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NON-ADIABATIC GRAVITATIONAL COLLAPSE OF A SUPERDENSE STAR

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The relativistic equations governing the non-adiabatic shear-free collapse of massive superdense stars in the presence of dissipative forces producing heat flow in the background of space–times of the Vaidya–Tikekar *ansatz* with associated physical three-spaces that have the three-spheroidal geometry are formulated. It is shown how the system can be used to examine the development and progress of the collapse during subsequent epochs until the radiating star becomes a black hole.

Keywords: Neutron stars; general relativity.

1. Introduction

The relativistic models of compact stars are usually studied by integrating numerically the appropriate set of Einstein's field equations on the basis of an established equation of state for its matter content. The precise nature of the behavior of matter in the interior of compact stars, like neutron stars, are strange stars not known with certainty and does not have reliable information about the equation of state of matter for such stars. One is led to stipulate assumptions of general nature for solving the relevant highly nonlinear system of Einstein's field equations. The approach suggested by Vaidya and Tikekar[1] and Tikekar[2] of stipulating compact three-spheroidal geometry to the interior physical space of the compact relativistic stars is very useful for describing easily surveyable models of such stars. The general closed form solutions of Einstein's field equations in the context of the spheroidal geometry have been investigated by Maharaj and Leach,[3] Mukharjee *et al.*[4] and

Gupta and Jasim.[5] The usefulness of another similar geometric *ansatz* has also been examined in this context.[6,7] An important feature of this approach is that the formulation admits models of compact stars with mass exceeding the limiting value of 3.20 $M_{\odot}$ of maximum mass for Neutron stars obtained by Rhodes and Ruffini.[8] Massive compact stars exceeding this limit are expected to be unstable and may collapse when their equilibrium is disturbed.

Gravity, thermal processes, and the processes of generation and transport of energy determine the interior structure of any compact star. When the gravitational attraction is counterbalanced by the repulsive fluid pressure of matter, the interior of a star will be in equilibrium. When this equilibrium is lost, the stellar structure will begin to collapse under its gravity and the thermal processes which are ignorable on a short time-scale will begin to play significant roles on astronomical scale. The gravitational collapse of spherical dust cloud for adiabatic flow under simplifying conditions was first studied by Oppenheimer and Snyder.[9] Later Vaidya[10,11] and Lindquist *et al.*[12] studied outgoing radiation from collapsing spherical bodies in general relativistic formalism. Misner and Sharp[13,14] studied the relationship of the outgoing radiation with the collapsing matter in the interior. Santos carried out a realistic analysis of non-adiabatic matter in radiating spherical bodies collapsing under self gravity, which was based on relativistic models of shear-free collapsing fluids with heat flow suggested by Glass.[15] Santos[16] and co workers (de Oliveira *et al.*,[17–19] Herrera *et al.*,[20,21] and Maharaj and Govinder[22]) used Glass' formulation to propose relativistic models for studying the shear-free collapse of a radiating star in general and with the help of analytic models based on specific solutions of a related system of relativistic equations. Other aspects like non-adiabatic charged collapse,[23] density inhomogeneity and local anisotropy of spherical gravitational collapse,[24] simplifying assumption such as geodesic fluid[25–27] *etc.* lead that to interesting inferences about the nature and evolution of the collapsing star have been shown to promote a better understanding of the evolution of the collapse. A natural extension of these studies, of examining the effects of shear and pressure anisotropy on the collapse process, have been done by Lake,[28] and Herrera and Santos.[29,30] A limitation of this approach in the study of the temperature distribution in collapsing configuration, expressed by Maartens,[31] is that the transport of energy within the collapsing configuration takes place through heat flux across spherical shells that are related with temperature gradient governed through a parabolic equation — a process which is expected to violate causality. The more appropriate approach that incorporates relativistically consistent causal thermodynamics in modeling the collapse of nonrotating radiating stars consists of invoking hyperbolic dissipative theory (Joseph and Preziosi,[32] Jou *et al.*,[33] Anile *et al.*[34] and L. Herrera *et al.*[35,36]). The relaxing effects on the temperature in a radiating collapsing star based on hyperbolic causal theories of dissipation and their impact on the evolution of the star have been examined by Govender *et al.*,[37,38] Martinez[39] and Di Prisco *et al.*[40–42]

We have set up the relativistic equations governing the non-adiabatic shear-free collapse of a massive superdense star, comprising of a radially collapsing fluid with pressure anisotropy on the background of a space–time of Vaidya–Tikekar *ansatz*, in the presence of dissipative forces that produce heat flow adhering to the non-causal approach of heat dissipative process which is believed to be a fair approximation in certain situations of physical relevance. In the Vaidya–Tikekar approach of modeling superdense stars the information about the equation of state of matter content is related with geometrical parameters. The influence of the geometrical parameters on the collapse is expected to give information about the role of equation of state on the collapse of stellar objects. The collapse of specific models of stars of almost the same mass, with different density variations, is expected to be of help in this regard.

In Sec. 2, Einstein's field equations governing the non-adiabatic shear-free collapsing fluid with pressure anisotropy in the presence of heat flow is obtained on the background of space–times with physical space having three-spheroidal geometry. The field equations for collapse of fluid with anisotropic pressure are considered for exploring the possibility of the development of pressure anisotropy during the collapse of a compact spherical stellar configuration in which the equilibrium is maintained by isotropic pressure before the collapse begins.

