Cylindrically symmetric inhomogeneous cosmological models with viscous fluid and varying Λ

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Cylindrically symmetric non-static cosmological models representing a bulk viscous fluid distribution have been obtained which are inhomogeneous and anisotropic. Without assuming any *adhoc* law, we obtain a cosmological constant as a decreasing function of time. Various physical and geometrical features of the models are also discussed.

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1 Introduction

The so-called cosmological constant Λ has played numerous roles in the framework of general relativity. Introduced by Einstein [1] in his field equations, the part played by the Λ-term was to provide a repulsive counter to the gravitational attraction in a static universe. In de Sitter's model [2], Λ led to an expanding universe. Subsequently the discovery of Hubble [3] that the universe is expanding and the realization that the models developed by Friedman [4, 5] describe the expanding universe without recourse to the Λ-term led Einstein to discard that from his field equations.

In recent time the Λ-term has interested theoreticians and observers for various reasons. The nontrivial role of the vacuum in the early universe generates a Λ-term that leads to the inflationary phase. Observationally, this term provides an additional parameter to accommodate conflicting data on the values to the Hubble constant, the deceleration parameter, the density parameter and the age of the universe (Gunn and Tinsley [6], Wampler and Burke [7]). Assuming that Λ owes its origin to vacuum interactions, as suggested in particular by Sakharov [8], it follows that it would in general be a function of space and time coordinates, rather than being a strict constant. In a homogeneous universe Λ will be at most time dependent. Such a model was considered by Peebles and Ratra [9]. In considering the nature of local massive objects, however, the space dependence of Λ cannot be ignored (Narlikar *et al.* [10], Ray *et al.* [11]). Narlikar *et al.* [10] has studied

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the effects of the spatial variation of the cosmological constant Λ on the spacetime geometry within and outside a massive object.

A very important problem in cosmology is that of the cosmological constant, the present value of which is infinitesimally small ($\Lambda \leq 10^{-56}\,\mathrm{cm}^{-2}$). However, it is believed that the smallness of the value of Λ at the present epoch is because of the universe being so old [12]. This suggest that Λ cannot be a constant. It will rather be a variable, dependent on coordinates, either on space or on time or on both. Some of the recent discussions on the cosmological constant "problem" and on cosmology with a time-varying cosmological constant by Ratra and Peebles [13], Dolgov [14]–[16] and Sahni and Starobinsky [17] point out that in the absence of any interaction with matter or radiation, the cosmological constant remains a "constant", however, in the presence of interactions with matter or radiation, a solution of Einstein equations and the assumed equation of covariant conservation of stress-energy with a time-varying Λ can be found. For these solutions, conservation of energy requires decrease in the energy density of the vacuum component to be compensated by a corresponding increase in the energy density of matter or radiation. Earlier researchers on this topic, are contained in Zeldovich [18], Bertolami [19], Weinberg [20], Berman *et al.* [21] and Carroll, Press, and Turner [22]. The recent developments on this topic are reviewed by Peebles [23] and Padmanbhan [24].

Recent observations by Perlmutter *et al.* [25] and Riess *et al.* [26] strongly favour a significant and positive Λ. Their finding arise from the study of more than 50 type Ia supernovae with redshifts in the range $0.10 \leq z \leq 0.83$ and suggest Friedman models with negative pressure matter such as a cosmological constant, domain walls or cosmic strings (Vilenkin [27], Garnavich *et al.* [28]). Recently, Carmeli and Kuzmenko [25] have shown that the cosmological relativity theory (Behar and Carmeli [30]) predicts the value $\Lambda = 1.934 \times 10^{-35}\,\mathrm{s}^{-2}$ for the cosmological constant. This value of Λ is of the same order as mentioned previously. This value of Λ is in excellent agreement with the measurements recently obtained by the High-Z Supernova Team and Supernova Cosmological Project (Garnavich *et al.* [28], Perlmutter *et al.* [25], Riess *et al.* [26], Schmidt *et al.* [31]). The main conclusion of these works is that the expansion of the universe is accelerating.

Inhomogeneous cosmological models play an important role in understanding some essential features of the universe such as the formation of galaxies during the early stages of evolution and process of homogenization. The earlier attempts to the construction of such models have been done by Tolman [32] and Bondi [33] who considered spherically symmetric models. Inhomogeneous plane symmetric models were considered by Taub [34] and later by Tomimura [35], Szekeres [36], Collins and Szafron [37], Szafron, and Collins [38]. Recently, Senovilla [39] obtained a new class of exact solutions of Einstein's equation without big-bang singularity representing a cylindrically symmetric inhomogeneous cosmological model filled with perfect fluid which is smooth and regular everywhere satisfying energy and causality conditions. Later Ruis and Senovilla [40] have separated out a fairly large class of singularity-free models through a comprehensive study of general cylindrically symmetric metric with separable function of r and t as metric coefficients. Dadhich *et al.* [41] have established a link between the FRW model and the singularity free family by deducing the later through a natural and simple inhomogenization and anisotropization of the former. Recently, Patel *et al.* [42] presented a general class of inhomogeneous cosmological models filled with non-thermalized perfect fluid by assuming that the background spacetime admits two space-like commuting Killing vectors and has separable metric coefficients. Bali and Tyagi [43] obtained a plane symmetric inhomogeneous cosmological models of perfect fluid distribution with electromagnetic field. Recently, Pradhan *et al.* [44–46] have investigated a plane and a cylindrically symmetric inhomogeneous viscous fluid cosmological models with electromagnetic fields.

A realistic treatment of the problem requires the consideration of material distribution other than the perfect fluid. It is well known that at an earlier stage of the universe when the radiation in the form of photons as well as neutrinos decoupled from matter, it behaved like a viscous fluid. Misner [47] has studied the effect of viscosity on the evolution of cosmological models. A number of authors have discussed cosmological solutions with bulk viscosity in various context [48–55]. It is therefore considerable interest to study also inhomogeneous cosmological models for viscous fluid distributions.

Recently, Kilinc and Yavuz [56] obtained some solutions for cylindrically symmetric inhomogeneous cosmological models with viscous fluid. Motivated by the situations discussed above, in this paper we shall focus upon the problem of establishing a formalism for studying the general relativistic evolution in presence of bulk viscosity in an expanding universe. The viscosity coefficient of bulk viscous fluid is assumed to be a power function of mass density. *Without assuming any adhoc law*, we obtain a cosmological constant as a decreasing function of time which is supported by results from recent Ia supernovae observations. This paper is organized as follows. The metric and the field equations are presented in Section 2. In Section 3 we deal with the solution of the field equation in presence of bulk viscous fluid. The Subsections 3.1, 3.2, and 3.3 contain the three cases as $ma/(m+1)$ is greater than, equal to and less than zero respectively. Some physical aspects of these models are given in Section 4. Finally in Section 5 concluding remarks will be given.

