

Detection of gravitational waves from inspiraling compact binaries using a network of interferometric detectors*

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Abstract

We formulate the data analysis problem for the detection of the Newtonian waveform from an inspiraling compact-binary by a network of arbitrarily oriented and arbitrarily distributed laser interferometric gravitational wave detectors. We obtain for the first time the relation between the optimal statistic and the magnitude of the network correlation vector, which is constructed from the matched network-filter. This generalizes the calculation reported in an earlier work (gr-qc/9906064), where the detectors are taken to be coincident.

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I. INTRODUCTION

Inspiraling compact binaries form prime candidates for detection by earth-based interferometric gravitational-wave (GW) detectors owing to the well understood waveform (chirp) emitted by them. Searching for chirps using a network of such detectors is gaining importance due to (a) its superior sensitivity *vis a vis* that of a constituent detector [1] and (b) improving feasibility for a real-time computational search. Here, we formulate the problem of how to optimally detect the Newtonian chirp using a network of arbitrarily orientated and arbitrarily located detectors. This extends a similar study in Ref. [2] of a network with coincident detectors.

We use the maximum likelihood method for optimizing the detection problem.[3] A single likelihood ratio (LR) is deduced for the entire network. A super-threshold value for the maximized likelihood ratio (MLR) implies a detection. The MLR is obtained by maximizing the LR over the eight parameters that determine the Newtonian chirp: the distance r to the binary, the initial phase δ of the waveform, the polarization angle ψ , the inclination angle ϵ of the binary orbit, the time of arrival t_a at a fiducial detector (fide), the source-direction angles $\{\phi, \theta\}$, and the chirp time ξ . In principle, this can always be done numerically using a grid in the eight dimensional parameter space. In practice, however, such a strategy is computationally unfeasible and wasteful. We show that maximization of the LR over four parameters, $\{r, \delta, \psi, \epsilon\}$, can be performed analytically using the symmetries in detector responses. This allows us to scan this parameter subspace *continuously*. Further, the Fast Fourier Transform (FFT) can be used to maximize LR over t_a , as in the case of a single detector. Such an analytic maximization and the FFT allow us to save substantially on computational costs. Numerical maximization is required over the remaining parameters, $\{\phi, \theta, \xi\}$, which we discuss in a future work. Here, we follow the convention laid out in Ref. [2].

II. THE SIGNAL

There are four distinctly different reference frames of interest, associated with the source, wave, fide, and a representative detector in the network. Physical quantities in these frames are related by orthogonal transformations, \mathcal{O}_k , which are defined in terms of three sets of Euler angles that specify the orientation of one frame with respect to another.[4] Let \mathbf{x} be an arbitrary three-dimensional real vector. Then, $\mathbf{x}_{\text{wave}} = \mathcal{O}_1(\psi, \epsilon, 0)\mathbf{x}_{\text{source}}$, $\mathbf{x}_{\text{fide}} = \mathcal{O}_2(\phi, \theta, 0)\mathbf{x}_{\text{wave}}$, and $\mathbf{x}_{\text{detector}} = \mathcal{O}_3^{-1}(\alpha, \beta, \gamma)\mathbf{x}_{\text{fide}}$. Here, the source axes have been chosen in accordance with Ref. [5].

The wave tensor w_{ij} associated with any source can be expanded in terms of the STF-2 tensors \mathcal{Y}_{2n}^{ij} in an arbitrary frame as [2]:

$$w^{ij}(t) = \sqrt{\frac{2\pi}{15}} \left[(h_+(t) - ih_\times(t)) T_2^n \mathcal{Y}_{2n}^{ij} + (h_+(t) + ih_\times(t)) T_{-2}^n \mathcal{Y}_{2n}^{ij} \right] , \quad (2.1)$$

where i, j denote spatial indices, and h_+ and h_\times are the two GW polarizations in the transverse-traceless gauge, as measured in some given frame. The expansion coefficients $T_{\pm 2}^n$ are the Gel'fand functions,[2] which depend on the Euler angles through which one

must rotate that frame into the frame in which w^{ij} is being analyzed. The above form suggests the definitions, $e_L^{ij} = \sqrt{8\pi/15} T_2^n \mathcal{Y}_{2n}^{ij}$ and $e_R^{ij} = \sqrt{8\pi/15} T_{-2}^n \mathcal{Y}_{2n}^{ij}$, for the left- and right-circular polarization tensors, respectively. They obey, $e_L^{ij*} = e_R^{ij}$, $e_{L,R}^{ij} e_{L,R}^{*ij} = 1$, and $e_{L,R}^{ij} e_{R,L}^{*ij} = 0$, in any frame. Thus,

