Evolution of transonicity in an accretion disc

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Abstract. For inviscid, rotational accretion flows driven by a general pseudo-Newtonian potential on to a Schwarzschild black hole, the only possible fixed points are saddle points and centre-type points. For the specific choice of the Newtonian potential, the flow has only two critical points, of which the outer one is a saddle point while the inner one is a centre-type point. A restrictive upper bound is imposed on the admissible range of values of the angular momentum of sub-Keplerian flows through a saddle point. These flows are very unstable to any deviation from a necessarily precise boundary condition. The difficulties against the physical realisability of a solution passing through the saddle point have been addressed through a temporal evolution of the flow, which gives a non-perturbative mechanism for selecting a transonic solution passing through the saddle point. An equation of motion for a real-time perturbation about the stationary flows reveals a very close correspondence with the metric of an acoustic black hole, which is also an indication of the primacy of transonicity.

PACS numbers: 98.62.Mw, 47.40.Hg, 47.10.Fg

1. Introduction

In accretion processes critical (transonic) flows — flows which are regular through a critical point — are of great interest [1]. Transonicity would imply that the bulk velocity of the flow would be matched by the speed of acoustic propagation in the accreting fluid. In this situation a subsonic-to-supersonic transition or vice versa can take place in the flow either continuously or discontinuously. In the former case the flow is smooth and regular through a critical point for transonicity (more specially, this can be a sonic point, at which the velocity of the bulk flow exactly matches the speed of sound), while in the latter case, there will arise a shock [2]. The possibility of both kinds of transition happening in an accreting system is very much real, and much effort has been made so far in studying these phenomena. For accretion on to a black hole especially, the argument that the inner boundary condition at the event horizon will lead to the exhibition of transonic properties in the flow, has been established well [1].

A paradigmatic astrophysical example of a transonic flow is the Bondi solution in steady spherically symmetric accretion [1, 3]. What is striking about the Bondi solution is that while the question of its realisability is susceptible to great instabilities arising from infinitesimal deviations from an absolutely precise prescription of the boundary condition in the steady limit of the hydrodynamic flow, it is easy to lock on to the Bondi solution when the temporal evolution of the flow is followed [4]. This dynamic and non-perturbative selection mechanism of the transonic solution agrees quite closely with Bondi's conjecture that it is the criterion of minimum energy that will make the transonic solution the favoured one [3, 5].

The appeal of the spherically symmetric flow, however, is limited by its not accounting for the fact that in a realistic situation, the infalling matter would be in possession of angular momentum — and hence the process of infall should lead to the formation of what is known as an accretion disc, i.e. the accretion process would be axisymmetric in nature. A substantial body of work over the past many years, has argued well the case for transonicity in axisymmetric accretion [1, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. Since in either of the two kinds of astrophysical flows — spherically symmetric or axisymmetric — transonic solutions pass through a critical point, it should be important to have a direct understanding of the nature of the critical points of the flow (and, of course, the physical solutions which pass through them). Without having to take recourse to the conventional approach of numerically integrating the governing non-linear flow equations, an alternative approach would be to adopt mathematical methods from the study of dynamical systems [19]. In accretion literature one would come across some relatively recent works which have indeed made use of the techniques of dynamical systems [4, 20, 21]. Through this approach a complete and mathematically rigorous prescription can be made for the exclusive nature of the critical points in axisymmetric pseudo-Schwarzschild accretion, and, in consequence, the possible behaviour of the flow solutions passing through those points and their immediate neighbourhood. In conjunction with a knowledge of the boundary conditions for physically feasible inflow solutions, this makes it possible to form an immediate qualitative notion of the solution topologies. What is more, this study has been shown to set a more restrictive condition on the specific angular momentum of sub-Keplerian flows passing through the critical points.

While mentioning these general issues, it must be stressed however, that the specific purpose of the present work is to examine for the case of disc accretion some aspects of the feasibility of its transonic solutions, particularly their stability against the choice of a boundary condition and their long-time evolutionary properties on large length scales. To do so it would be necessary first to have a clear notion of at least the qualitative features of the phase portrait of the steady flow solutions, and its critical points. For the pedagogically simple and particular case of an axisymmetric flow driven by the classical Newtonian potential, it has been shown here that there are only two critical points for the steady flow, of which, for realistic boundary conditions, the outer one is a saddle point, while the inner one is a centre-type point. As an aside, it should be interesting here to note that this apparently simplistic scenario is identically reproduced for an axisymmetric accretion flow, under a fully rigorous general relativistic formalism, for certain values of the flow parameters [18]. That these qualitatively physical conclusions about the flow could be drawn without falling back on the usual practice of a numerical integration of the steady flow equations, amply demonstrates the simplicity, the elegance and the power of the dynamical systems approach that has been adopted to study the thin accretion disc.

Having gained an understanding of the nature of the critical points in the phase portrait, the question that is then taken up is about the preference of the accreting system for any particular velocity profile in the stationary limit of the flow, and a selection criterion thereof. In this regard the transonic solution has always been the favoured candidate. However, it is common knowledge from the study of dynamical systems, that a flow solution passing through a saddle point (the transonic flow in this case) cannot be realised physically [19]. To address this issue satisfactorily it must be appreciated that the real physical flow is not static in nature, but has an explicit time-dependence. With respect to this point it is tempting to subject the steady flow solutions to small perturbations in real time, and then study their behaviour. This has been done elsewhere [21, 22] for the inviscid disc, and it has been shown that the steady inflow solutions of abiding interest are all stable under the influence of a linearised time-dependent perturbation on the mass inflow rate.

Since one way or the other, no direct conclusion could be drawn about the selection of a particular solution through a perturbative technique, one could then try to have an understanding of a true selection mechanism and the attendant choice of a particular solution, by studying the evolution of the accreting system through real time. A model analog shows that it is indeed possible in principle for the temporal evolution to allow for the selection of an inflow solution that passes through the saddle point, and under restricted conditions it has been demonstrated that the selection criterion conforms to Bondi's minimum energy argument, which is invoked to favour the transonic solution in the spherically symmetric case. Interestingly enough in this context, it has also been shown that although the perturbative study has offered no direct clue about the selection of a solution, the equation governing the propagation of an acoustic disturbance in the flow bears a close resemblance to the effective metric of an acoustic black hole. Use of this similarity has been made to argue that the flow would cross the acoustic horizon transonically.

