^{(m)}$ as

$$T_{\mu\nu}^{(m)} = \frac{2}{\kappa}\left(F_{\mu\lambda}F_{\nu}^{\lambda} - \frac{\alpha}{4} g_{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma} \right), \tag{13}$$

where $\alpha = \dfrac{2}{1+k}$, and $F_{\mu\nu}$ can be considered as electromagnetic field, given by

$$F_{\mu\nu} = \frac{[k(1+k)m'(v,r)]^{1/2}}{r}(\delta_{\mu}^{0}\delta_{\nu}^{1} - \delta_{\mu}^{1}\delta_{\nu}^{0}), \tag{14}$$

which satisfies the Maxwell field equations,

$$F_{[\mu\nu,\,\lambda]} = 0,\ F_{\mu\nu,\,\lambda}g^{\nu\lambda} = J_{\mu}. \tag{15}$$

It is clear from equations (11) and (12) that to satisfy weak and strong energy conditions, we must have $g(v) \geq 0$, and to ensure dominant energy condition we would expect $\dot{m}(v,r) > 0$. (This is discussed in detail in Ref. [8, 11].)

Substituting $m(v, r)$ from equation (10) into equation (1), we can write Husain metric as

$$ds^2 = -\left(1 - \frac{2f(v)}{r} + \frac{2g(v)}{(2k-1)r^{2k}}\right)dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{16}$$

It can be observed that above metric is asymptotically flat for $k > ½$ and cosmological for $k < ½$ [11].

Analysis of the structure of the singularity is initiated by a study of transverse radial null geodesic equation

$$\frac{dr}{dv} = \frac{1}{2}\left[1 - \frac{2f(v)}{r} + \frac{2g(v)}{(2k-1)r^{2k}}\right]. \tag{17}$$

In general, above equation does not yield analytic solution. However if $f \propto v$ and $g(v) \propto r^{2k}$, this equation becomes homogeneous and can be solved in terms of elementary functions [12]. In the next section we shall discuss the nature of singularities arising in both the types of solutions (i.e. in asymptotically flat as well as in cosmological).

2.1. Asymptotically flat solution

First we consider asymptotically flat solution (k > ½). In metric (13) if we choose $k = 1$, then it becomes charged Vaidya solution, this case we have already discussed in [13]. Here we consider the solution for ½ < k < 1. Let us take $k = \frac{2}{3}$.

In order to simplify the calculations we choose $2f = \lambda\, v$ and $g(v) = \mu\, v^{4/3}$ where λ, μ are positive constants. With this choice of the functions, metric (16) becomes

$$ds^2 = -\left(1 - \frac{\lambda v}{r} + \frac{6\mu v^{4/3}}{r^{4/3}}\right)dv^2 + 2dvdr + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \tag{18}$$

It can be observed that the metric (18) is self-similar admitting a homothetic killing vector ξ^a given by

$$\xi^a = v\frac{\partial}{\partial v} + r\frac{\partial}{\partial r}, \tag{19}$$

which satisfies

$$L_\xi g_{ab} = \xi_{a;b} + \xi_{b;a} = 2g_{ab}, \tag{20}$$

where L denotes the Lie-derivative.

The equation for outgoing radial null geodesics can be found by putting $ds^2 = 0$ in equation (18) as

$$\frac{dr}{dv} = \frac{1}{2}\left(1 - \frac{\lambda v}{r} + \frac{6\mu v^{4/3}}{r^{4/3}}\right). \tag{21}$$

It can be observed that the above differential equation has singularity at $r = 0$, $v = 0$.

To discuss the nature of this singularity, we analyze the outgoing radial null geodesics terminating at the singularity in the past.

We follow the technique described in Ref. [14].

Let

$$X_0 = \lim_{\substack{v\to 0\\ r\to 0}} X = \lim_{\substack{v\to 0\\ r\to 0}} \frac{v}{r}. \tag{22}$$

Hence equation (21) can be written as

$$X_0 = \lim_{\substack{v \to 0 \\ r \to 0}} \frac{dv}{dr} = \frac{2}{1 - \lambda X_0 + 6\mu X_0^{4/3}}. \tag{23}$$

i.e.

$$6X_0^{7/3} - \lambda X_0^{2} + X_0 - 2 = 0. \tag{24}$$

In order for the singularity at $v = 0$, $r = 0$ to be naked, radial null geodesics should be able to propagate outward, starting from the singularity. The variable X can be interpreted as the tangent to the outgoing geodesics, hence if equation (24) has atleast one positive and real root, then the singularity could be naked. If the equation (24) has no real and positive root, then the collapse ends in to a black hole.