The space–time in the exterior region of the collapsing star witnesses, the presence of radiation and is appropriately described by the Vaidya metric for a radiating star. The boundary conditions which match the interior metric of the collapsing fluid with the exterior metric of a radiating star and their implications are examined in Sec. 3. An exact solution obtained in Sec. 4 is subsequently used in Secs. 4.1 and 4.2 to examine the progress of collapse of a superdense star of mass 3.24 $M_\odot$, by determining the temperature distribution, size and time of formation of the black hole in the case of two specific model stars of radius 17 km with almost the same compactification parameter MG/c^2a.

2. Field Equations of Interior Space–Time

The interior space–time of a static superdense star in accordance with the Vaidya–Tikekar approach is described by the metric

$$ds^2 = e^{\nu(r')}dt^2 - \left\{ \frac{1 - K\left(\dfrac{r'}{R}\right)^2}{1 - \left(\dfrac{r'}{R}\right)^2} dr'^2 + r'^2 d\Omega^2 \right\},$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the metric on a two-sphere. The $t = \text{const}$ hypersurfaces of the space–time are three-dimensional spheroids immersed in a four-dimensional Euclidean space. The geometrical parameters R and K respectively measure the sphericity and departure of the spheroidal three-space from sphericity.

We choose the new radial variable $r = r'/R$ so that the space–time metric can be expressed in the form

$$ds^2 = e^{\nu(r)}dt^2 - R^2 \left\{ \frac{1 - Kr^2}{1 - r^2} dr^2 + r^2 d\Omega^2 \right\},$$

indicating that R can be interpreted as a scale factor of an inhomogeneous spherically symmetric three-space. We stipulate that the interior space–time of the collapsing star is described by the metric

$$ds^2 = e^{\nu(t,r)}dt^2 - R(t)^2 \left\{ \frac{1 - Kr^2}{1 - r^2} dr^2 + r^2 d\Omega^2 \right\} \tag{1}$$

and the physical three-space is obtained as $t =$ const. A section of the space–time continues to be a three-spheroidal space which is regular for $r < 1$ and $K < 1$.

We stipulate that the physical content of the space–time will be in the form described by[43]

$$T^i_j = \left(\rho + \frac{p}{c^2}\right) u^i u_j - \frac{p}{c^2}\delta^i_j + \pi^i_j + q^i u_j + q_j u^i, \tag{2}$$

where ρ denotes matter density, p isotropic fluid pressure, u^i components of unit timelike velocity field of fluid, q^i spacelike radial heat flux field (u^i and q_i are mutually orthogonal) and π^i_j represents anisotropic pressure tensor. In the frame of a comoving observer following fluid motion $u^i = (0, 0, 0, e^{-\nu/2})$, the tensor π^i_j will have components

$$\pi^i_j = \frac{\sqrt{3}}{c^2} A_o(r,t) \left[C^i C_j + \frac{1}{3}(\delta^i_j - u^i u_j) \right], \tag{3}$$

where

$$C^i = \left(\sqrt{\left[\frac{1 - r^2}{R^2(1 - Kr^2)} \right]}, 0, 0, 0 \right)$$

and $A_o(r,t)$ denotes the magnitude of π^i_j. The heat flux vector will have the expression

$$q_i = \left(-qR^2 \left[\frac{(1 - Kr^2)}{1 - r^2} \right], 0, 0, 0 \right) \tag{4}$$

and magnitude $q = q(r,t)$. The presence of anisotropy will imply the existence of different pressures along radial and transverse spatial directions p_r and $p_\perp$ defined by the relations

$$p_r = p + \frac{2A_o}{\sqrt{3}}, \quad p_\perp = p - \frac{A_o}{\sqrt{3}}. \tag{5}$$

Einstein's field equations $G^i_j = -(8\pi G/c^2)T^i_j$ relating the dynamical variables with the geometric parameters in the interior of the collapsing distribution lead to the following system of four equations[44]:

$$\frac{8\pi G}{c^2}\rho = \frac{1}{R^2}\left[\frac{1-K}{1-Kr^2}\right]\left(1+\frac{2}{1-Kr^2}\right)+\frac{3e^{-\nu}\dot{R}^2}{R^2}, \tag{6}$$

$$\frac{8\pi G}{c^4}p_r = \frac{1}{R^2}\left[\frac{1-r^2}{1-Kr^2}\right]\left(\frac{\nu'}{r}+\frac{1}{r^2}\right)-\frac{1}{r^2R^2}-\frac{\dot{R}^2e^{-\nu}}{R^2}\left(1-\frac{R\dot{\nu}}{\dot{R}}+\frac{2R\ddot{R}}{\dot{R}^2}\right), \tag{7}$$

$$\begin{aligned}\frac{8\pi G}{c^4}p_\perp = {} & \frac{1}{R^2}\left[\frac{1-r^2}{1-Kr^2}\right]\left(\frac{\nu''}{2}+\frac{\nu'^2}{4}+\frac{\nu'}{2r}-\frac{(1-K)}{(1-r^2)(1-Kr^2)}\left(1+\frac{r\nu'}{2}\right)\right)\\ & -\frac{\dot{R}^2e^{-\nu}}{R^2}\left(1-\frac{R\dot{\nu}}{\dot{R}}+\frac{2R\ddot{R}}{\dot{R}^2}\right),\end{aligned} \tag{8}$$

$$\frac{8\pi G}{c^2}q = -\frac{\dot{R}}{R^3}\left[\frac{1-r^2}{1-Kr^2}\right]e^{-\nu/2}\nu'. \tag{9}$$