2. The equations of the flow and its fixed points

It is a standard practice to consider a thin, rotating, axisymmetric, inviscid steady flow, with the condition of hydrostatic equilibrium imposed along the transverse direction [23, 24]. The two equations which determine the drift in the radial direction are Euler's equation,

$$v\frac{\mathrm{d}v}{\mathrm{d}r} + \frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} + \phi'(r) - \frac{\lambda^2}{r^3} = 0 \tag{1}$$

and the equation of continuity,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\rho v r H\right) = 0 \tag{2}$$

in which $\phi(r)$ is the generalised pseudo-Newtonian potential driving the flow (with the prime denoting a spatial derivative), λ is the conserved angular momentum of the flow, P is the pressure of the flowing gas, and $H \equiv H(r)$ is the local thickness of the disc [24], respectively.

The pressure, P, is prescribed by an equation of state for the flow. As a general polytropic it is given as $P = K\rho^\gamma$, in which K is a measure of the entropy in the flow and γ is the polytropic exponent. The function H in (2) will be determined according to the way P has been prescribed [24], while transonicity in the flow will be measured by scaling the bulk velocity of the flow with the help of the local speed of sound, given as $c_{\mathrm{s}} = (\partial P/\partial\rho)^{1/2}$.

With the polytropic relation thus specified for P, it is a straightforward exercise to set down in terms of the speed of sound, c_{s}, a first integral of (1) as,

$$\frac{v^2}{2} + nc_{\mathrm{s}}^2 + \phi(r) + \frac{\lambda^2}{2r^2} = \mathcal{E} \tag{3}$$

in which $n = (\gamma - 1)^{-1}$, and the integration constant $\mathcal{E}$ is the Bernoulli constant. Before moving on to find the first integral of (2) it should be important to derive the functional form of H. Assumption of hydrostatic equilibrium in the vertical direction deliver this form to be

$$H = c_{\mathrm{s}}\left(\frac{r}{\gamma\phi'}\right)^{1/2} \tag{4}$$

with the help of which, the first integral of (2) could be recast as

$$c_{\mathrm{s}}^{2(2n+1)}\frac{v^2 r^3}{\phi'} = \frac{\gamma}{4\pi^2}\dot{\mathcal{M}}^2 \tag{5}$$

where $\dot{\mathcal{M}} = (\gamma K)^n \dot{m}$ [1] with $\dot{m}$, an integration constant itself, being physically the matter flow rate.

To obtain the critical points of the flow, it should be necessary first to differentiate both (3) and (5), and then, on combining the two resulting expressions, to arrive at

$$\left(v^2 - \beta^2 c_{\mathrm{s}}^2\right) \frac{\mathrm{d}}{\mathrm{d}r}(v^2) = \frac{2v^2}{r} \left[\frac{\lambda^2}{r^2} - r\phi' + \frac{\beta^2 c_{\mathrm{s}}^2}{2} \left(3 - r\frac{\phi''}{\phi'}\right)\right] \tag{6}$$

with $\beta^2 = 2(\gamma+1)^{-1}$. The critical points of the flow will be given by the condition that the entire right hand side of (6) will vanish along with the coefficient of $\mathrm{d}(v^2)/\mathrm{d}r$ on the left hand side. Explicitly written down, following some rearrangement of terms, this will give the two critical point conditions as,

$$v_{\mathrm{c}}^2 = \beta^2 c_{\mathrm{sc}}^2 = 2\left[r_{\mathrm{c}}\phi'(r_{\mathrm{c}}) - \frac{\lambda^2}{r_{\mathrm{c}}^2}\right] \left[3 - r_{\mathrm{c}}\frac{\phi''(r_{\mathrm{c}})}{\phi'(r_{\mathrm{c}})}\right]^{-1} \tag{7}$$

with the subscript c labelling critical point values.

To fix the critical point coordinates, v_{c} and r_{c}, in terms of the system constants, one would have to make use of the conditions given by (7) along with (3), to obtain

$$\frac{2\gamma}{\gamma-1}\left[r_{\mathrm{c}}\phi'(r_{\mathrm{c}}) - \frac{\lambda^2}{r_{\mathrm{c}}^2}\right] \left[3 - r_{\mathrm{c}}\frac{\phi''(r_{\mathrm{c}})}{\phi'(r_{\mathrm{c}})}\right]^{-1} + \phi(r_{\mathrm{c}}) + \frac{\lambda^2}{2r_{\mathrm{c}}^2} = \mathcal{E} \tag{8}$$

from which it is easy to see that solutions of r_{c} may be obtained in terms of λ and $\mathcal{E}$ only, i.e. $r_{\mathrm{c}} = f_1(\lambda, \mathcal{E})$. Alternatively, r_{c} could be fixed in terms of λ and $\dot{\mathcal{M}}$. By making use of the critical point conditions in (5) one could write

$$\frac{4\pi^2\beta^2 r_{\mathrm{c}}^3}{\gamma\phi'(r_{\mathrm{c}})} \left(\frac{2}{\beta^2}\left[r_{\mathrm{c}}\phi'(r_{\mathrm{c}}) - \frac{\lambda^2}{r_{\mathrm{c}}^2}\right] \left[3 - r_{\mathrm{c}}\frac{\phi''(r_{\mathrm{c}})}{\phi'(r_{\mathrm{c}})}\right]^{-1}\right)^{2(n+1)} = \dot{\mathcal{M}}^2 \tag{9}$$

with the obvious implication being that the dependence of r_{c} will be given as $r_{\mathrm{c}} = f_2(\lambda, \dot{\mathcal{M}})$. Comparing these two alternative means of fixing r_{c}, the next logical step would be to say that for the fixed points, and for the solutions passing through them, it should suffice to specify either $\mathcal{E}$ or $\dot{\mathcal{M}}$ [1].