2 Field equations

We consider the metric in the form

$$\mathrm{d}s^2 = A^2(\mathrm{d}x^2 - \mathrm{d}t^2) + B^2\mathrm{d}y^2 + C^2\mathrm{d}z^2\,, \tag{1}$$

where the metric potentials A, B, and C are functions of x and t both. The energy momentum tensor is taken as

$$T_{ik} = (p + \rho - \xi\theta)v_i v_k + (p - \xi\theta)g_{ik} - 2\eta\sigma_{ik}\,, \tag{2}$$

where ρ being the energy density, p the isotropic pressure, θ expansion of the fluid velocity congruence v^i, ξ bulk viscosity coefficient, η shear viscosity coefficient, σ_{ik}

shear tensor and v_i is the flow vector satisfying the relation

$$v^i v_i = -1\,. \tag{3}$$

Here the coordinates are considered to be comoving so that $v^1 = v^2 = v^3 = 0$ and $v^4 = 1/A$. The components of the Einstein's tensor from the field equations (in gravitational units $c = 1$, $G = 1$) read as

$$G_{ik} \equiv R_{ik} - \tfrac{1}{2} R g_{ik} = T_{ik} + \Lambda g_{ik}\,, \tag{4}$$

for the line element (1) has been set up as

$$\begin{aligned} G_{11} \equiv & -\frac{B_{44}}{B} - \frac{C_{44}}{C} + \frac{A_4}{A}\left(\frac{B_4}{B} + \frac{C_4}{C}\right) + \frac{A_1}{A}\left(\frac{B_1}{B} + \frac{C_1}{C}\right) + \frac{B_1 C_1}{BC} - \frac{B_4 C_4}{BC} \\ & = \frac{2A\eta}{3}\left(\frac{B_4}{B} + \frac{C_4}{C} - \frac{2A_4}{A}\right) + A^2(p - \xi\theta) + \Lambda A^2\,, \end{aligned} \tag{5}$$

$$G_{14} \equiv \frac{B_{14}}{B} + \frac{C_{14}}{C} - \frac{A_1}{A}\left(\frac{B_4}{B} + \frac{C_4}{C}\right) - \frac{A_4}{A}\left(\frac{B_1}{B} + \frac{C_1}{C}\right) = 0\,, \tag{6}$$

$$\begin{aligned} G_{22} \equiv & \frac{B^2}{A^2}\left[\frac{A_{11}}{A} + \frac{C_{11}}{C} - \frac{A_{44}}{A} - \frac{C_{44}}{C} - \left(\frac{A_1}{A}\right)^2 - \left(\frac{A_4}{A}\right)^2\right] \\ & = \frac{2B^2\eta}{3A^2}\left(\frac{A_4}{A} + \frac{C_4}{C} - \frac{2B_4}{B}\right) + B^2(p - \xi\theta) + \Lambda B^2\,, \end{aligned} \tag{7}$$

$$\begin{aligned} G_{33} \equiv & \frac{C^2}{A^2}\left[\frac{A_{11}}{A} + \frac{B_{11}}{B} - \frac{A_{44}}{A} - \frac{B_{44}}{B} - \left(\frac{A_1}{A}\right)^2 - \left(\frac{A_4}{A}\right)^2\right] \\ & = \frac{2C^2\eta}{3A^2}\left(\frac{A_4}{A} + \frac{B_4}{B} - \frac{2C_4}{C}\right) + C^2(p - \xi\theta) + \Lambda C^2\,, \end{aligned} \tag{8}$$

$$\begin{aligned} G_{44} \equiv & \frac{B_{11}}{B} + \frac{C_{11}}{C} - \frac{A_1}{A}\left(\frac{B_1}{B} + \frac{C_1}{C}\right) - \frac{A_4}{A}\left(\frac{B_4}{B} + \frac{C_4}{C}\right) + \frac{B_1 C_1}{BC} - \frac{B_4 C_4}{BC} \\ & = -\rho A^2 + \Lambda\,. \end{aligned} \tag{9}$$

The suffixes 1 ans 4 at the symbols A, B, and C denote differentiation with respect to x and t, respectively.

3 Solutions of the field equations

Equations (5)–(9) represent a system of five equations in eight unknowns A, B, C, ρ, p, ξ, η, and Λ. To get a determinate solution, we need three extra conditions. Let us consider that

$$\begin{aligned} A &= f(x)\lambda(t)\,, \\ B &= g(x)\mu(t)\,, \\ C &= g(x)\nu(t)\,, \end{aligned} \tag{10}$$

and

$$\frac{\lambda_4}{\lambda} = k \quad \text{(constant, say)}. \tag{11}$$

Using Eqs. (10) and (11) in Eqs. (5)–(9), we obtain

$$\frac{2f_1g_1}{fg} + \left(\frac{\mu_4}{\mu} + \frac{\nu_4}{\nu}\right)k + \left(\frac{g_1}{g}\right)^2 - \frac{\mu_4\nu_4}{\mu\nu} - \frac{\mu_{44}}{\mu} - \frac{\nu_{44}}{\nu}$$

$$= f^2\lambda^2(p-\xi\theta) + \frac{2}{3}f\lambda\eta\left(\frac{\mu_4}{\mu} + \frac{\nu_4}{\nu} - 2k\right) + \Lambda, \tag{12}$$

$$\frac{g_1\nu_4}{g\nu} - k\left(\frac{2g_1}{g}\right) + \frac{g_1\mu_4}{g\mu} - \frac{f_1}{f}\left(\frac{\mu_4}{\mu} + \frac{\nu_4}{\nu}\right) = 0, \tag{13}$$

$$\frac{g_{11}}{g} - \frac{\nu_{44}}{\nu} - \frac{\lambda_{44}}{\lambda} - k^2 + \frac{f_{11}}{f} - \left(\frac{f_1}{f}\right)^2$$

$$= \frac{2}{3}f\lambda\eta\left(\frac{\nu_4}{\nu} - \frac{2\mu_4}{\mu} + k\right) + f^2\lambda^2(p-\xi\theta) + \Lambda f^2\lambda^2, \tag{14}$$

$$\frac{g_{11}}{g} - \frac{\mu_{44}}{\mu} - \frac{\lambda_{44}}{\lambda} - k^2 + \frac{f_{11}}{f} - \left(\frac{f_1}{f}\right)^2$$

$$= -\frac{2}{3}f\lambda\eta\left(\frac{2\nu_4}{\nu} - \frac{2\mu_4}{\mu} + k\right) + f^2\lambda^2(p-\xi\theta) + \Lambda f^2\lambda^2, \tag{15}$$

$$\frac{2f_1g_1}{fg} + \left(\frac{\mu_4}{\mu} + \frac{\nu_4}{\nu}\right)k - \left(\frac{g_1}{g}\right)^2 + \frac{\mu_4\nu_4}{\mu\nu} - \frac{2g_{11}}{g} = f^2\lambda^2\rho - \Lambda. \tag{16}$$