Here and in what follows, a prime and a dot denote differentiation with respect to r and t respectively. From Eqs. (8) and (7), the following equation

$$\begin{aligned}\frac{8\pi G}{c^4}\sqrt{3}A_o &= \frac{8\pi G}{c^4}(p_r-p_\perp)\\ &= -\frac{1}{R^2}\left[\frac{1-r^2}{1-Kr^2}\right]\left(\frac{\nu''}{2}+\frac{\nu'^2}{4}-\frac{\nu'}{2r}-\frac{1}{r^2}\right.\\ &\quad \left.-\frac{(1-K)}{(1-r^2)(1-Kr^2)}\left(1+\frac{r\nu'}{2}\right)\right)-\frac{1}{r^2R^2}\end{aligned} \tag{10}$$

determines the measure of anisotropy of pressure $p_r - p_\perp$ at all points of the interior region.

3. Exterior Space–Time and Matching Conditions

The space–time in the exterior region of the collapsing stellar configuration, denoted as $\mathcal{M}_+$, will be filled with radiation flowing outward along radial direction. It is appropriately described by the Vaidya metric

$$ds^2 = \left(1-\frac{2M(v)}{y}\right)dv^2 + 2dvdy - y^2d\Omega^2, \tag{11}$$

with the energy–momentum tensor

$$T^i_j = \epsilon\,\zeta^i\zeta_j \tag{12}$$

describing its physical content. Here ϵ represents the energy density of radiation with null flow vector $\zeta_i = (0,0,0,1)$.

Let $\mathcal{M}_-$ denote the space–time in the interior of the collapsing star which is separated from the exterior by a timelike three-dimensional space–time Σ which represents at any instance the boundary separating the exterior manifold $\mathcal{M}_+$ from the interior $\mathcal{M}_-$. The intrinsic metric on Σ will be

$$ds^2 = d\tau^2 - \mathcal{R}^2(\tau)d\Omega^2. \tag{13}$$

We stipulate the boundary conditions smoothly joining the interior and exterior manifold $\mathcal{M}_-$ and $\mathcal{M}_+$ across Σ as

$$(ds_-^2)_\Sigma = (ds_+^2)_\Sigma = (ds^2)_\Sigma, \tag{14}$$

$$K_{ij}^- = K_{ij}^+, \tag{15}$$

where

$$K_{ij}^\pm = -n_\alpha^\pm \frac{\partial^2 x_\pm^\alpha}{\partial\xi^i \partial\xi^j} - n_\alpha^\pm \Gamma_{\beta\gamma}^\alpha \frac{\partial x_\pm^\beta}{\partial\xi^i}\frac{\partial x_\pm^\gamma}{\partial\xi^j} \tag{16}$$

denote extrinsic curvatures of Σ in $\mathcal{M}_\pm$ respectively. Here $\Gamma_{\beta\gamma}^\alpha$ are Christoffel symbols, $n_\alpha^\pm$ the unit normal vectors to Σ in $\mathcal{M}_+$ and $\mathcal{M}_-$, x^α are the coordinates in $\mathcal{M}_+$ and $\mathcal{M}_-$ and ξ^i are the intrinsic coordinates on Σ. Equations (14) and (15) are Darmois[45] conditions, implying the continuity of the first and second fundamental form. Israel[46] relaxed these conditions by allowing discontinuities in the second fundamental form. The content of the star before the collapse begin is not a perfect gas, hence the conditions in the form (14) and (15) need to be relaxed to admit the discontinuity in density $\rho(r)$ across the static boundary Σ_s.

We state the explicit expressions for the normal vectors

$$n_\alpha^- = \left(R\left[\frac{1-Kr^2}{1-r^2}\right]^{1/2}, 0, 0, 0\right), \quad n_\alpha^+ = \frac{dv}{d\tau}\left(1, 0, 0, -\frac{dy}{dv}\right). \tag{17}$$

The boundary conditions (14) imply the following relations

$$\frac{dt}{d\tau} = e^{-\nu(r_\Sigma, t)/2}, \quad r_\Sigma R(t) = \mathcal{R}(\tau), \quad y = \mathcal{R}(\tau) \tag{18}$$

and

$$\left(\frac{dv}{d\tau}\right)_\Sigma^{-2} = \left(1 - \frac{2M(v)}{y} + 2\frac{dy}{dv}\right)_\Sigma. \tag{19}$$

The extrinsic curvatures K_{ij} of Σ are found to have the following explicit expressions

$$K_{\tau\tau}^- = -\left[\frac{1}{2R}\left(\frac{1-r^2}{1-Kr^2}\right)^{1/2}\nu'\right]_\Sigma,$$

$$K_{\theta\theta}^- = \left[rR\left(\frac{1-r^2}{1-Kr^2}\right)^{1/2}\right]_\Sigma,$$

$$K_{\tau\tau}^+ = \left[\frac{d^2v}{d\tau^2}\left(\frac{dv}{d\tau}\right)^{-1} - \frac{M(v)}{y^2}\frac{dv}{d\tau}\right]_\Sigma,$$

$$
\begin{aligned}
K^{+}_{\theta\theta} &= \left[y\frac{dy}{d\tau} + y\frac{dv}{d\tau}\left(1 - \frac{2M}{y}\right)\right]_{\Sigma},\\
K^{\pm}_{\phi\phi} &= \sin^2\theta K^{\pm}_{\theta\theta},\\
K^{-}_{ij} &= K^{+}_{ij} = 0 \quad \text{for } i \neq j.
\end{aligned}
\tag{20}
$$