3. Nature of the fixed points : A dynamical systems study

The equations governing the flow in an accreting system are in general first-order non-linear differential equations. There is no standard prescription for a rigorous mathematical analysis of these equations. Therefore, for any understanding of the behaviour of the flow solutions, a numerical integration is in most cases the only recourse. On the other hand, an alternative approach could be made to this question, if the governing equations are set up to form a standard first-order autonomous dynamical system [19]. This is a very usual practice in general fluid dynamical studies [25], and short of carrying out any numerical integration, this approach allows for gaining physical insight into the behaviour of the flows to a surprising extent. As a first step towards this end, for the stationary polytropic flow, as given by (6), it should be necessary to parametrise this equation and set up a coupled autonomous first-order dynamical system as [19]

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(v^2) = 2v^2\left[\frac{\lambda^2}{r^2} - r\phi' + \frac{\beta^2 c_{\mathrm{s}}^2}{2}\left(3 - r\frac{\phi''}{\phi'}\right)\right]$$
$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = r\left(v^2 - \beta^2 c_{\mathrm{s}}^2\right) \tag{10}$$

in which τ is an arbitrary mathematical parameter. With respect to accretion studies in particular, this kind of parametrisation has been reported before [4, 20, 21], but the present treatment, as far as disc accretion is concerned, will much more thoroughly highlight some serious questions about the feasibility of transonic flows.

The critical points have themselves been fixed in terms of the flow constants. About these fixed point values, upon using a perturbation prescription of the kind $v^2 = v_{\mathrm{c}}^2 + \delta v^2$, $c_{\mathrm{s}}^2 = c_{\mathrm{sc}}^2 + \delta c_{\mathrm{s}}^2$ and $r = r_{\mathrm{c}} + \delta r$, one could derive a set of two autonomous first-order linear differential equations in the δv^2 — δr plane, with δc_{s}^2 itself having to be first expressed in terms of δr and δv^2, with the help of (5) — the continuity equation — as

$$\frac{\delta c_{\mathrm{s}}^2}{c_{\mathrm{sc}}^2} = -\frac{\gamma - 1}{\gamma + 1}\left(\frac{\delta v^2}{v_{\mathrm{c}}^2} + \left[3 - r_{\mathrm{c}}\frac{\phi''(r_{\mathrm{c}})}{\phi'(r_{\mathrm{c}})}\right]\frac{\delta r}{r_{\mathrm{c}}}\right). \tag{11}$$

The resulting coupled set of linear equations in δr and δv^2 will be given as

$$\frac{1}{2v_{\mathrm{c}}^2}\frac{\mathrm{d}}{\mathrm{d}\tau}(\delta v^2) = \frac{\mathcal{A}}{2}\left(\frac{\gamma - 1}{\gamma + 1}\right)\delta v^2 - \left[\frac{2\lambda^2}{r_{\mathrm{c}}^3} + \phi'(r_{\mathrm{c}}) + r_{\mathrm{c}}\phi''(r_{\mathrm{c}}) + \frac{\beta^2 c_{\mathrm{sc}}^2}{2}\frac{\phi''(r_{\mathrm{c}})}{\phi'(r_{\mathrm{c}})}\mathcal{B} + \frac{\beta^2 c_{\mathrm{sc}}^2}{2}\left(\frac{\gamma - 1}{\gamma + 1}\right)\frac{\mathcal{A}^2}{r_{\mathrm{c}}}\right]\delta r$$
$$\frac{1}{r_{\mathrm{c}}}\frac{\mathrm{d}}{\mathrm{d}\tau}(\delta r) = \frac{2\gamma}{\gamma + 1}\delta v^2 - \mathcal{A}\left(\frac{\gamma - 1}{\gamma + 1}\right)\frac{v_{\mathrm{c}}^2}{r_{\mathrm{c}}}\delta r \tag{12}$$

in which

$$\mathcal{A} = r_{\mathrm{c}}\frac{\phi''(r_{\mathrm{c}})}{\phi'(r_{\mathrm{c}})} - 3\,, \qquad \mathcal{B} = 1 + r_{\mathrm{c}}\frac{\phi'''(r_{\mathrm{c}})}{\phi''(r_{\mathrm{c}})} - r_{\mathrm{c}}\frac{\phi''(r_{\mathrm{c}})}{\phi'(r_{\mathrm{c}})}.$$

Trying solutions of the kind $\delta v^2 \sim \exp(\Omega\tau)$ and $\delta r \sim \exp(\Omega\tau)$ in (12), will deliver the eigenvalues Ω — growth rates of δv^2 and δr — as

$$\Omega^2 = \frac{2r_{\mathrm{c}}\phi'(r_{\mathrm{c}})\beta^2 c_{\mathrm{sc}}^2}{(\gamma + 1)}\left(\left[(\gamma - 1)\mathcal{A} - 2\gamma(4 + \mathcal{A}) + 2\gamma\mathcal{B}\left(1 + \frac{3}{\mathcal{A}}\right)\right] - \frac{\lambda^2}{\lambda_{\mathrm{K}}^2(r_{\mathrm{c}})}\left[4\gamma + (\gamma - 1)\mathcal{A} + 2\gamma\mathcal{B}\left(1 + \frac{3}{\mathcal{A}}\right)\right]\right) \tag{13}$$

where $\lambda_{\mathrm{K}}(r)$ is the local Keplerian angular momentum, expressed as $\lambda_{\mathrm{K}}^2(r) = r^3\phi'(r)$.

Once the position of a critical point, r_{c}, has been ascertained, it is then a straightforward task to find the nature of that critical point by using r_{c} in (13). Since it has been discussed in Section 2 that r_{c} is a function of λ and $\mathcal{E}$ (or $\dot{\mathcal{M}}$) for polytropic flows, it effectively implies that Ω^2 can, in principle, be rendered as a function of the flow parameters. A generic conclusion that can be drawn about the critical points from the form of Ω^2 in (13), is that for a conserved pseudo-Schwarzschild axisymmetric flow driven by any potential, the only admissible critical points will be saddle points and centre-type points. For a saddle point, $\Omega^2 > 0$, while for a centre-type point, $\Omega^2 < 0$. Once the behaviour of all the physically relevant critical points has been understood in this way, a complete qualitative picture of the flow solutions passing through these points (if they are saddle points), or in the neighbourhood of these points (if they are centre-type points), can be constructed, along with an impression of the direction that these solutions can have in the phase portrait of the flow [19].

A further interesting point that can be appreciated from the derived form of Ω^2, is related to the admissible range of values for a sub-Keplerian flow passing through a saddle point. It is self-evident that for this kind of flow, $(\lambda/\lambda_{\mathrm{K}})^2 < 1$ [6]. However, a look at (13) will reveal that a more restrictive upper bound on $\lambda/\lambda_{\mathrm{K}}$ can be imposed under the requirement that $\Omega^2 > 0$ for a saddle point, and this restriction will naturally be applicable to solutions which pass through such a point. This is entirely a physical conclusion, and yet its establishing has been achieved through a mathematical parametrisation of a dynamical system.