From Eq. (13), we get

$$\frac{\frac{g_1}{g}}{\frac{f_1}{f}} = \frac{\frac{\mu_4}{\mu} + \frac{\nu_4}{\nu}}{\frac{\mu_4}{\mu} + \frac{\nu_4}{\nu} - 2k} = m \quad \text{(constant, say)}. \tag{17}$$

Equation (17) leads to

$$\frac{g_1}{g} = m\frac{f_1}{f}, \tag{18}$$

which on integration gives

$$g = \alpha f^m, \tag{19}$$

where α is constant of integration and

$$\frac{\mu_4}{\mu} = \frac{2mk}{m-1} - \frac{\nu_4}{\nu}. \tag{20}$$

From Eqs. (14) and (15), we get

$$\frac{\mu_{44}}{\mu} - \frac{\nu_{44}}{\nu} = 2\left(\frac{\mu_4}{\mu} - \frac{\nu_4}{\nu}\right)f\lambda\eta. \tag{21}$$

From Eqs. (12) and (14), we get

$$\frac{2f_1g_1}{fg}+\left(\frac{g_1}{g}\right)^2-\frac{g_{11}}{g}-\frac{f_{11}}{f}+\left(\frac{f_1}{f}\right)^2=-k\left(\frac{\mu_4}{\mu}+\frac{\nu_4}{\nu}\right)$$
$$+\frac{\mu_4\nu_4}{\mu\nu}+\frac{\mu_{44}}{\mu}-\frac{\lambda_{44}}{\lambda}+k^2+2f\lambda\eta\left(\frac{\mu_4}{\mu}-k\right). \tag{22}$$

Eliminating η between Eqs. (21) and the left-hand side of (22), we obtain

$$\frac{2f_1g_1}{fg}+\left(\frac{g_1}{g}\right)^2-\frac{g_{11}}{g}-\frac{f_{11}}{f}+\left(\frac{f_1}{f}\right)^2=-k\left(\frac{\mu_4}{\mu}+\frac{\nu_4}{\nu}\right)+\frac{\mu_4\nu_4}{\mu\nu}$$
$$+\frac{\mu_{44}}{\mu}-\frac{\lambda_{44}}{\lambda}+k^2+\left(\frac{\mu_4}{\mu}-k\right)\left[\frac{\frac{\mu_{44}}{\mu}-\frac{\nu_{44}}{\nu}}{\frac{\mu_4}{\mu}-\frac{\nu_4}{\nu}}\right]=a, \tag{23}$$

where a is constant. Using Eq. (18) in Eq. (23), we obtain

$$ff_{11}-\frac{3m+1}{m+1}f_1^2+\frac{a}{m+1}f^2=0. \tag{24}$$

Substituting $f(x)=u(x)^{-1/(2m)}$ in Eq. (24) reduces to

$$u_{11}-\frac{2ma}{(m+1)^2}u=0, \tag{25}$$

which on integration leads to

$$f(x)=\begin{cases}\left[c_1\mathrm{e}^{[\sqrt{2ma}/(m+1)]x}+c_2\mathrm{e}^{-[\sqrt{2ma}/(m+1)]x}\right]^{-1/(2m)} & \text{when } ma/(m+1)>0,\\ (c_1+c_2x)^{-1/(2m)} & \text{when } ma/(m+1)=0,\\ \left[c_1\cos\left(\frac{\sqrt{2ma}}{m+1}x\right)+c_2\sin\left(\frac{\sqrt{2ma}}{m+1}x\right)\right]^{-1/(2m)} & \text{when } ma/(m+1)<0,\end{cases} \tag{26}$$

where c_1 and c_2 are constants of integration. From Eqs. (19) and (26), we obtain

$$g(x)=\alpha\begin{cases}\left(c_1\mathrm{e}^{[\sqrt{2ma}/(m+1)]x}+c_2\mathrm{e}^{-[\sqrt{2ma}/(m+1)]x}\right)^{-1/2} & \text{when } ma/(m+1)>0,\\ (c_1+c_2x)^{-1/2} & \text{when } ma/(m+1)=0,\\ \left(c_1\cos\frac{\sqrt{2ma}}{m+1}x+c_2\sin\frac{\sqrt{2ma}}{m+1}x\right)^{-1/2} & \text{when } ma/(m+1)<0.\end{cases} \tag{27}$$

Using Eq. (20) in the right-hand side of Eq. (23), we have

$$\mu\mu_{44}-\mu_4^2+\frac{a(m-1)}{k}\mu\mu_4-ma\mu^2=0. \tag{28}$$

Substituting $\mu = \mathrm{e}^{z}$ in Eq. (28), we obtain

$$z_{11} + \frac{(m-1)}{k} a z_1 - ma = 0\,, \tag{29}$$

which on integration leads to

$$\mu = \mathrm{e}^{c_3 + c_4 \mathrm{e}^{[(1-m)/k]at} + [mk/(m-1)]t}\,, \tag{30}$$

and from Eq. (20) we obtain

$$\nu = \mathrm{e}^{-c_3 - c_4 \mathrm{e}^{[(1-m)/k]at} + [mk/(m-1)]t}\,, \tag{31}$$

where c_3 and c_4 are constants of integration. Here we consider three cases according to the values of $ma/(m+1)$. We discuss each case and its consequences separately below in this paper.

3.1 Case (1): $ma/(m+1) > 0$

In this case the geometry of the spacetime (1) reduces to the form

$$\mathrm{d}s^2 = \mathrm{e}^{2kt} M^{-1/m} \left[\mathrm{d}x^2 - \mathrm{d}t^2\right] + M^{-1}\left[\mathrm{e}^{L + [2mk/(m-1)]t}\mathrm{d}y^2 + \mathrm{e}^{-L + [2mk/(m-1)]t}\mathrm{d}z^2\right], \tag{32}$$

where

$$M = c_1 \mathrm{e}^{[\sqrt{2ma}/(m+1)]x} + c_2 \mathrm{e}^{-[\sqrt{2ma}/(m+1)]x}\,,$$

$$L = 2c_3 + 2c_4 \mathrm{e}^{[(1-m)a/k]t}\,.$$

The expression for shear viscosity η, density ρ and pressure p, for model (32) are given by

$$\eta = \frac{\dfrac{2mk}{1-m} - \dfrac{(1-m)a}{k}}{2\mathrm{e}^{kt} M^{-1/(2m)}}\,, \tag{33}$$