In view of the first four lines of Eq. (20), the boundary conditions ensuring continuity of extrinsic curvatures across Σ imply the following relations

$$\left[\frac{d^2v}{d\tau^2}\left(\frac{dv}{d\tau}\right)^{-1} - \frac{M(v)}{y^2}\frac{dv}{d\tau}\right]_{\Sigma} = -\left[\frac{1}{2R}\left(\frac{1-r^2}{1-Kr^2}\right)^{1/2}\nu'\right]_{\Sigma}, \tag{21}$$

$$\left[y\frac{dy}{d\tau} + y\frac{dv}{d\tau}\left(1 - \frac{2M(v)}{y}\right)\right]_{\Sigma} = \left[rR\left(\frac{1-r^2}{1-Kr^2}\right)^{1/2}\right]_{\Sigma}. \tag{22}$$

Equations (18) and (22) determine the mass contained within the spherical region in $\mathcal{M}_+$ as

$$M(v) = \left[\frac{rR}{2}\left(1 - \left[\frac{1-r^2}{1-Kr^2}\right] + r^2\dot{R}^2e^{-\nu}\right)\right]_{\Sigma}. \tag{23}$$

The mass $m(r,t)$ of the collapsing matter of the star that is contained within the spherical region of radius $r < r_\Sigma$ will be

$$m(r,t) = \frac{rR}{2}\left[1 - \left(\frac{1-r^2}{1-Kr^2}\right) + r^2\dot{R}^2e^{-\nu}\right]. \tag{24}$$

Equations (6) and (9) determine the gradient of the mass function as

$$\frac{\partial m}{\partial r} = \frac{4\pi G}{c^2}\left[\rho r^2R^3 + qr^3\dot{R}R^4e^{-\nu/2}\left(\frac{1-Kr^2}{1-r^2}\right)\right]. \tag{25}$$

In general it is observed that $m(r,t)$ will be an increasing function of r, since from Eqs. (6) and (9) it follows that $\partial m/\partial r \geq 0$ in the region $\mathcal{M}_-$ with $K < 1$. The mass $M(v)$ contained within the spherical region of radius $r > r_\Sigma$ has the following explicit relation

$$M(v) = M_o(t) + \left[\int_0^{r_\Sigma}\frac{\partial m}{\partial r}dr\right]_{\Sigma} \tag{26}$$

where $M_o(t)$ is an arbitrary function of time. We set $M_o(t) = 0$ which ensures regularity at the center of the collapsing configuration.

On using Eqs. (18) and (19) to eliminate $dv/d\tau$ and $d^2v/d\tau^2$ and using field equations (7) and (9) to eliminate ν', one finds that the condition (21) leads to the relation

$$\left(\frac{p_r}{c^2}\right)_{\Sigma} = \left\{qR\left[\frac{1-Kr^2}{1-r^2}\right]^{1/2}\right\}_{\Sigma}. \tag{27}$$

It shows how the heat flux across the star boundary is directly linked with the fluid pressure along the radial direction across the spherical boundary surface of the collapsing star. In the absence of non-adiabatic dissipative forces, Eq. (27) reduces to the condition $(p_r)_\Sigma = 0$, which the anisotropic matter content of a spherical star collapsing with empty exterior is expected to comply with.

When the equilibrium of the fluid is lost it begins to collapse. We set $R(t) = 1$ and $dR/dt = 0$ at the instance the collapse of the star begins from its equilibrium state and use the suffix 's' to denote the respective dynamical variables of the star configuration in its static state in the beginning of the collapse. We have the following expressions for

$$\frac{8\pi G}{c^2}\rho_s = \left[\frac{1-K}{1-Kr^2}\right]\left(1 + \frac{2}{1-Kr^2}\right), \tag{28}$$

$$\frac{8\pi G}{c^4}(p_r)_s = \left[\frac{1-r^2}{1-Kr^2}\right]\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2}, \tag{29}$$

$$\begin{aligned}\frac{8\pi G}{c^4}(p_\perp)_s = \left[\frac{1-r^2}{1-Kr^2}\right]\Bigg(&\frac{\nu''}{2} + \frac{\nu'^2}{4} + \frac{\nu'}{2r} \\ &- \frac{(1-K)}{(1-r^2)(1-Kr^2)}\left(1 + \frac{r\nu'}{2}\right)\Bigg),\end{aligned} \tag{30}$$

$$\begin{aligned}\frac{8\pi G}{c^4}(p_r - p_\perp)_s = -\left[\frac{1-r^2}{1-Kr^2}\right]\Bigg(&\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\nu'}{2r} - \frac{1}{r^2} \\ &- \frac{(1-K)}{(1-r^2)(1-Kr^2)}\left(1 + \frac{r\nu'}{2}\right)\Bigg) - \frac{1}{r^2},\end{aligned} \tag{31}$$