To appreciate the practical usefulness of the method developed so far, it should be worthwhile to use the simple example of the Newtonian potential, $\phi = -GM/r$ as a special case. From (8) it will then be possible to find the spatial coordinates of the fixed points, which will be given by the two roots of a quadratic equation, whose solution will be given by

$$r_{\mathrm{c}} = \frac{(5-3\gamma)GM}{10(\gamma-1)\mathcal{E}}\left[1 \pm \sqrt{1 - \frac{10(5-\gamma)(\gamma-1)\lambda^2\mathcal{E}}{(5-3\gamma)^2(GM)^2}}\right] \tag{14}$$

from which a conclusion that can be drawn is that for critical conditions to be obtained, there will have to be real solutions for r_{c}, and this could only be achieved if the condition

$$\lambda < \frac{(5-3\gamma)GM}{\sqrt{10(5-\gamma)(\gamma-1)\mathcal{E}}} \tag{15}$$

were to be maintained.

The properties of these two critical points can be further analysed with the help of the eigenvalues of the stability matrix associated with the critical points. Upon using the Newtonian potential, $\phi = -GM/r$, in (13), it would be easy to arrive at

$$\Omega^2 = 2\beta^2 c_{\mathrm{sc}}^2 \left(\frac{5-\gamma}{\gamma+1}\right)\frac{GM}{r_{\mathrm{c}}}\left[\zeta - \frac{\lambda^2}{\lambda_{\mathrm{K}}^2(r_{\mathrm{c}})}\right] \tag{16}$$

where $\zeta = (5-3\gamma)(5-\gamma)^{-1}$ and $\lambda_{\mathrm{K}}^2(r_{\mathrm{c}}) = GMr_{\mathrm{c}}$.

The term in the square brackets in (16) will determine the sign of Ω^2. Knowing that two adjacent fixed points cannot be of the same nature [19], i.e. they will differ in their respective signs of Ω^2, it is now evident that for the signs of all other factors in (16) remaining always unchanged, if r_{c} is the length coordinate of the outer fixed point, for which $(\lambda/\lambda_{\mathrm{K}})^2 < \zeta$, then Ω^2 will be positive and the fixed point will be a saddle point, while if r_{c} gives the inner fixed point, for which $(\lambda/\lambda_{\mathrm{K}})^2 > \zeta$, then Ω^2 will be negative and the fixed point will be centre-type. It is also remarkable that this apparently simplistic scenario of two fixed points, with the outer one being a saddle point and the inner one being centre-type, can be identically reproduced in the case of a proper general relativistic flow on to a Schwarzschild black hole, under certain ranges of the flow parameters [18]. All the flows of physical interest in these cases are governed by the boundary condition that at large distances the drift velocity becomes vanishingly small, while the speed of sound approaches a constant value. This knowledge of the boundary condition, in conjunction with the nature of the fixed points, makes it possible to understand what the phase trajectories would look like. In a manner of speaking, the outer fixed point, which is a saddle point, may be dubbed the "sonic point" because one of the two trajectories passing through it is a transonic inflow solution, rising from subsonic values far away from the saddle point to attain supersonic values on length scales which are less than that of the saddle point. Transonicity is to be attained when the drift velocity equals the speed of the propagation of an acoustic disturbance, which, for this system is given by $\sqrt{2}(\gamma+1)^{-1/2}c_{\mathrm{s}}$ [22]. Dwelling on this last point in somewhat greater detail, it may be noted that in some earlier works [2, 23] the speed of acoustic propagation has been scaled as an "effective" speed of sound. This scaling has been done by a constant factor that arises due to the chosen geometry of the thin disc flow (especially for a disc in vertical hydrostatic equilibrium, with the local disc height being a function of the acoustic velocity). This leads to the Mach number of the flow being scaled accordingly, and the critical condition for the flow will consequently be achieved when this scaled Mach number becomes unity. In the present treatment, on the other hand, the conventional definition of the Mach number has been adhered to, and as a result criticality will occur when the Mach number assumes the value $\sqrt{2}(\gamma+1)^{-1/2}$. Either approach is entirely equivalent to the other, if one is mindful of the fact that the *exact* sonic condition (where the bulk flow velocity matches the local *unscaled* speed of sound) and the critical condition differ by a constant scaling factor only. This distinction disappears for the case of a spherically symmetric flow and for an axisymmetric flow with a constant disc height. On the other hand, the situation becomes radically more complicated for the case of a fully general relativistic flow. Here the critical Mach number is not a global constant, but is a local function of the critical point coordinates [18], all of which, of course, makes it very much difficult a mathematical exercise to fix the critical point coordinates, calculate the slope of solutions passing through the critical points (in terms of physical relevance these points should be saddle points), and then numerically integrate these solutions under appropriate boundary conditions.

It has been mentioned earlier that the particular properties of a saddle point in the phase portrait of thin disc flow solutions will also have a bearing on the possible range of values that the constant specific angular momentum, λ, may be allowed to have. For solutions passing through the saddle point, it can be easily recognised from (7) that the flow will be sub-Keplerian [1, 6]. In that case the condition, $(\lambda/\lambda_{\mathrm{K}})^2 < 1$, shall hold good. In addition to this, for the specific case represented by (16), the saddle-type behaviour of the outer fixed point would also imply that there would be another upper bound on λ, given by $(\lambda/\lambda_{\mathrm{K}})^2 < \zeta$. For the admissible range of the polytropic index γ, the possible range of ζ would be $0 < \zeta < 1/2$. This would then imply that the latter bound on λ would be more restrictive, as compared to the former.