$$\rho = \frac{M^{1/m}}{\mathrm{e}^{2kt}}\left[\frac{m(3m-2)k^2}{(m-1)^2} + \frac{2ma}{(m+1)^2} + \frac{(2-7m)aN^2}{2(m+1)^2 M^2}\right.$$
$$\left. - \left\{\frac{(1-m)a}{k}\right\}^2 c_4^2 \mathrm{e}^{[2(1-m)a/k]t} + \Lambda\right], \tag{34}$$

$$p - \xi\theta = \frac{M^{1/m}}{\mathrm{e}^{2kt}}\left[\frac{2N^2}{(m+1)M^2} - \frac{1}{3}ma - \frac{10m^2k^2}{3(m-1)^2}\right.$$
$$\left. + \frac{m^2-2}{m+1} - 2\left\{\frac{(1-m)a}{k}\right\}^2 c_4^2 \mathrm{e}^{[2(1-m)a/k]t} - 2\Lambda\right], \tag{35}$$

where

$$N = c_1 \mathrm{e}^{[\sqrt{2ma}/(m+1)]x} - c_2 \mathrm{e}^{-[\sqrt{2ma}/(m+1)]x}\,.$$

Here θ is the scalar of expansion calculated for the flow vector v^i as

$$\theta = \frac{(3m-1)kM^{1/(2m)}}{(m-1)\mathrm{e}^{kt}} . \tag{36}$$

For the simplification of ξ, we assume that the fluid obeys an equation of state of the form

$$p = \gamma\rho , \tag{37}$$

where γ is a constant $(0 \le \gamma \le 1)$.

Thus, given $\xi(t)$ we can solve the cosmological parameters. In most of the investigations involving bulk viscosity is assumed to be a simple power function of the energy density [51–53]

$$\xi(t) = \xi_0\rho^n , \tag{38}$$

where ξ_0 and n are constants. For large value of ρ, n is quite small and Santos *et al.* [54] suggested to get more realistic models n lies in the regime $0 \le n \le \frac{1}{2}$. For small density, n may even be equal to unity as used in Murphy's work for simplicity [57]. Also if $n = 1$, Eq. (38) may correspond to a radiative fluid [58]. On using (38) in (35), we obtain

$$p - \xi_0\rho^n\theta = \frac{M^{1/m}}{\mathrm{e}^{2kt}}\left[\frac{2N^2}{(m+1)M^2} - \frac{1}{3}ma - \frac{10m^2k^2}{3(m-1)^2} \right.$$

$$\left. + \frac{m^2-2}{m+1} - 2\left\{\frac{(1-m)a}{k}\right\}^2 c_4^2\mathrm{e}^{[2(1-m)a/k]t} - 2\Lambda\right] . \tag{39}$$

For simplicity and realistic models for physical importance, we consider the following two cases:

3.1.1 Model I: Solutions for $\xi = \xi_0$

When $n = 0$, Eq. (38) reduces to $\xi = \xi_0$. With the use of Eqs. (34), (36), and (37), Eq. (39) reduces to

$$(2+\gamma)\rho = \frac{(3m-1)k\xi_0M^{1/(2m)}}{(m-1)\mathrm{e}^{kt}} + \frac{M^{1/m}}{\mathrm{e}^{2kt}}\left[\frac{N^2}{(m+1)M^2}\left\{2 + \frac{(2-7m)a}{(m+1)}\right\}\right.$$

$$\left. + \frac{4m(2m-3)k^2}{3(m-1)^2} + \frac{4ma}{(m+1)^2} + \frac{m^2-2}{m+1} - \frac{1}{3}ma - 4\left\{\frac{(1-m)a}{k}\right\}^2 c_4^2\mathrm{e}^{[2(1-m)a/k]t}\right] . \tag{40}$$

Eliminating $\rho(t)$ between Eqs. (34) and (40), we obtain

$$(2+\gamma)\Lambda = \frac{(3m-1)k\xi_0\mathrm{e}^{kt}}{(m-1)M^{1/(2m)}} + \frac{N^2}{(m+1)M^2}\left\{2 - \frac{(2-7m)a\gamma}{2(m+1)}\right\} + \frac{m^2-2}{m+1} - \frac{1}{3}ma -$$

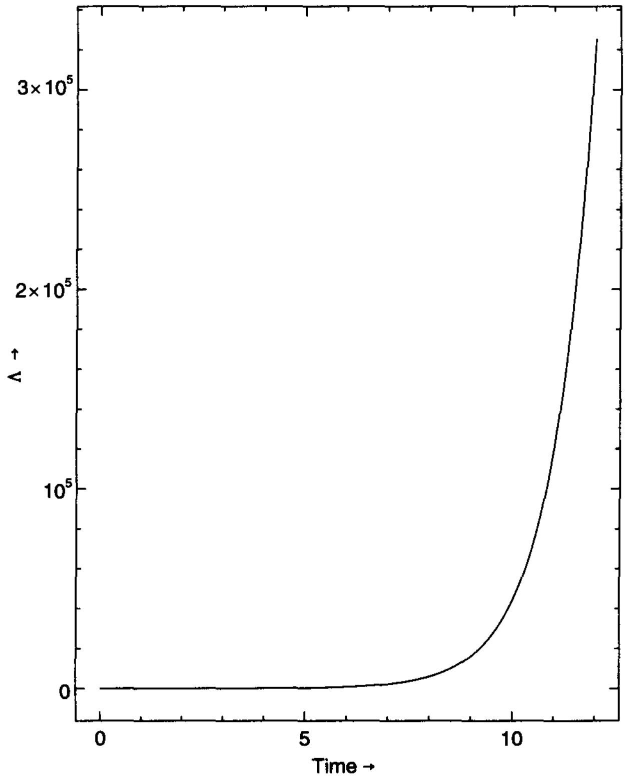

Fig. 1. The plot of cosmological constant Λ versus time τ for the model $\xi = \xi_0$ with parameters $\gamma = 0.5$, $m = 2.0$, $a = 2.0$, $k = 1$.

$$-\frac{mk^2}{3(m-1)^2}\{10m+3(3m-2)\gamma\}-\frac{2ma\gamma}{(m+1)^2}+(\gamma-2)\left\{\frac{(1-m)a}{k}\right\}^2 c_4^2 \mathrm{e}^{[2(1-m)a/k]t}. \tag{41}$$

From Eq. (41), with varying all parameters in the best choice, it is clear from Fig. 1 that Λ remains constant for initial time and at later stages increases rapidly. Hence don't seem to be viable for more consideration. So this case shall not be considered for study.