$$w^{ij}(t) = \text{Re} \left[(h_+(t) + ih_\times(t)) e_R^{ij} \right] \equiv 2\kappa \text{Re} \left[R(t) e_R^{ij} \right] , \quad (2.2)$$

where $\kappa = \sqrt{\xi}/r$ (up to a normalization factor) and $R \equiv (h_+(t) + ih_\times(t))/(2\kappa)$. For a chirp, we define R in the source frame.[5] Then $h_{+,\times}$ are GW amplitudes for a face-on binary (i.e., for $\epsilon = 0$), and R depends only on $\{\delta, t_a, \xi\}$.

The response amplitude (i.e., the signal) in the I -th detector is the scalar product $s^I = w^{ij} d_{ij}^I$, which depends on projections of $e_{L,R}^{ij}$ onto the I -th detector tensor, d_{ij}^I . One such projection defines the extended beam-pattern function:

$$F^I = e_L^{ij} d_{ij}^I \equiv T_2^p(\psi, \epsilon, 0) D_p^I , \quad p = \pm 2 , \quad (2.3)$$

which corresponds to the left-circular polarization. Above,

$$D_p^I \equiv \sqrt{\frac{8\pi}{15}} T_p^n(\phi, \theta, 0) d_{ij}^I \mathcal{Y}_{2n}^{ij} = ig^I T_p^n(\phi, \theta, 0) (T_{2n}^{I*} - T_{-2n}^{I*}) , \quad (2.4)$$

where $T_{\pm 2n}^I = T_{\pm 2n}(\alpha_I, \beta_I, \gamma_I)$ and $d_{ij}^I = g^I (n_{1i}^I n_{1j}^I - n_{2i}^I n_{2j}^I)$, with $\mathbf{n}_{1,2}^I$ being unit vectors along the two arms of the I -th interferometer, respectively. Also, g^I is the detector's noise power spectral density.[6] Then,

$$s^I(t) = 2\kappa \text{Re} (F^{I*} R^I) \equiv 2\kappa \text{Re} (F^{I*} S^I e^{i\delta}) \quad (2.5)$$

where R^I is defined via Eq. (2.2) and S^I is independent of δ .

III. THE OPTIMAL NETWORK STATISTIC

Under the Neyman-Pearson decision criterion,[3] the optimal network statistic is the network LR, λ . If the noise in each detector is additive and independent of the noise in any other detector in the network, then λ reduces to a product of the individual detector LR's.[2] Further, for Gaussian noise,[7] the logarithmic likelihood ratio (LLR), $\ln \lambda$, simplifies to the following sum of LLR's of N individual detectors [2]:

$$\ln \lambda = \sum_{I=1}^N \langle s_I, x_I \rangle_I - \frac{1}{2} \sum_{I=1}^N \langle s_I, s_I \rangle_I = \mathbf{b} \sum_{I=1}^N \langle z_I, x_I \rangle_I - \frac{1}{2} \mathbf{b}^2 , \quad (3.1)$$

where $\mathbf{b} \equiv 2\kappa (\sum_{I=1}^N \|F_I\|^2)^{1/2}$ and $z_I = s_I/\mathbf{b}$. Above, r appears only in \mathbf{b} .

Maximizing $\ln \lambda$ with respect to \mathbf{b} and δ gives, $\ln \lambda|_{\hat{\mathbf{b}}, \hat{\delta}} = \left| \sum_{I=1}^N Q_I C_I^* \right|^2 / 2$, where $Q_I \equiv 2\kappa F_I/\mathbf{b}$ and $C_I^* \equiv \langle S_I, x_I \rangle_I$. [2] This shows that the network vector \mathbf{S} , with S_I 's as its components, is the matched network-filter. Also, $\ln \lambda|_{\hat{\mathbf{b}}, \hat{\delta}}$ is a function of six parameters,

namely, $\{\psi, \epsilon, t_a, \phi, \theta, \xi\}$. To extend these results to the case where the detectors are arbitrarily located, note that the dependence of $\ln \lambda|_{\hat{\mathbf{b}}, \hat{\delta}}$ on $\{\psi, \epsilon\}$ can be isolated. This is because the network vector \mathbf{Q} , with Q^I 's as its components, is:

$$\mathbf{Q} = \|\mathbf{F}\|^{-1} \left(T_2^{-2}(\psi, \epsilon, 0) \mathbf{D}_{-2} + T_2^2(\psi, \epsilon, 0) \mathbf{D}_2 \right) \equiv Q^{-2} \hat{\mathbf{D}}_{-2} + Q^2 \hat{\mathbf{D}}_2 \quad , \quad (3.2)$$