$$\frac{8\pi G}{c^2}(q)_s = 0. \tag{32}$$

As the collapse begins from the state of equilibrium it follows that at that instance $[(p_r)_s]_\Sigma = 0$. The expressions for the dynamical variables subsequently obey Einstein's field equations (6)–(9). It is observed that these equations have the following form

$$\frac{8\pi G}{c^2}\rho R^2 = \frac{8\pi G}{c^2}(\rho)_s + 3e^{-\nu}\dot{R}^2, \tag{33}$$

$$\frac{8\pi G}{c^4}p_r R^2 = \frac{8\pi G}{c^4}(p_r)_s - \dot{R}^2 e^{-\nu}\left(1 - \frac{R\dot{\nu}}{\dot{R}} + \frac{2R\ddot{R}}{\dot{R}^2}\right), \tag{34}$$

$$\frac{8\pi G}{c^4}p_\perp R^2 = \frac{8\pi G}{c^4}(p_\perp)_s - \dot{R}^2 e^{-\nu}\left(1 - \frac{R\dot{\nu}}{\dot{R}} + \frac{2R\ddot{R}}{\dot{R}^2}\right), \tag{35}$$

$$\frac{8\pi G}{c^4}(p_r - p_\perp)R^2 = \frac{8\pi G}{c^4}(p_r - p_\perp)_s = 0. \tag{36}$$

The last equation implies that if the matter content of the star is a perfect fluid with isotropic pressure when the collapse begins, the collapse continues without developing any anisotropy in its pressure. The condition (27), in view of the field equations (7) and (9), is expressible in the form

$$\left\{\dot{R}^2\left(1-\frac{R\dot{\nu}}{\dot{R}}+\frac{2R\ddot{R}}{\dot{R}^2}\right)-\dot{R}e^{\nu/2}\nu'\left[\frac{1-r^2}{1-Kr^2}\right]^{1/2}-\frac{8\pi G}{c^4}(p_r)_s e^{\nu}\right\}_{\Sigma}=0. \quad (37)$$

We have investigated the subsequent collapse by stipulating that as the collapse continues $\nu(r,t)=[\nu(r)]_s$. Since $[(p_r)_s]_\Sigma=0$, Eq. (37) can be cast into the form

$$\dot{R}^2+2R\ddot{R}-2\alpha_o\dot{R}=0, \quad (38)$$

where

$$2\alpha_o=\left[e^{\nu/2}\nu'\left[\frac{1-r^2}{1-Kr^2}\right]^{1/2}\right]_{\Sigma}. \quad (39)$$

The differential equation (38) possesses a first integral

$$\dot{R}=-\frac{2\alpha_o}{C\sqrt{R}}(1-C\sqrt{R}) \quad (40)$$

that determines the evolution of the parameter $R(t)$. In fact it admits

$$t-t_o=\frac{1}{\alpha_o}\left(\frac{R}{2}+\frac{\sqrt{R}}{C}+\frac{\ln|(1-C\sqrt{R})|}{C^2}\right) \quad (41)$$

as its general solution, with C and t_o as constants of integration. Without any loss of generality we set $C=1$ and parameterize t so that the solution is expressible in the form

$$\dot{R}=-\frac{2\alpha_o}{\sqrt{R}}(1-\sqrt{R}), \quad (42)$$

and

$$t=\frac{1}{\alpha_o}\left(\frac{R}{2}+\sqrt{R}+\ln(1-\sqrt{R})\right). \quad (43)$$

It follows that when $t=-\infty, R(t)=1$, and for $R(t)=0, t=0$. We interpret it to imply that the collapse of the configuration begins in the remote past from the static equilibrium state. When the collapse began the star boundary was a spherical surface of radius r_s. Subsequently it is a shrinking spherical surface of radius $r_o(t)=r_sR(t)=r_\Sigma$, across which the heat is flowing as outward radiation.

In the next section we shall study the evolution of the collapse by considering two specific star models with content in the form of perfect fluid, and with almost the same mass and size on different geometrical backgrounds.

4. Exact Solution and Specific Models

The interior space–time, Eq. (1), of the star is static when the collapse begins with $R(t) = 1$. The metric potential $e^{\nu(r,\ t)} = e^{\nu(r)}$ is determined by the pressure isotropy condition $T_1^1 = T_2^2$ which reads[2]

$$(1 - K + KZ^2)\frac{d^2\psi}{dZ^2} - KZ\frac{d\psi}{dZ} + K(K-1)\psi = 0, \tag{44}$$

where $Z = \sqrt{1 - r^2}$ and $\psi = e^{\nu/2}$. Exact solutions of this equation obtained for the choice of the geometric parameters $K = -2, -7$ have been known to provide easily tractable physically plausible relativistic models of superdense stars in equilibrium. Following Rhodes and Ruffini,[8] we stipulate that the surface density of matter of a neutron star is $2 \times 10^{14}\,\mathrm{gms\,cm^{-3}}$. Introducing the density variation parameter $\lambda =$ surface density/central density, we considered two specific models (from those proposed in Tikekar[2] and Vaidya and Tikekar[1]) for the choices of $K = -2, -7$ of superdense stars of mass about $3.24\,\mathrm{M}_\odot$, with boundary radius a of the order of 17–18 kms and examined the nature of evolution of the collapse of stars of these models. It is generally believed that neutron stars with masses exceeding $3.2\,\mathrm{M}_\odot$ will be highly unstable and will collapse when their equilibrium is lost.[47] In Table 1, we have specified geometric parameters: K, R, mass and size parameters M, a, density variation and compactification parameters $\lambda, M\,\mathrm{km}/a$ and the metric variable $e^{\nu/2} = \psi$ for the two star models chosen.