3.1.2 Model II: Solutions for $\xi = \xi_0 \rho$

When $n = 1$, Eq. (38) reduces to $\xi = \xi_0\rho$. With the use of Eqs. (34), (36), and (37), Eq. (39) reduces to

$$\left[2+\gamma-\frac{(3m-1)k\xi_0 M^{1/(2m)}}{(m-1)\mathrm{e}^{kt}}\right]\rho=\frac{M^{1/m}}{\mathrm{e}^{2kt}}\left[\frac{N^2}{(m+1)M^2}\left\{2+\frac{(2-7m)a}{(m+1)}\right\}+\right.$$

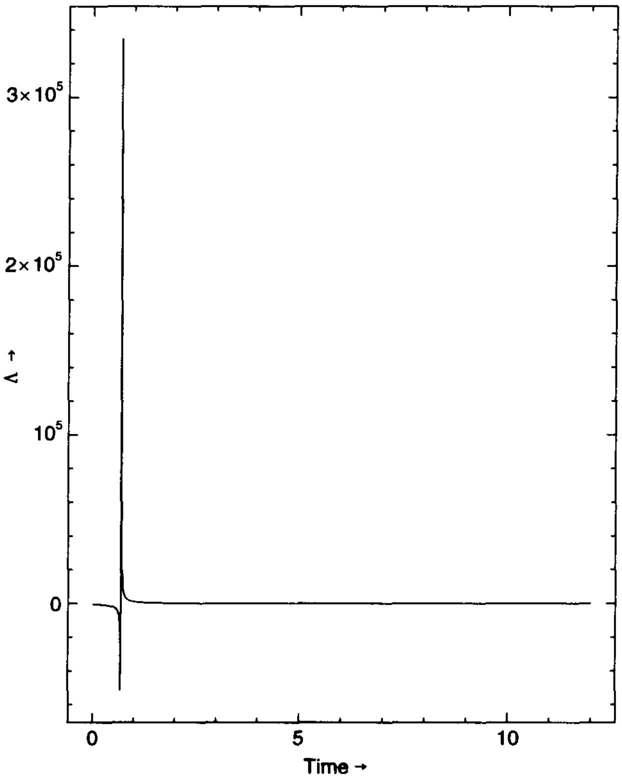

Fig. 2. The plot of cosmological constant Λ versus time τ for the model $\xi = \xi_0\rho$ with parameters $\gamma = 0.5$, $m = 2.0$, $a = 2.0$, $k = 2$.

$$+\frac{4m(2m-3)k^2}{3(m-1)^2} + \frac{4ma}{(m+1)^2} + \frac{m^2-2}{m+1} - \frac{1}{3}ma - 4\left\{\frac{(1-m)a}{k}\right\}^2 c_4^2 \mathrm{e}^{[2(1-m)a/k]t}\Bigg]. \tag{42}$$

Eliminating $\rho(t)$ between Eqs. (34) and (42), we obtain

$$\left[2+\gamma-\frac{(3m-1)k\xi_0 M^{1(2m)}}{(m-1)\mathrm{e}^{kt}}\right]\Lambda = \frac{(3m-1)k\xi_0 M^{1/(2m)}}{(m-1)\mathrm{e}^{kt}}\left[\frac{(2-7m)aN^2}{2(m+1)^2M^2}\right.$$

$$\left.+\frac{m(3m-2)k^2}{(m-1)^2} + \frac{2ma}{(m+1)^2} - \left\{\frac{(1-m)a}{k}\right\}^2 c_4^2 \mathrm{e}^{[2(1-m)a/k]t}\right]$$

$$+\frac{N^2}{(m+1)M^2}\left\{2-\frac{(2-7m)a\gamma}{2(m+1)}\right\} + \frac{m^2-2}{m+1} - \frac{1}{3}ma$$

$$-\frac{mk^2}{3(m-1)^2}\left\{10m+3(3m-2)\gamma\right\} - \frac{2ma\gamma}{(m+1)^2} + (\gamma-2)\left\{\frac{(1-m)a}{k}\right\}^2 c_4^2 \mathrm{e}^{[2(1-m)a/k]t}. \tag{43}$$

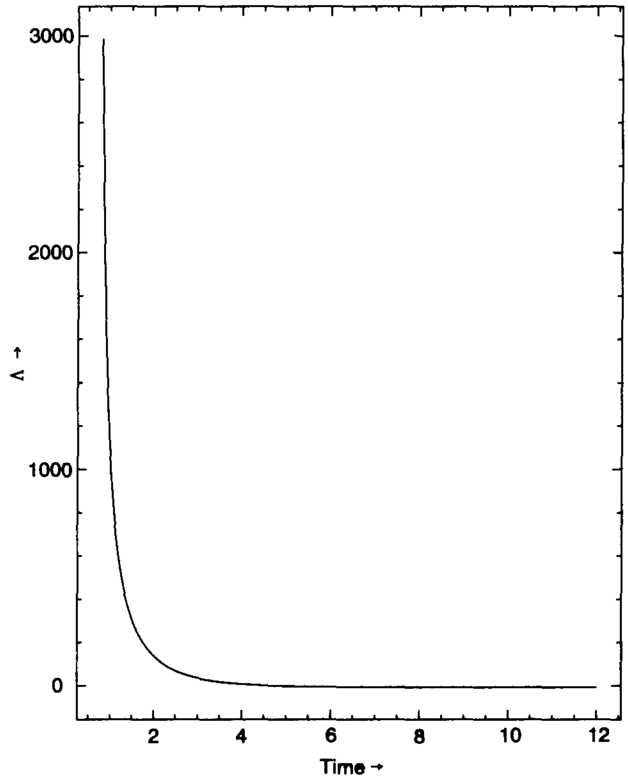

Fig. 3. The plot of cosmological constant Λ versus time τ for the model $\xi = \xi_0$ with parameters $\gamma = 0.5$, $m = 2.0$, $a = 2.0$, $k = 2$. This an inset view of showing Λ decreasing with time

From Eq. (43), we observe that the cosmological constant is a decreasing function of time and it approaches a small positive value at late times, which is supported by the results from recent type Ia supernovae observations. From Fig. 2, it is seen that at early stage the Λ-term oscillates and then decreases at later time. Figure 3 shows this behaviour of decreasing Λ-term as time progresses but remains small positive.