If we substitute $X_0 = y^3$, then equation (24) becomes

$$6\mu y^7 - \lambda y^6 + y^3 - 2 = 0 \tag{25}$$

It can be checked from the *Theory of equations* that above equation has atleast three positive roots.

In particular, for $\lambda = 0.1$ and $\mu = 0.001$, one of the positive roots to the equation (25) is $y = 2.066909$. Using this value in $X_0 = y^3$, we get $X_0 = 8.830069$, which ensures that the singularity is naked.

In Tipler's sense [15], singularity is said to be strong curvature singularity if collapsing volume elements crushed to zero at the singularity.

According to Clarke and Krolak criteria [16], singularity is strong if

$$\psi = \lim_{k \to 0} k^2 R_{ab} K^a K^b > 0, \tag{26}$$

where K_a is tangent to the geodesic, R_{ab} is the Ricci tensor and k is an affine parameter.

In this case we find that

$$\psi = \frac{\lambda - 8\mu X_0^{1/3}}{\left(1 - \lambda X_0 + 6\mu X_0^{4/3}\right)^2}. \tag{27}$$

Thus along radial null geodesics strong curvature condition is satisfied if

$\lambda - 8\,\mu X_0^{1/3} > 0.$

Substituting $\lambda = 0.1$, $\mu = 0.001$ and $X_0 = 8.830069$ in equation (27) we find that $\lambda - 8\mu\, X_0^{\ 1/3} > 0$, which implies naked singularity arising in this case is of strong curvature type.

2. 2. Cosmological solution

Next we consider the collapse in cosmological solution ($k < ½$). We take $k = ¼$ and choose $2\, f(v) = \lambda v$ and $g(v) = \mu v^{½}$. With these choices, the metric in equation (16) becomes

$$ds^2 = -\left(1 - \frac{\lambda v}{r} - \frac{4\mu v^{1/2}}{r^{1/2}}\right) dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{28}$$

Equation of outgoing radial null geodesics, then can be written as

$$\frac{dv}{dr} = \frac{2}{1 - \frac{\lambda v}{r} - \frac{4\mu v^{1/2}}{r^{1/2}}}\,. \tag{29}$$

Which gives

$$X_0 = \lim_{\substack{v\to 0 \\ r\to 0}} \frac{dv}{dr} = \frac{2}{1 - \lambda X_0 - 4\mu X_0^{\ 1/2}}\,, \tag{30}$$

which implies

$$\lambda X_0^{\ 2} + 4\mu X_0^{\ 3/2} - X_0 + 2 = 0\,. \tag{31}$$

With the substitution $X_0 = y^2$, above equation becomes

$$\lambda y^4 + 4\mu y^3 - y^2 + 2 = 0. \tag{32}$$

One of the positive roots of the above equation for $\lambda = 0.1$ and $\mu = 0.001$ is $y = 2.3053675$, from which we obtain $X_0 = 5.3147193$.

Thus this solution also gives a naked singularity. Again the value of ψ, defined in equation (26) is found to be

$$\psi = \frac{\lambda + \frac{2\mu}{\sqrt{X_0}}}{(1 - \lambda X_0 - 4\mu X_0^{\ 1/2})^{1/2}} > 0\ , \tag{33}$$

which shows that the singularity is strong.

3. Husain solution in toroidal, cylindrical and planar spacetime

In this section, we generalize the non-spherical collapse in Ref. [17] to Husain solution. Following the work given in Ref. [8, 17, 18] , it can be seen that Einstein field equation (3) has also the solution

$$ds^2 = -\left[\alpha^2 r^2 - \frac{qm(v,r)}{r}\right]dv^2 + 2dvdr + r^2(d\theta^2 + d\phi^2) \ , \tag{34}$$

where we have taken $q\ m(v, r)$ as

$$qm(v,r) = qf(v) - \frac{qg(v)}{(2k-1)r^{2k-1}}, \qquad k \neq ½$$

$$= q\,f(v) + \ q\,g(v)\ ln\,r\ , \qquad k = ½. \tag{35}$$

v is the Eddigton advanced coordinate and $\alpha = \sqrt{-\frac{\Lambda}{3}}$.

σ in this case is given by

$$\sigma = \frac{q\dot{m}}{\kappa r^2} . \tag{36}$$

Coordinates θ, ϕ describe the two dimensional zero-curvature space generated by the two dimensional commutative Lie group G_2 of isometries [17]. Referring [17, 18], we write the topology of two dimensional space:

Topology of toroidal model is SxS, cylindrically symmetric model has topology RxS; while planar symmetrical model has RxR. Ranges for θ and ϕ in these models are

i) Toroidal: $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq 2\pi$.

ii) Cylindrical: $-\infty < \theta < \infty$, $0 < \phi < 2\pi$.

iii) Planer: $-\infty < \theta < \infty$, $-\infty < \phi < \infty$.