The fluid distributions in the interior before the collapse begins are found to comply with the physical requirements such as $\rho > 0, p > 0, \rho > p/c^2$ and $dp/d\rho < c^2$. The space–time metric in the interior of the collapsing star after the collapse begins is described by Eq. (1), with $R(t)$ determined by Eq. (43), as a decreasing function of time. As there is no equation of state for the matter content of the star formulated it is essential to check that the above criteria for physical plausibility are not violated as the collapse proceeds. The expression for the matter density (6) clearly indicates that $\rho > 0$ throughout, as the collapse continues. We checked the validity of the requirements $p > 0, \rho > p/c^2$ and $dp/d\rho < c^2$ for the selected star models using graphical methods for different values of $R(t)$. In Figs. 1 and 2, we have indicated graphically the nature of $\rho(r)$, $p(r)$ for (i) $R(t) = 0.29$ for $K = -2$ and (ii) $R(t) = 0.306$ for $K = -7$ star models. It has been observed that for the condition $R(t) \geq 0.29$ in the case (i), and $R(t) \geq 0.306$ in the case (ii), the conditions for physical plausibility $p > 0, \rho > p/c^2$ are fulfilled throughout the

Table 1. Initial data of two star models conceived from Refs. 1 and 2.

K	λ	a km	R km	M/a	$M/\mathrm{M}_\odot$	$e^{\nu/2} = \psi$
-2	0.494	18.02	34.45	0.2652	3.24	$0.546Z\left(1 - \frac{4}{9}Z^2\right) + 0.999\left(1 - \frac{2}{3}Z^2\right)^{3/2}$
-7	0.35	17.33	47.44	0.2764	3.2474	$2.83Z\left(1 - \frac{7}{8}Z^2\right)^{3/2} - 0.71\left(1 - \frac{7}{2}Z^2 + \frac{49}{24}Z^4\right)$

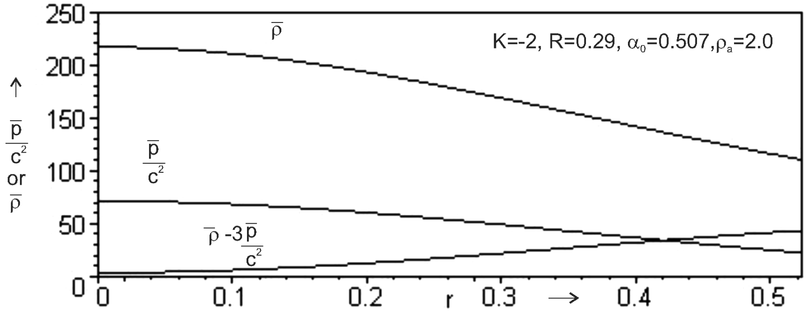

Fig. 1. The plots of $\overline{\rho}, \overline{p}/c^2$ and $\overline{\rho} - 3\overline{p}/c^2$ for $K = -2$ where $k = 8\pi G/c^2, k\rho = \overline{\rho}$ and $kp = \overline{p}$ fulfill the strong energy condition, and $\overline{\rho}$ and $\overline{p}/c^2$ decrease radially outward.

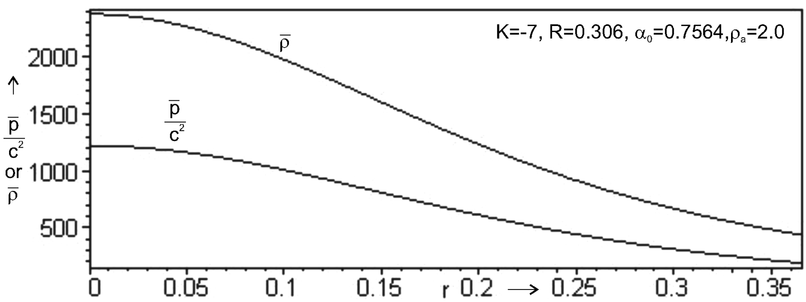

Fig. 2. The plots of $\overline{\rho}$ and $\overline{p}/c^2$ for $K = -7$ show the fulfilment of the weak energy condition. $\overline{\rho}$ and $\overline{p}/c^2$ maintain their decreasing nature in the radially outward direction. It is also observed that when the value of R increases toward $R = 1$ from $R = 0.306$, the same conclusions can be drawn.

interior. The strong energy condition $\rho > 3p/c^2$ is only observed to be violated in the case $K = -7$.