3.2 Case (2): $ma/(m+1) = 0$

In this case the geometry of the spacetime (1) reduces to the form

$$\begin{aligned} \mathrm{d}s^2 = {} & \mathrm{e}^{2kt}(c_1 + c_2 x)^{-1/m}\left[\mathrm{d}x^2 - \mathrm{d}t^2\right] + (c_1 + c_2 x)^{-1} \\ & \times \left[\mathrm{e}^{L+[2mk/(m-1)]t}\mathrm{d}y^2 + \mathrm{e}^{L-[2mk/(m-1)]t}\mathrm{d}z^2\right] . \end{aligned} \tag{44}$$

The expression for shear viscosity η, density ρ and pressure p, for model (44) are given by

$$\eta = \frac{1}{2\mathrm{e}^{kt}}\left[\frac{2mk}{1-m} - \frac{(1-m)a}{k}\right](c_1 + c_2x)^{1/(2m)}, \tag{45}$$

$$\rho = \frac{(c_1 + c_2x)^{1/m}}{\mathrm{e}^{2kt}}\left[\frac{m(3m-2)k^2}{(m-1)^2} + \frac{(2-7m)ac_2^2}{4m(c_1+c_2x)^2} \right.$$
$$\left. - \left\{\frac{(1-m)a}{k}\right\}^2 c_4^2\mathrm{e}^{[2(1-m)a/k]t} + \Lambda\right], \tag{46}$$

$$p - \xi\theta = \frac{(c_1 + c_2x)^{1/m}}{\mathrm{e}^{2kt}}\left[-\frac{10m^2k^2}{3(m-1)^2} + \frac{(1+m)c_2^2}{m(c_1+c_2x)^2} \right.$$
$$\left. + (m-1) - \frac{1}{3}ma - 2\left\{\frac{(1-m)a}{k}\right\}^2 c_4^2\mathrm{e}^{[2(1-m)a/k]t} - 2\Lambda\right] \tag{47}$$

and θ is the scalar of expansion calculated for the flow vector v^i as

$$\theta = \frac{(3m-1)k(c_1+c_2x)^{1/(2m)}}{(m-1)\mathrm{e}^{kt}}. \tag{48}$$

3.2.1 Model I : Solutions for $\xi = \xi_0$

When $n = 0$, Eq. (38) reduces to $\xi = \xi_0$. In this case Eq. (47) with the use of Eqs. (37), (46), and (48) reduces to

$$(2+\gamma)\rho = \frac{(3m-1)k\xi_0(c_1+c_2x)^{1/(2m)}}{(m-1)\mathrm{e}^{kt}} + \frac{(c_1 + c_2x)^{1/m}}{\mathrm{e}^{2kt}}\left[\frac{4m(2m-3)k^2}{3(m-1)^2} \right.$$
$$\left. + \frac{(4-5m)c_2^2}{2m(c_1+c_2x)^2} + (m-1) - \frac{1}{3}ma - 4\left\{\frac{(1-m)a}{k}\right\}^2 c_4^2\mathrm{e}^{[2(1-m)a/k]t}\right]. \tag{49}$$

Eliminating $\rho(t)$ between Eqs. (46) and (49), we obtain

$$(2+\gamma)\Lambda = \frac{(3m-1)k\xi_0\mathrm{e}^{kt}}{(m-1)(c_1+c_2x)^{1/2m}} - \frac{mk^2}{3(m-1)^2}\{10m + 3(3m-2)\gamma\}$$
$$+ \frac{c_2}{4m(c_1+c_2x)^2}\{4(1+m) - (2-7m)\gamma\} + (m-1) - \frac{1}{3}ma$$
$$+ (\gamma - 2)\left\{\frac{(1-m)a}{k}\right\}^2 c_4^2\mathrm{e}^{[2(1-m)a/k]t}. \tag{50}$$

3.2.2 Model II : Solutions for $\xi = \xi_0\rho$

When $n = 1$, Eq. (38) reduces to $\xi = \xi_0\rho$. In this case Eq. (47) with the use of Eqs. (37), (46), and (48) reduces to

$$\left[2+\gamma-\frac{(3m-1)k\xi_0(c_1+c_2x)^{1/(2m)}}{(m-1)\mathrm{e}^{kt}}\right]\rho=\frac{(c_1+c_2x)^{1/m}}{\mathrm{e}^{2kt}}\left[\frac{4m(2m-3)k^2}{3(m-1)^2}\right.$$
$$\left.+\frac{(4-5m)c_2^2}{2m(c_1+c_2x)^2}+(m-1)-\frac{1}{3}ma-4\left\{\frac{(1-m)a}{k}\right\}^2 c_4^2\mathrm{e}^{[2(1-m)a/k]t}\right]. \tag{51}$$

Eliminating $\rho(t)$ between Eqs. (46) and (51), we obtain

$$\left[2+\gamma-\frac{(3m-1)k\xi_0(c_1+c_2x)^{1/(2m)}}{(m-1)\mathrm{e}^{kt}}\right]\Lambda=\frac{(3m-1)k\xi_0(c_1+c_2x)^{1/(2m)}}{(m-1)\mathrm{e}^{kt}}$$
$$\times\left[\frac{m(3m-2)k^2}{(m-1)^2}+\frac{(2-7m)c_2^2}{4m(c_1+c_2x)^2}-\left\{\frac{(1-m)a}{k}\right\}^2 c_4^2\mathrm{e}^{[2(1-m)a/k]t}\right]$$
$$+(m-1)-\frac{1}{3}ma-\frac{mk^2}{3(m-1)^2}\left\{10m+3(3m-2)\gamma\right\}+\frac{c_2}{4m(c_1+c_2x)^2}$$
$$\times\left\{4(1+m)-(2-7m)\gamma\right\}+(\gamma-2)\left\{\frac{(1-m)a}{k}\right\}^2 c_4^2\mathrm{e}^{[2(1-m)a/k]t}. \tag{52}$$

It is observed from Eqs. (50) and (52) that Λ remains constant at all times. Thus no further interest in this case. It has been pointed out by various studies that constant value correspond to no interaction with either matter or radiation.

3.3 Case (3): $ma/(m+1) < 0$

In this case the geometry of the spacetime (1) reduces to the form

$$\mathrm{d}s^2=\mathrm{e}^{2kt}P^{-1/m}\left[\mathrm{d}x^2-\mathrm{d}t^2\right]+P^{-1}\left[\mathrm{e}^{L+[2mk/(m-1)]t}\mathrm{d}y^2+\mathrm{e}^{-L+[2mk/(m-1)]t}\mathrm{d}z^2\right], \tag{53}$$

where

$$P=c_1\cos\frac{\sqrt{2ma}}{m+1}x+c_2\sin\frac{\sqrt{2ma}}{m+1}x\,.$$

The model (53) is characterized by distribution

$$\eta=\frac{1}{2\mathrm{e}^{kt}}\left[\frac{2mk}{1-m}-\frac{(1-m)a}{k}\right]P^{1/(2m)}, \tag{54}$$

$$\rho=\frac{P^{1/m}}{\mathrm{e}^{2kt}}\left[\frac{m(3m-2)k^2}{(m-1)^2}-\frac{2ma}{(m+1)^2}+\frac{(2-7m)aQ^2}{2(m+1)^2P^2}-\right.$$