Saddle points are, however, inherently unstable, and to make a solution pass through such a point, after starting from an outer boundary condition, will entail an infinitely precise fine-tuning of that boundary condition [4]. This can be demonstrated through simple arguments. Going back to (12), the coupled set of linear differential equations in δv^2 and δr can be set down as

$$\frac{\mathrm{d}\left(\delta v^2\right)}{\mathrm{d}\left(\delta r\right)} = \frac{\mathrm{d}\left(\delta v^2\right)/\mathrm{d}\tau}{\mathrm{d}\left(\delta r\right)/\mathrm{d}\tau} = \frac{\mathcal{Q}_1\delta v^2 + \mathcal{Q}_2\delta r}{\mathcal{Q}_3\delta v^2 + \mathcal{Q}_4\delta r} \tag{17}$$

in which the constant coefficients $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$ and $\mathcal{Q}_4$ are to be determined simply by inspection of (12). It is also to be easily seen that $\mathcal{Q}_1 = -\mathcal{Q}_4$. This makes the integration of (17) a staightforward exercise that yields

$$\mathcal{Q}_2\left(\delta r\right)^2 + 2\mathcal{Q}_1\left(\delta v^2\right)\left(\delta r\right) - \mathcal{Q}_3\left(\delta v^2\right)^2 + \mathcal{C} = 0 \tag{18}$$

with $\mathcal{C}$ being an integration constant. Generally speaking (18) is the equation of a conic section in the δv^2 — δr plane. If the origin of this plane were to be considered to have been shifted to the saddle point, then the condition for solutions passing through the origin, i.e. $\delta v^2 = \delta r = 0$, would be $\mathcal{C} = 0$, which reduces (18) to a pair of straight lines intersecting each other through the origin itself. All other solutions in the vicinity of the origin will, therefore, be hyperbolic in nature, a fact that is given by the condition $(\mathcal{Q}_1^2 + \mathcal{Q}_2\mathcal{Q}_3) > 0$. For the case of $\phi = -GM/r$, this contention can be verified completely analytically, and this shows that even a very minute deviation from a precise boundary condition for transonicity (i.e. a boundary condition that will generate solutions to pass *only* through the origin, $\delta v^2 = \delta r = 0$) will take the stationary solution far away from a transonic state. This extreme sensitivity of transonic solutions to boundary conditions is entirely in keeping with the nature of saddle points. It may be imagined that in a proper astrophysical system such precise fulfillment of a boundary condition will make the transonic solution well-nigh physically non-realisable. Indeed, this difficulty, for any kind of accreting system, is readily appreciated by anyone trying to carry out a numerical integration of (1) to generate the transonic solutions, which can only be obtained when the numerics is first biased in favour of transonicity by using the saddle point condition itself as the boundary condition for numerical integration.

There is another aspect to the physical non-realisability of transonic solutions, although this is somewhat mathematical in nature. Using the condition $\mathcal{C} = 0$ will make it easy to express δv^2 in terms of δr and vice versa. Going back to the set of linear equations given by (12) and choosing the second one of the two equations (the choice of the first would also have led to the same result), one gets

$$\frac{\mathrm{d}\left(\delta r\right)}{\mathrm{d}\tau} = \pm\sqrt{\mathcal{Q}_1^2 + \mathcal{Q}_2\mathcal{Q}_3}\ \delta r \tag{19}$$

which can be integrated for both the roots from an arbitrary initial value of $\delta r = [\delta r]_{\mathrm{in}}$ lying anywhere on the transonic solution, to a point $\delta r = \epsilon$, with ϵ being very close to the critical point given by $\delta r = 0$. With this it can be shown that

$$\tau = \pm \frac{1}{\sqrt{\mathcal{Q}_1^2 + \mathcal{Q}_2 \mathcal{Q}_3}} \int_{[\delta r]_{\rm in}}^{\epsilon} \frac{{\rm d}\,(\delta r)}{\delta r} = \pm \frac{1}{\sqrt{\mathcal{Q}_1^2 + \mathcal{Q}_2 \mathcal{Q}_3}} \ln \left| \frac{\epsilon}{[\delta r]_{\rm in}} \right| \tag{20}$$

from which it is easy to see that $|\tau| \longrightarrow \infty$ for $\epsilon \longrightarrow 0$. This implies that the critical point may be reached along either of the separatrices, only after $|\tau|$ has become infinitely large. This divergence of the parameter τ indicates that in the stationary regime, solutions passing through the saddle point are not actual solutions, but separatrices of various classes of solutions [19]. This fact, coupled with the sensitivity of the stationary transonic solutions to the choice of an outer boundary condition, makes their feasibility a seriously questionable matter.

4. Dynamic evolution as a selection mechanism for transonicity

In Section 3 it has been demonstrated that generating a stationary solution through a saddle point (a transonic solution) will be impossible, physically speaking. Nevertheless, in accretion studies transonicity is not a matter of doubt. The key to this paradox lies in considering explicit time-dependence in the flow. This happens because the two stationary equations (1) and (2), as it is very much evident from their form, are invariant under the transformation $v \longrightarrow -v$ (i.e. inflows and outflows are twin solutions of the same set of stationary equations), and it is this invariance that gives birth to the saddle point, which is an intersection point of the transonic inflow and outflow solutions. This invariance breaks down as soon as time-dependent terms are introduced in the governing equations, something that is very easy to see from

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} + \frac{1}{\rho} \frac{\partial P}{\partial r} + \phi'(r) - \frac{\lambda^2}{r^3} = 0. \tag{21}$$

This will obviously imply that a choice of inflows ($v < 0$) or outflows ($v > 0$) has to be made right at the very beginning (at $t = 0$), and solutions generated thereafter will be free of all adverse conditions that one may associate with the presence of a saddle point in the stationary flow.

In trying to make a time-dependent study, one could also assure oneself that a linearised perturbative analysis in real time cannot give any conclusive insight about the accreting system showing any preference for any particular solution. For inviscid, axisymmetric accretion, this fact has been clearly established [21, 22]. One may then say that any selection mechanism based on explicit time-dependence has to be non-perturbative and evolutionary in character.

However, even for the simple inviscid disc system, with explicit time-dependence taken into consideration, the equations of a compressible flow cannot be integrated analytically. Hence, to have any appreciation of the time-evolutionary selection of the critical solution, it should be instructive to consider an analogous model situation first. The model system being introduced here, describes the dynamics of a field $y(x, t)$ that is analogous to a steadily accreting system with two fixed points, as has been discussed so far. The details of the dynamic selection of the critical solutions in the model problem have been presented in the Appendix.

To the extent that this model has been a good representative of the true physical situation, the whole argument for a time-dependent and non-perturbative method of selection developed in the Appendix, may now be extended to the actual problem of thin disc accretion. For a steady accretion disc many previous works have upheld the case for transonicity, although without explicitly addressing the issue of what special physical criterion may select the transonic solution to the exclusion of all other possible solutions. For the case of disc accretion on to black holes, Liang and Thompson [26] make a clear point by saying that "the solution for the radial drift velocity of thin disk accretion onto black holes must be transonic, and is analogous to the critical solution in spherical Bondi accretion, except for the presence of angular momentum." For the inviscid and thin disc at least, this is indeed a most crucial analogy, which will make it possible to invoke all the physical arguments used to uphold transonicity in spherically symmetric flows.