4.1. *Radius and time of formation of a black hole*

If r_s denotes the radius of the static model star when the collapse begins, the radius of the contracting configuration will subsequently be a decreasing function of time t as $r_o(t) = r_s R(t) = r_\Sigma$, the radius of the two-spherical boundary Σ . The mass enclosed within the collapsing configuration $m = m(r,t)$ will vary with time t. The boundary conditions provide the relation between the mass $M(v)$ enclosed within a spherical boundary of radius $r > r_\Sigma$ and m_s the mass of the star when the collapse begins:

$$M(v) = [2{\alpha_o}^2 r^3 e^{-\nu}(1-\sqrt{R})^2 + m_s\ R]_\Sigma, \tag{45}$$

where

$$m_s = \left[\frac{(1-K)r^3}{2(1-Kr^2)}\right]_\Sigma. \tag{46}$$

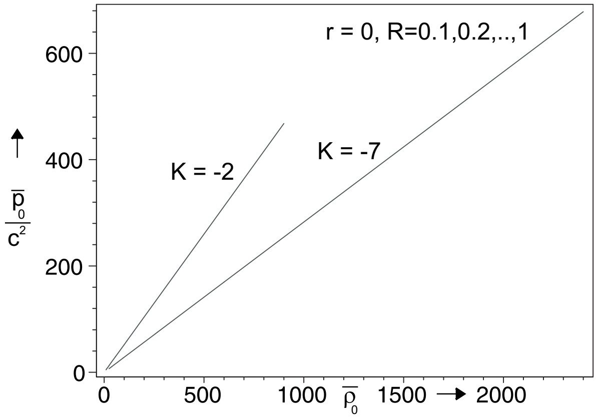

Fig. 3. The plot of $\overline{p}_o/c^2$ against $\overline{\rho}_o$ for $K = -2$ and $K = -7$ at the center $r = 0$ showing $\frac{d\overline{p}}{d\overline{\rho}} < c^2$.

Following Lindquist, Schwartz and Misner,[12] we write the total luminosity of the star observed by a remotely placed observer as

$$L_\infty = \lim_{y\to\infty,\, \frac{dy}{d\tau}=0} 4\pi y^2 \epsilon = -\frac{dM}{dv}, \tag{47}$$

where $L = 4\pi y^2\epsilon$ denotes the luminosity for the observer situated on Σ, the boundary of the collapsing star. The use of the boundary condition (22) leads to the expression

$$L_\infty = \left\{ \frac{\alpha_o(1-\sqrt{R})r^2 e^{-\nu/2}[(1-K)\sqrt{R} - {\nu'}^2(1-r^2)(1-\sqrt{R})](rR-2M)}{(1-Kr^2)R^2\left[\left(\frac{1-r^2}{1-Kr^2}\right)^{1/2} - r\dot{R}e^{-\nu/2}\right]} \right\}_\Sigma \tag{48}$$

for the apparent luminosity L_∞. After the collapsing configuration becomes a black hole at $t = t_{\mathrm{BH}}$, the observer at infinity ceases to receive any radiation and subsequently $L_\infty = 0$. In Eq. (48) this happens when (i) $R(t) = 1$ and (ii) $[r\ R(t_{\mathrm{BH}})]_\Sigma = r_s\ R(t_{\mathrm{BH}}) = 2M_\Sigma$. Condition (i) relates to the instance when the collapse begins. There is no heat flux across the boundary and so no radiation emission at the beginning. The condition (ii) corresponds to the epoch when the configuration crosses its Schwarzschild radius and becomes a black hole. On using Eq. (45) we find that this condition is equivalent to

$$[rR_{\mathrm{BH}}]_\Sigma = 2[2{\alpha_o}^2 r^3 e^{-\nu}(1-\sqrt{R_{\mathrm{BH}}})^2 + m_s R_{\mathrm{BH}}]_\Sigma, \tag{49}$$

a quadratic equation for $\sqrt{R_{\rm BH}}$, determining it as

$$\sqrt{R_{\rm BH}} = \left[\frac{2\alpha_o e^{-\nu/2} r}{2\alpha_o e^{-\nu/2} r + \sqrt{1 - \frac{2m_s}{r}}}\right]_\Sigma . \tag{50}$$

The radius of the configuration, when it becomes a black hole, is $r_{\rm BH} = r_s R_{\rm BH}$ while the time $t_{\rm BH}$ of the black hole formation follows from Eqs. (50) and (43) as

$$t_{\rm BH} = \frac{1}{\alpha_o}\left(\frac{R_{\rm BH}}{2} + \sqrt{R_{\rm BH}} + \ln(1 - \sqrt{R_{\rm BH}})\right). \tag{51}$$

4.2. *Temperature distribution in the collapsing configuration*

The fluid which makes up the collapsing star has to fulfil the thermodynamic relations[48]

$$q^i = -K_o(g^{ij} + u^i u^j)(T_{,j} + T u_{j;k} u^k), \tag{52}$$

where $K_o > 0$ and T denote the thermal conductivity and the temperature of the fluid.

Equation (52), also known as the Landau–Eckart Law is the relativistic version of the Maxwell–Fourier equation for heat transfer when dissipative forces are present. As it was pointed out earlier, it leads to a parabolic equation for heat diffusion a process which does not rule out violations of causality. This difficulty is avoided in the hyperbolic theory of dissipation. However, Eq. (52) is a fair approximation useful in specific situations of physical relevance.