$$-\left\{\frac{(1-m)a}{k}\right\}^2 c_4^2 \mathrm{e}^{[2(1-m)a/k]t} + \Lambda\Bigg], \tag{55}$$

$$p - \xi\theta = \frac{P^{1/m}}{\mathrm{e}^{2kt}}\Bigg[\frac{2Q^2}{(m+1)P^2} - \frac{1}{3}ma - \frac{10m^2k^2}{3(m-1)^2}$$

$$+\frac{m^2a^2}{m+1} - 2\left\{\frac{(1-m)a}{k}\right\}^2 c_4^2 \mathrm{e}^{[2(1-m)a/k]t} - 2\Lambda\Bigg], \tag{56}$$

where

$$Q = c_1 \cos\frac{\sqrt{2ma}}{m+1}x - c_2 \sin\frac{\sqrt{2ma}}{m+1}x\,.$$

Here θ is the scalar of expansion calculated for the flow vector v^i as

$$\theta = \frac{(3m-1)kP^{1/(2m)}}{(m-1)\mathrm{e}^{kt}}\,. \tag{57}$$

3.3.1 Model I : Solutions for $\xi = \xi_0$

When $n = 0$, Eq. (38) reduces to $\xi = \xi_0$. In this case Eq. (56) with the use of Eqs. (37), (55), and (57) reduces to

$$(2+\gamma)\rho = \frac{(3m-1)k\xi_0 P^{1/(2m)}}{(m-1)\mathrm{e}^{kt}} + \frac{P^{1/m}}{\mathrm{e}^{2kt}}\Bigg[\frac{4m(2m-3)k^2}{3(m-1)^2} + \frac{m^2a^2}{m+1} - \frac{1}{3}ma$$

$$-\frac{4ma}{(m+1)^2} + \frac{2Q^2}{(m+1)P^2}\left\{1 + \frac{(2-7m)a}{2(m+1)}\right\} - 4\left\{\frac{(1-m)a}{k}\right\}^2 c_4^2 \mathrm{e}^{[2(1-m)a/k]t}\Bigg]. \tag{58}$$

Eliminating $\rho(t)$ between Eqs. (55) and (58), we obtain

$$(2+\gamma)\Lambda = \frac{(3m-1)k\xi_0 \mathrm{e}^{kt}}{(m-1)P^{1/(2m)}} - \frac{mk^2}{3(m-1)^2}\{10m + 3(3m-2)\gamma\} + \frac{m^2a^2}{m+1} - \frac{1}{3}ma$$

$$+\frac{2ma\gamma}{(m+1)^2} + \frac{Q^2}{(m+1)P^2}\left\{2 - \frac{(2-7m)a\gamma}{2(m+1)}\right\} + (\gamma-2)\left\{\frac{(1-m)a}{k}\right\}^2 c_4^2 \mathrm{e}^{[2(1-m)a/k]t}. \tag{59}$$

3.3.2 Model II : Solutions for $\xi = \xi_0\rho$

When $n = 1$, Eq. (38) reduces to $\xi = \xi_0\rho$. In this case Eq. (56) with the use of Eqs. (37), (55), and (57) reduces to

$$\left[2 + \gamma - \frac{(3m-1)k\xi_0 P^{1/(2m)}}{(m-1)\mathrm{e}^{kt}}\right]\rho = \frac{P^{1/m}}{\mathrm{e}^{2kt}}\Bigg[\frac{4m(2m-3)k^2}{3(m-1)^2} + \frac{m^2a^2}{m+1} -$$

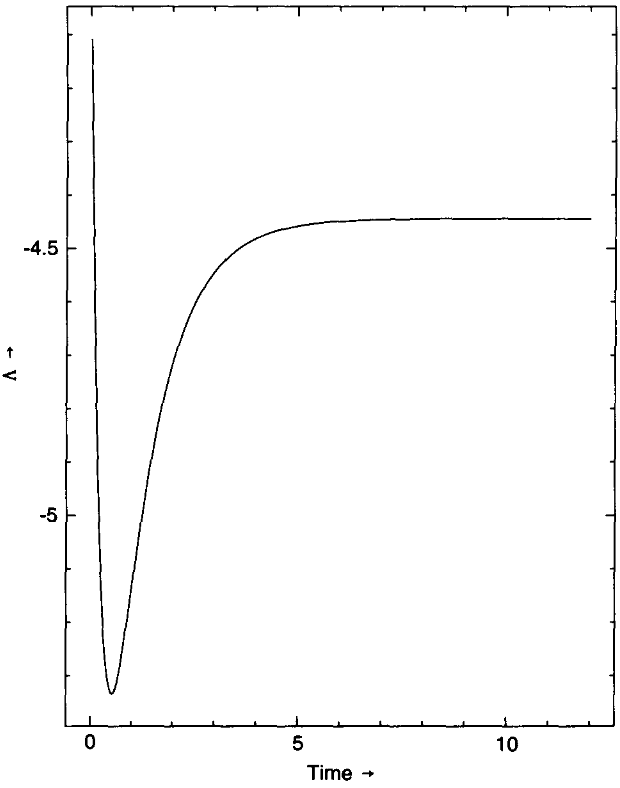

Fig. 4. The plot of cosmological constant Λ versus time τ for the model $\xi = \xi_0 \rho$ with parameters $\gamma = 0.5$, $m = 2.0$, $a = -2.0$, $k = -1$.

$$\left. - \frac{4ma}{(m+1)^2} - \frac{1}{3}ma + \frac{Q^2}{(m+1)P^2}\left\{2+\frac{(2-7m)a}{(m+1)}\right\} - 4\left\{\frac{(1-m)a}{k}\right\}^2 c_4^2 \mathrm{e}^{[2(1-m)a/k]t}\right]. \tag{60}$$

Eliminating $\rho(t)$ between Eqs. (55) and (60), we obtain

$$\begin{aligned}\left[2+\gamma-\frac{(3m-1)k\xi_0 P^{1/(2m)}}{(m-1)\mathrm{e}^{kt}}\right]\Lambda = \frac{(3m-1)k\xi_0 P^{1/(2m)}}{(m-1)\mathrm{e}^{kt}} & \left[\frac{m(3m-2)k^2}{(m-1)^2}\right. \\ \left. - \frac{2ma}{(m+1)^2} + \frac{(2-7m)aQ^2}{2(m+1)^2P^2} - \left\{\frac{(1-m)a}{k}\right\}^2 c_4^2 \mathrm{e}^{[2(1-m)a/k]t}\right] & \\ - \frac{mk^2}{3(m-1)^2}\left\{10m + 3(3m-2)\gamma\right\} + \frac{m^2a^2}{m+1} - \frac{1}{3}ma & \\ + \frac{2ma\gamma}{(m+1)^2} + \frac{Q^2}{(m+1)P^2}\left\{2-\frac{(2-7m)a\gamma}{2(m+1)}\right\} + (\gamma-2)\left\{\frac{(1-m)a}{k}\right\}^2 c_4^2 \mathrm{e}^{[2(1-m)a/k]t}. &\end{aligned} \tag{61}$$