Transonicity is a settled fact [3, 5] in spherically symmetric accretion, and this has been so because at every spatial point in the flow, velocity evolves at a much greater rate through time, in comparison with density, and this, therefore, excludes the possibility of matter accumulating in regions close to the surface of the accretor. As a result gravity wins over pressure at small distances and the system is naturally driven towards selecting the transonic solution. This sort of a situation will be more so obtained for the case of matter accreting on to a black hole, and it then raises the question of whether or not a likewise time-evolutionary mechanism should be at work for the selection of the critical inflow solution in an inviscid and thin accretion disc. This connection need not be so obvious, considering the fact that thin disc accretion involves, apart from a very different flow geometry, a whole range of different physical phenomena, not invoked for spherically symmetric flows.

To have any analytical appreciation of how the temporal evolution acts as a selection mechanism in disc accretion, it should be instructive to prescribe the Newtonian potential for ϕ in (21), as well as recast the pressure term in it by the polytropic relation, $P = K\rho^{\gamma}$. All of this will lead to

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} + K\gamma\rho^{\gamma-2}\frac{\partial \rho}{\partial r} + \frac{GM}{r^2} - \frac{\lambda^2}{r^3} = 0 \tag{22}$$

for which it should be noted that the question of generating transonicity in the inflow solution is not going to be affected overmuch by the specific choice of the simple Newtonian potential for ϕ, especially so in the vicinity of the far-off outer critical (saddle) point. It would be easy to see that to have a solution pass through this critical point, the temporal evolution should proceed in such a manner, that the inflow velocity would increase much faster in time than density at distances on the scale of the saddle point, where, for sub-Keplerian flows, gravity, going as r^{-2}, dominates the rotational effects, which go as r^{-3}.

But what should be the key physical criterion guiding this dynamic selection of the transonic solution? It is to be stressed once again that the selection principle would very likely be the same as in the spherically symmetric case, where the transonic solution is chosen by dint of its corresponding to a configuration of the lowest possible energy [5]. To have an understanding of this, an analytical treatment of (22) may be carried out, with the density gradient being neglected as a working approximation. On large length scales, where the flow is highly subsonic, this is an especially effective approximation, and it will allow for treating the evolution of the velocity field to be largely independent of the density evolution. The resulting non-linear partial differential equation for velocity is then to be integrated by the method of characteristics [27] (the mathematical details of this method have been outlined for the model problem presented in the Appendix) to get a solution that can be written as

$$\frac{v^2}{2} - \frac{GM}{r} + \frac{\lambda^2}{2r^2} = \mathcal{F}\left[\frac{1}{r_0 + r\left(1 + v/a\right)} \exp\left(\frac{vr}{ar_0} - \frac{at}{r_0}\right)\right] \tag{23}$$

in which $r_0 = GM/a^2$, and a itself is an integration constant deriving from the spatial part of the characteristic equations, and, therefore, should, in general, be a function of time. The form of the function $\mathcal{F}$ is to be determined from the physically realistic initial condition $v = 0$ at $t = 0$ for all r, which will render $\mathcal{F}$ as

$$\mathcal{F}(\xi) = -\frac{GM\xi}{1 - \xi r_0} + \frac{\lambda^2 \xi^2}{2\left(1 - \xi r_0\right)^2}. \tag{24}$$

On examining the argument of $\mathcal{F}$ in (23) and then studying its long-time behaviour, it will be seen from (24) that for $t \longrightarrow \infty$, the selected solution will correspond to the condition

$$\frac{v^2}{2} - \frac{GM}{r} + \frac{\lambda^2}{2r^2} = 0. \tag{25}$$

It can now be seen that prior to the evolution, the system had no bulk motion and that the radial drift velocity was given flatly everywhere by $v = 0$. This, of course, gives the condition that initially the total specific mechanical energy of the system was zero. Then at $t = 0$ both a gravitational mechanism is activated in this system and some angular momentum is imparted to it. This will induce a potential $-GM/r$ everywhere, and at the same time start a rotational motion, respectively. The system will then start evolving in time, with the velocity v at each point in space evolving temporally according to (23). Finally the system will restore itself to a steady state in such a manner that the total specific mechanical energy at the end of the evolution (for $t \longrightarrow \infty$) will remain the same as at the beginning (at $t = 0$), a condition that is given by (25), whose left hand side gives the sum of the specific kinetic energy, the specific gravitational potential energy and the specific rotational energy, respectively. This sum is zero, and therefore, under the given initial condition, this must be the steady state corresponding to the minimum possible total specific energy of the system. Hence, this is the configuration that is dynamically and non-perturbatively selected. It is now conceivable that if the pressure term were to be taken into account, then the solution that would be dynamically selected, would be the one that would pass through the saddle point, since, as Bondi had analogously conjectured for spherically symmetric accretion [3], this would be the one to satisfy the criterion of minimum energy.

Interestingly enough, the issue of the primacy of transonicity can also be addressed from a very different, and somewhat unlikely, perspective. It has been mentioned earlier that carrying out a linearised perturbative analysis in real time on stationary flows will not indicate any particular solution to be favoured over all the others. This line of thinking may now be subjected to a closer scrutiny.

Under the condition of hydrostatic equilibrium in the vertical direction, the time-dependent generalisation of the governing equations for an axisymmetric pseudo-Schwarzschild disc is given by (21), as well as

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(\Sigma v r\right) = 0 \tag{26}$$

in which the surface density of the thin disc Σ, is to be expressed as $\Sigma \cong \rho H$ [24]. Making use of (4) and the polytropic relation $P = K\rho^{\gamma}$, (26) can be rendered as

$$\frac{\partial}{\partial t}\left[\rho^{(\gamma+1)/2}\right] + \frac{\sqrt{\phi^{\prime}}}{r^{3/2}}\frac{\partial}{\partial r}\left[\rho^{(\gamma+1)/2} v \frac{r^{3/2}}{\sqrt{\phi^{\prime}}}\right] = 0. \tag{27}$$