Since $q^i = (q, 0, 0, 0)$, the temperature gradient law implies

$$q = \frac{K_o}{R^2} e^{\nu/2}\left[\frac{1 - r^2}{1 - Kr^2}\right]\frac{d}{dr}(Te^{-\nu/2}). \tag{53}$$

The field equation (9) then relates the temperature of the fluid with the metric function $\nu(r)$

$$\frac{8\pi G}{c^2} K_o \frac{d}{dr}(Te^{-\nu/2}) = -\frac{\dot{R}}{R} e^{-\nu}\nu'. \tag{54}$$

We assume that the temperature dependence of thermal conductivity K_o is expressible as

$$K_o = \beta_o T^{\Omega_o} \geq 0, \tag{55}$$

where β_o and Ω_o are constants. On integration,the field equation in the form (54) relates T with $R(t)$:

$$T^{(1+\Omega_o)} = d(t)e^{(1+\Omega_o)\nu/2} - 2\alpha_o\left(\frac{1+\Omega_o}{2+\Omega_o}\right)\left(\frac{c^2}{4\pi G\beta_o}\right)\left(\frac{1-\sqrt{R}}{\sqrt{R}R}\right)e^{-\nu/2}, \tag{56}$$

where $d(t)$ is an arbitrary function of time. The effective surface temperature seen by an external observer follows from the expression

$$[T^4]_\Sigma = \left[\left(\frac{1}{\pi\delta r^2 R^2}\right)\left(\frac{1-r^2}{1-Kr^2}\right)\right]_\Sigma L_\infty. \tag{57}$$

Here $\delta = \pi^2\mathcal{B}^4/(15h^3)$ for photons with $\mathcal{B}$ and h denoting respectively Boltzmann and Planck constants. Following Misner and Sharp,[14] we set $\Omega_o = 3$. Equations (56) and (57) then determine the arbitrary function $d(t)$ giving

$$\begin{aligned} d(t)[e^{2\nu}]_\Sigma = &\left[\left(\frac{2c^2\alpha_o}{5\pi G\beta_o}\right)\left(\frac{1-\sqrt{R}}{\sqrt{R}R}\right)e^{-\nu/2}\right]_\Sigma \\ &+ \left[\left(\frac{1}{\pi\delta r^2 R^2}\right)\left(\frac{1-r^2}{1-Kr^2}\right)\right]_\Sigma L_\infty. \end{aligned} \tag{58}$$

The temperature distribution throughout the interior of the collapsing configuration is then given by the relation

$$T^4 = d(t)e^{2\nu} - \left[\left(\frac{2c^2\alpha_o}{5\pi G\beta_o}\right)\left(\frac{1-\sqrt{R}}{\sqrt{R}R}\right)e^{-\nu/2}\right]. \tag{59}$$

It follows that the surface temperature of the collapsing body $T_\Sigma \to 0$ as $t \to -\infty$ at the beginning of the collapse and at the stage of formation of black hole $[rR(t)]_\Sigma \to 2M_\Sigma$.

5. Discussion and Conclusions

A neutron star with mass exceeding 3.2 $M_\odot$ is expected to collapse under its own gravity when its equilibrium is lost and its state of collapse at any instance t is related to its state in equilibrium before the collapse begins by Eqs. (33)–(36). These equations clearly indicate that if the interior content is in the form of perfect fluid before collapse begins, the collapse continues without developing any anisotropy in pressure. If the matter content of the star that sets in before the collapse is in the form of an imperfect fluid with anisotropic pressure, the pressures in radial and transverse directions will continue to differ as the collapse proceeds.

The function $R(t)$ which acts as a scale factor is a real-valued decreasing function of time with range $[0, 1]$ for a collapsing star. The boundary radius of the configuration at any epoch will be $r_o(t) = r_s R(t) = r_\Sigma$ after the collapse sets in. When the boundary surface of the contracting configuration crosses its Schwarzschild radius it becomes a black hole. In Table 2 the time $t_{\rm BH}$ of formation of black hole and the mass parameter $m_{\rm BH}$ of the black hole formed are determined using Eqs. (51), (24) and (46) for the collapsing configuration of classes (i) $K = -2$ and (ii) $K = -7$, and they have been given. The boundary radius $a_{\rm BH}$ in kms and the mass of the black hole formed $M_{\rm BH}/M_\odot$ in the solar mass unit are also indicated in Table 2. The mass parameter $m_{\rm BH}$ refers to the system of units, rendering $R(t) = 1$ at the time the collapse begins.

Table 2. Data obtained in the prescribed models.

K	λ	$r_s = a/R$	$R_{\rm BH}(t)$	$m_{\rm BH}$	$t_{\rm BH}$	$a_{\rm BH}$ Kms	$M_{\rm BH}/{\rm M}_\odot$
-2	0.494	0.523	0.2814	0.0736	-0.1674	5.0708	1.7192
-7	0.35	0.3653	0.3058	0.0559	-0.1313	5.2995	1.7973

The model configurations of both classes have almost the same mass and size when the collapse sets in. However the configuration of class (i) $K = -2$ crosses its Schwarzschild radius considerably earlier than the configuration of class (ii) $K = -7$. Though the configurations of the two classes are of almost the same mass, size and mass density at their boundary surfaces, the central mass density of the configuration of class (i) $K = -2$ is considerably lower than the central mass density of the configuration of class (ii) $K = -7$, in view of the different values of their density variation parameter λ. Accordingly, the density variation, which in a way describes the nature of distribution of matter, with the configurations, will be expected to influence the collapse of the stars of the same mass and size. Our observation indicates that configuration with lower matter density in the central region may be expected to collapse into its Schwarzschild horizon at an early epoch.

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