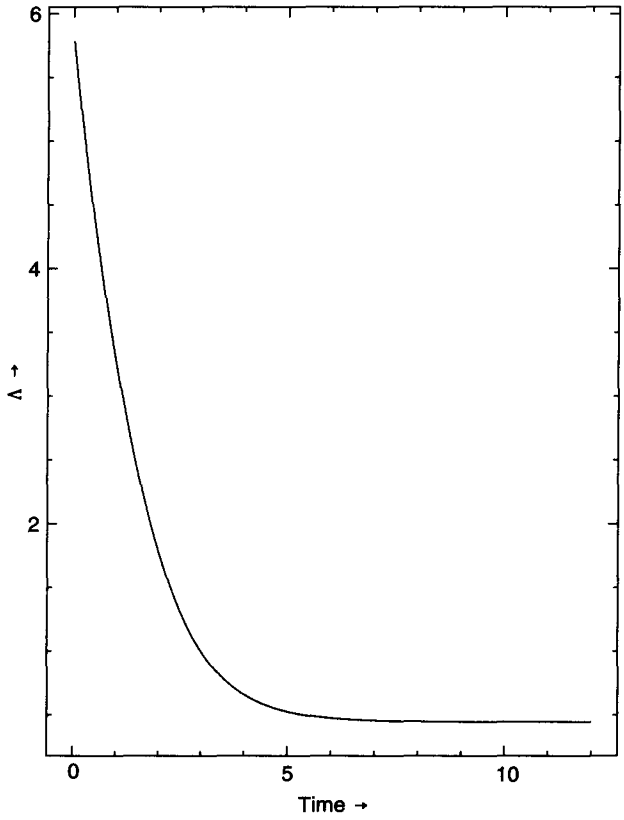

Fig. 5. The plot of cosmological constant Λ versus time τ for the model $\xi = \xi_0 \rho$ with parameters $\gamma = 0.5$, $m = 0.5$, $a = -2.0$, $k = 1.0$

From Eq. (59), it is observed that the Λ first decreases, reaches a deeper negative value, and then increases and becomes a constant negative value. Since we are giving a representative case by proper set of parameters, it will hold (see Fig. 4). A negative cosmological constant adds to the attractive gravity of matter; therefore, universe with a negative cosmological constant is invariably doomed to recollapse. A positive cosmological constant resists the attractive gravity of matter due to a negative pressure. For most universe, the positive cosmological constant eventually dominates over the attraction of matter and drives the universe to expand exponentially. From Eq. (61), we observe that the cosmological constant is a decreasing function of time and approaches a small positive value at late times, which is supported by the results from recent type Ia supernovae observations (see Fig. 5). One of the remarkable feature of model II is that it has possibility to incorporate earlier cases with appropriate set of parameters. We do not discuss it in more details. This could be related with some mechanism of interaction of Λ with either matter or radiation.

4 Some physical and geometrical features of the models

For σ/θ, we have

$$\frac{\sigma}{\theta} = \frac{1}{\sqrt{3}} \frac{\left[k^2 + \frac{k^2m^2}{(m-1)^2} - \frac{2mk^2}{m-1} + \frac{3}{k}(1-m)^2a^2c_4^2 e^{[2(1-m)a/k]t}\right]^{1/2}}{\frac{2}{k}(1-m)ac_4 e^{(1-m)a/k}t + k}, \tag{62}$$

$$\lim_{t\to 0} \frac{\sigma}{\theta} = \frac{1}{\sqrt{3}} \frac{\left[k^2 + \frac{k^2m^2}{(m-1)^2} - \frac{2mk^2}{m-1} + \frac{3}{k}(1-m)^2a^2c_4^2\right]^{1/2}}{\frac{2}{k}(1-m)ac_4 + k}, \tag{63}$$

$$\lim_{t\to\infty} \frac{\sigma}{\theta} = \frac{1}{2}, \tag{64}$$

Hence all models do not approach isotropy for large values of t. The rotation ω^2 is identically zero. For the acceleration

$$\dot{v}^i = \left(\frac{A_1}{A}, 0, 0, 0\right). \tag{65}$$

Thus we have obtained

$$\dot{v}^i = -\frac{\sqrt{2ma}}{2m(m+1)} \frac{\left(c_1 e^{\sqrt{2ma}/(m+1)} - c_2 e^{\sqrt{2ma}/(m+1)}\right)}{\left(c_1 e^{\sqrt{2ma}/(m+1)} + c_2 e^{\sqrt{2ma}/(m+1)}\right)} \quad \text{when} \quad ma/(m+1) > 0\,, \tag{66}$$

$$\dot{v}^i = -\frac{c_2}{2m(c_1 + c_2 x)} \quad \text{when} \quad ma/(m+1) = 0\,, \tag{67}$$

$$\dot{v}^i = -\frac{\sqrt{2ma}}{2m(m+1)} \frac{\left(c_1 \cos\frac{\sqrt{2ma}}{m+1} - c_2 \sin\frac{\sqrt{2ma}}{m+1}\right)}{\left(c_1 \cos\frac{\sqrt{2ma}}{m+1} + c_2 \sin\frac{\sqrt{2ma}}{m+1}\right)} \quad \text{when} \quad ma/(m+1) < 0\,. \tag{68}$$

Our solutions represent expanding, shearing and non-rotating universe.

5 Concluding remarks

We have obtained a new class of cylindrically symmetric inhomogeneous cosmological models of bulk viscous fluid as the source of matter. Generally the models represent expanding, shearing and non-rotating universe in which the flow vector is geodetic. In all these models, we observe that they do not approach isotropy for large values of time.

The cosmological constants in all models given in Subsubsections 3.2.2 and 3.4.2 are decreasing functions of time and they all approach a small value as time increases (*i.e.* the present epoch). The values of cosmological "constant" for these models are found to be small and positive which are supported by the results from recent supernovae Ia observations recently obtained by the High-Z Supernova Team and Supernova Cosmological Project (Garnavich *et al.* [28], Perlmutter *et al.* [25], Riess *et al.* [26], Schmidt *et al.* [31]). Our strong point of these models is that it incorporates matter density naturally and so makes feasible a model which can incorporate the physical constraints. In future, we would discuss elsewhere the model of Subsubsection 3.4.2 which seems to have broader features to incorporate physical constraints.

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