Defining a new variable $f = \rho^{(\gamma+1)/2} v r^{3/2}/\sqrt{\phi^{\prime}}$, it is quite obvious from the form of (27) that the stationary value of f will be a constant, f_0, which can be closely identified with the matter flux rate. This follows a similar approach to spherically symmetric flows established in earlier works [28, 29]. The present treatment, of course, pertains to a disc flow being driven by a general pseudo-Newtonian potential, $\phi(r)$. In this system, a perturbation prescription of the form $v(r,t) = v_0(r) + v^{\prime}(r,t)$ and $\rho(r,t) = \rho_0(r) + \rho^{\prime}(r,t)$, will give, on linearising in the primed quantities,

$$\frac{f^{\prime}}{f_0} = \left[\left(\frac{\gamma+1}{2}\right)\frac{\rho^{\prime}}{\rho_0} + \frac{v^{\prime}}{v_0}\right] \tag{28}$$

with the subscript 0 denoting stationary values in all cases. From (27), it then becomes possible to set down the density fluctuations $\rho^{\prime}$, in terms of $f^{\prime}$ as

$$\frac{\partial \rho^{\prime}}{\partial t} + \beta^2 \frac{v_0 \rho_0}{f_0}\left(\frac{\partial f^{\prime}}{\partial r}\right) = 0 \tag{29}$$

with $\beta^2 = 2(\gamma+1)^{-1}$, as before. Combining (28) and (29) will then render the velocity fluctuations as

$$\frac{\partial v^{\prime}}{\partial t} = \frac{v_0}{f_0}\left(\frac{\partial f^{\prime}}{\partial t} + v_0 \frac{\partial f^{\prime}}{\partial r}\right) \tag{30}$$

which, upon a further partial differentiation with respect to time, will give

$$\frac{\partial^2 v^{\prime}}{\partial t^2} = \frac{\partial}{\partial t}\left[\frac{v_0}{f_0}\left(\frac{\partial f^{\prime}}{\partial t}\right)\right] + \frac{\partial}{\partial t}\left[\frac{v_0^2}{f_0}\left(\frac{\partial f^{\prime}}{\partial r}\right)\right]. \tag{31}$$

From (21) the linearised fluctuating part could be extracted as

$$\frac{\partial v^{\prime}}{\partial t} + \frac{\partial}{\partial r}\left(v_0 v^{\prime} + c_{s0}^2 \frac{\rho^{\prime}}{\rho_0}\right) = 0 \tag{32}$$

with c_{s0} being the speed of sound in the steady state. Differentiating (32) partially with respect to t, and making use of (29), (30) and (31) to substitute for all the first and second-order derivatives of v' and ρ', will deliver the result

$$\frac{\partial}{\partial t}\left[\frac{v_0}{f_0}\left(\frac{\partial f'}{\partial t}\right)\right]+\frac{\partial}{\partial t}\left[\frac{v_0^2}{f_0}\left(\frac{\partial f'}{\partial r}\right)\right]+\frac{\partial}{\partial r}\left[\frac{v_0^2}{f_0}\left(\frac{\partial f'}{\partial t}\right)\right]+\frac{\partial}{\partial r}\left[\frac{v_0}{f_0}\left(v_0^2-\beta^2 c_{s0}^2\right)\frac{\partial f'}{\partial r}\right]=0 \quad (33)$$

all of whose terms can be ultimately rendered into a compact formulation that looks like

$$\partial_\mu\left(\mathrm{f}^{\mu\nu}\partial_\nu f'\right)=0 \quad (34)$$

in which the Greek indices are made to run from 0 to 1, with the identification that 0 stands for t, and 1 stands for r. An inspection of the terms on the left hand side of (33) will then allow for constructing the symmetric matrix

$$\mathrm{f}^{\mu\nu}=\frac{v_0}{f_0}\begin{pmatrix}1 & v_0\\ v_0 & v_0^2-\beta^2 c_{s0}^2\end{pmatrix}. \quad (35)$$

Now in Lorentzian geometry the d'Alembertian for a scalar in curved space is given in terms of the metric $\mathrm{g}_{\mu\nu}$ by [30, 31]

$$\Delta\psi \equiv \frac{1}{\sqrt{-\mathrm{g}}}\partial_\mu\left(\sqrt{-\mathrm{g}}\,\mathrm{g}^{\mu\nu}\partial_\nu\psi\right) \quad (36)$$

with $\mathrm{g}^{\mu\nu}$ being the inverse of the matrix implied by $\mathrm{g}_{\mu\nu}$. Comparing (34) and (36) it would be tempting to look for an exact equivalence between $\mathrm{f}^{\mu\nu}$ and $\sqrt{-\mathrm{g}}\,\mathrm{g}^{\mu\nu}$. This, however, cannot be done in a general sense. What can be appreciated, nevertheless, is that (34) gives an equation for f' which is of the type given by (36). The metrical part of (34), as given by (35), may then be extracted, and its inverse will incorporate the notion of a sonic horizon of an acoustic black hole when $v_0^2=\beta^2 c_{s0}^2$. This point of view does not make for a perfect acoustic analogue model, but it has some similar features to the metric of a wave equation for a scalar field in curved space-time, obtained through a somewhat different approach, in which the velocity of an an irrotational, inviscid and barotropic fluid flow is first represented as the gradient of a scalar function ψ, i.e. $\mathbf{v}=-\nabla\psi$, and then a perturbation is imposed on this scalar function [30, 31].

The discussion above indicates that the physics of supersonic acoustic flows closely corresponds to many features of black hole physics. This closeness of form is very intriguing, as well as instructive. For a black hole, infalling matter crosses the event horizon maximally, i.e. at the greatest possible speed. By analogy the same thing may be said of matter crossing the sonic horizon of a fluid flow. Indeed, this has been a long-standing conjecture for the case of spherically symmetric accretion on to a point sink [3, 5]. That this fact can be appreciated for the accretion problem through a perturbative result, as given by (33), is quite remarkable. This is because conventional wisdom would have it that one would be quite unable to have any understanding of the special status of any inflow solution solely through a perturbative technique [5]. It is the transonic solution that crosses the sonic horizon at the greatest possible rate, and the near similarity of form between (33) and (36) may very well be indicative of the primacy of the transonic solution. If such an insight were truly to be had with the help of the perturbation equation, then the perturbative linear stability analysis might not have been carried out in vain after all.