Depending upon the topology of the two dimensional space parameter q has different values. For torous model, $m(v, r)$ is mass and $q = 2/\pi$. For the cylindrical case, $m(v, r)$ is mass per unit length and $q = 4/\alpha$ and for planar symmetrical model, $m(v, r)$ is mass per unit area and $q = 2/\alpha^2$. The values of parameter q are taken from Arnowitt-Deser-Misner (ADM) masses of the corresponding static black holes [c.f. 17]. It is clear from equation (35) that the spacetime (34) is asymptotically anti-de-Sitter for $k > ½$ and it is cosmological for $k < ½$. We shall consider these cases one by one.

3.1. Asymptotically anti-de-Sitter solution

Let us take $k = 2/3$, and choose $q\, f(v) = \lambda v$, $q\, g(v) = \mu\, v^{4/3}$. Spacetime (34) then becomes

$$ds^2 = -\left[\alpha^2 r^2 - \frac{\lambda v}{r} + \frac{3\mu v^{4/3}}{r^{4/3}}\right] dv^2 + 2dvdr + r^2(d\theta^2 + d\phi^2)\,. \tag{37}$$

From above metric, the equation for outgoing radial null geodesics can be written as

$$\frac{dv}{dr} = \frac{2}{\alpha^2 r^2 - \frac{\lambda v}{r} + \frac{3\mu v^{4/3}}{r^{4/3}}}\,. \tag{38}$$

It can be easily checked that above differential equation has singularity at $v = 0$, $r = 0$. Let

$$X_0 = \lim_{\substack{v\to 0\\ r\to 0}} \frac{v}{r} = \lim_{\substack{v\to 0\\ r\to 0}} \frac{dv}{dr} = \frac{2}{-\lambda X_0 + 3\mu X_0^{\ 4/3}}\,. \tag{39}$$

i.e.

$$3\mu X_0^{\ 7/3} - \lambda X_0^{\ 2} - 2 = 0\,. \tag{40}$$

With the substitution $X_0^{1/3} = y$, above equation becomes

$$3\mu y^7 - \lambda y^6 - 2 = 0\,. \tag{41}$$

As there is a one sign change in the above algebric equation, it is clear by *theory of equations* that this equation must have atleast one positive and real root. Hence singularity arising in this case is naked. Again it can be checked by finding ψ defined in equation (26) that this singularity is strong curvature type.

3.2. Non-spherical cosmological solution

Choosing $k = ¼$, $q\, f(v) = \lambda\, v$, and $q\, g(v) = \mu\, v^{½}$ in equation (35), we write the metric (34) as

$$ds^2 = -\left[\alpha^2 r^2 - \frac{\lambda v}{r} - \frac{2\mu v^{1/2}}{r^{1/2}}\right]dv^2 + 2dvdr + r^2(d\theta^2 + d\phi^2)\,. \tag{42}$$

The equation for outgoing radial null geodesic then can be written as

$$\frac{dv}{dr} = \frac{2}{\alpha^2 r^2 - \frac{\lambda v}{r} - \frac{2\mu v^{1/2}}{r^{1/2}}}\,. \tag{43}$$

Let

$$X_0 = \lim_{\substack{v\to 0\\ r\to 0}} X = \lim_{\substack{v\to 0\\ r\to 0}} \frac{dv}{dr} = \frac{2}{-\lambda X_0 - 2\mu X_0^{\ 1/2}}\,. \tag{44}$$

i.e.

$$\lambda X_0^{\ 2} + 2\mu X_0^{\ 3/2} + 2 = 0 \tag{45}$$

As all the coefficients in equation (45) are positive, it can be argued by theory of equations that equation (45) cannot have positive real roots. In other words outgoing radial null geodesics having definite tangent at the singularity in the past are absent in this case, hence singularity is not naked. Thus collapse proceeds to form toroidal, cylindrical or planar black holes.

4. Conclusion

Cosmic censorship conjecture has become a challenging and most significant open problem in a general relativity. Despite of several attempts made by many researchers, this problem remains unproven till date. Keeping this fact in a mind we have studied Husain Solution analytically in detailed and found that this solution does admit naked singularities. Further it is found that these singularities are strong in Tipler sense. Generalizing the non-spherical collapse in Ref.[17] to Husain Solution, we have shown that spherical cosmological Husain Solution leads to a naked singularity while non-spherical cosmological collapse proceeds to form a toroidal, cylindrical or planar black hole. In other words, spherically symmetric cosmological Husain solution contradicts the cosmic censorship conjecture, while non-spherical cosmological Husain solution respects the C.C.C.

Acknowledgement: One of the authors (KDP) would like to thank IUCAA, Pune (India) for kind hospitality under Associateship program, where part of this work was done.

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