5. Concluding remarks

The difficulties against transonicity, arising from the choice of purely stationary equations, have been addressed through non-perturbative dynamics. This has been a relatively simple exercise to carry out in the Newtonian construct of space and time. It may be readily appreciated, however, that a rigorously general relativistic flow will not lend itself so easily to such a treatment as has been executed here for a pseudo-Newtonian system. Nevertheless, it should be helpful to note that particular pseudo-Newtonian systems [32] bear a very close resemblance to an actual general relativistic system [14]. This closeness is a reason to believe that the non-perturbative approach to the selection of transonicity demonstrated in this paper may also be very relevant for general relativistic flows.

As a final point, it may be mentioned that realistically speaking, a disc system should involve viscosity as a means of transporting angular momentum, to make infall a sustained global process. However, in that event, viscosity will also bring about dissipation of energy in the flow, something that will make the minimum energy criterion an ineffective instrument for identifying the transonic solution. Besides this, viscosity will violate Lorentzian invariance, which is very crucial for constructing an analogue gravity model. With the breakdown of the Lorentzian invariance, the quest for an acoustic analogue of a black hole becomes a most difficult one, especially when the fluid system under study is compressible in nature. For incompressible flows, on the other hand, some studies have considered the issue of an analogue gravity model vis-a-vis viscosity in a semi-quantitative manner. In analysing the outflow of a very shallow layer of water on a flat surface, it has been discussed that the formation of an abrupt hydraulic jump in the flow is entirely due to viscosity, and many features of the jump itself can be closely connected to an acoustic white hole [33, 34]. But more importantly the basic properties of surface gravity waves in a shallow layer of water remain unchanged in the vicinity of the jump, inspite of viscosity [35]. Going by this analogy, it is conceivable that the properties of an acoustic wave in a compressible flow (such as the accretion process is) will likewise remain unaffected. This argument, however, will not make it directly possible to establish precise relations like (34) and (35), which give a clear mathematical result to indicate that transonicity might be favoured. Yet transonicity will very likely be achieved because regardless of energy dissipation due to viscosity, the stationary matter flux rate will be conserved and the flow will continue to proceed at the greatest possible rate.

Acknowledgments

This research has made use of NASA's Astrophysics Data System. The authors acknowledge some useful discussions with Rajaram Nityananda and Paul J. Wiita.

Appendix

The model system that has been introduced here serves to show that non-realisable separatrices passing through the saddle point in the stationary regime, can behave like proper physical flows, when the dynamics, as opposed to the statics, is to be followed. The dynamics of the field $y(x,t)$ is described as

$$\frac{\partial y}{\partial t} + (y-x)\frac{\partial y}{\partial x} = y + 2x + x^2 \tag{A.1}$$

whose static limit leads to

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y+2x+x^2}{y-x} \tag{A.2}$$

and which, viewed as a dynamical system, is seen as

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = y + 2x + x^2$$
$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = y - x. \tag{A.3}$$

In the y — x space, the fixed points $(x_{\mathrm{c}}, y_{\mathrm{c}})$ are to be found at $(0,0)$ and $(-3,-3)$. A linear stability analysis of the fixed points in τ space gives the eigenvalues Ω, by $\Omega^2 = 1 + 2(x_{\mathrm{c}}+1)$. It is then easy to see that $(0,0)$ is a saddle point while $(-3,-3)$ is a centre-type point. This mathematical model is very much similar to the stationary inviscid accretion disc driven by the Newtonian potential, with two critical points, of which the outer one is a saddle point, while the inner one is centre-type. The integral curves of the model system are obtained from (A.2) as

$$y^2 - 2xy - 2x^2 - \frac{2}{3}x^3 = c \tag{A.4}$$

with the solutions passing through the saddle point being given by $c=0$.

To explore the temporal dynamics, and to obtain a solution to (A.1), it would be necessary to apply the method of characteristics [27]. This involves writing

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{y-x} = \frac{\mathrm{d}y}{y+2x+x^2}. \tag{A.5}$$

The subsequent task is to find two constants c_1 and c_2 from the above set and the general solution of (A.1) would then be given by $c_1 = F(c_2)$, where the function F is to be determined from the initial conditions. It is easy to see that one of the constants of integration is clearly the c of (A.4). Hence, writing $c_1 = c$, and using (A.4) in the first part of (A.5), will give

$$\int \mathrm{d}t = \pm \int \frac{\mathrm{d}x}{\sqrt{3x^2 + (2/3)\,x^3 + c}} \tag{A.6}$$

which solves the problem in principle. To put this in a usable form, the integration in (A.6) would have to be carried out. For small x (the most important region, since it is near the saddle), the x^3 term may be left out to a good approximation. Further, only the positive sign in the right hand side of (A.6) is to be chosen by the physical argument that the system is to evolve through a positive range of t (time) values. Integration of (A.6) will then lead to the result

$$\left(x + \sqrt{x^2 + c/3}\right) e^{-\sqrt{3}t} = c_2 \tag{A.7}$$

which will then make the solution of (A.1) look like

$$y^2 - 2xy - 2x^2 - \frac{2}{3}x^3 = F\left(\left[x + \sqrt{\frac{(y-x)^2}{3} - \frac{2}{9}x^3}\right] e^{-\sqrt{3}t}\right). \tag{A.8}$$

The evolution of the system is to be followed from the initial condition that $y = 0$ for all x at $t = 0$. Dropping the x^3 term again for small x, will allow for determining the form of the function F as $F(z) = -3(2-\sqrt{3})z^2$. The solution, consequently, becomes

$$y^2 - 2xy - 2x^2 - \frac{2}{3}x^3 = -3\left(2-\sqrt{3}\right)\left[x + \sqrt{\frac{(y-x)^2}{3} - \frac{2}{9}x^3}\right]^2 e^{-2\sqrt{3}t} \tag{A.9}$$

and as $t \longrightarrow \infty$, the steady solution that would be selected would be

$$y^2 - 2xy - 2x^2 - \frac{2}{3}x^3 = 0 \tag{A.10}$$

which is actually the equation for the separatrices. It is worth stressing the remarkable feature of this result. The evolution started under conditions far removed from transonicity. In fact, it started as $y = 0$ for all x (in the vicinity of the origin of coordinates) at $t = 0$. The evolution proceeded through a myriad of possible steady state solutions (all arguably stable under a linear stability analysis) and then in the stationary limit, selected the separatrices. This is a convincing demonstration that it is in principle possible for apparently non-realisable separatrices in the steady regime, to become eminently realisable physically, when the temporal evolution of the system is followed.

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