where \mathbf{D}_p define network vectors with D_p^I as their components; $\hat{\mathbf{D}}_p$ are their normalized counterparts. Thus, $Q_2 = \hat{\mathbf{D}}_2 \cdot \mathbf{Q} = Q^{+2} + Q^{-2} \hat{\mathbf{D}}_2 \cdot \hat{\mathbf{D}}_{-2}$. Hence, $\{\hat{\mathbf{D}}_2, \hat{\mathbf{D}}_{-2}\}$ define a two-dimensional complex plane, \mathcal{P} (the helicity space, a subspace of \mathcal{C}^N), on which a metric g_{pq} can be defined. Then, $Q_p = g_{pq} Q^q$ with $p, q = \pm 2$. The N -dimensional correlation vector \mathbf{C} , in general, lies outside \mathcal{P} . However, \mathbf{Q} lies totally in \mathcal{P} . Thus, the statistic reduces to

$$\ln \lambda|_{\hat{\mathbf{b}}, \hat{\delta}} = |\mathbf{Q} \cdot \mathbf{C}^*|^2 / 2 = |\mathbf{Q} \cdot \mathbf{C}_{\mathcal{P}}^*|^2 / 2 \quad , \quad (3.3)$$

where $\mathbf{C}_{\mathcal{P}}$ is the projection of \mathbf{C} onto the helicity space.

Maximization of $\ln \lambda|_{\hat{\mathbf{b}}, \hat{\delta}}$ over $\{\psi, \epsilon\}$ is achieved by aligning \mathbf{Q} along $\mathbf{C}_{\mathcal{P}}$. This requires that $Q^{\pm 2} = C_{\mathcal{P}}^{\pm 2} / \|\mathbf{C}_{\mathcal{P}}\|_{\mathcal{P}}$, which implies: $Q^{+2}/Q^{-2} = C_{\mathcal{P}}^{+2}/C_{\mathcal{P}}^{-2}$. Since the RHS above can take any value in the complex plane, we need to prove that Q^{+2}/Q^{-2} obeys the same property. To this end, note that,

$$\frac{Q^{+2}}{Q^{-2}} = \frac{T_2^{-2}(\psi, \epsilon, 0)}{T_2^2(\psi, \epsilon, 0)} = \left(\frac{1 - \cos \epsilon}{1 + \cos \epsilon} \right)^2 \exp(4i\psi) \quad , \quad (3.4)$$

which can indeed attain any value on the Argand plane. The values of ψ and ϵ that maximize the statistic are, $\hat{\psi} = \arg(x)/4$ and $\hat{\epsilon} = \cos^{-1}[(1 - \sqrt{\|x\|})/(1 + \sqrt{\|x\|})]$ where $x \equiv C_{\mathcal{P}}^{+2}/C_{\mathcal{P}}^{-2}$. Thus, the statistic maximized over these four parameters is,

$$\ln \lambda|_{\hat{\mathbf{b}}, \hat{\delta}, \hat{\psi}, \hat{\epsilon}} = \|\mathbf{C}_{\mathcal{P}}\|^2 / 2. \quad (3.5)$$

Let \hat{V}^{\pm} denote a pair of orthonormal, complex basis vectors on \mathcal{P} . Then,

$$\|\mathbf{C}_{\mathcal{P}}\|^2 = \|C^+\|^2 + \|C^-\|^2 = (c_0^+)^2 + (c_{\pi/2}^+)^2 + (c_0^-)^2 + (c_{\pi/2}^-)^2 \quad , \quad (3.6)$$

where $C^{\pm} = \mathbf{C}_{\mathcal{P}} \cdot \hat{V}^{\pm} = c_0^{\pm} + ic_{\pi/2}^{\pm}$. It can be verified that the network statistic is, therefore, a sum of the squares of four Gaussian random variables with constant variance. This simplifies the computation of thresholds and detection probabilities.

A network filter is constructed as follows: For a given ξ compute the Newtonian chirp for $t_a = 0$. Then, for a given direction $\{\phi, \theta\}$, use the appropriate time-delays with respect to fide to time-displace the chirp at each detector. This collection of time-displaced chirps constitute the network filter. Also, \hat{t}_a is obtained by shifting the network filter 'rigidly' on the time axis, which can be done efficiently using FFT. The bank of filters on $\{\phi, \theta, \xi\}$ can be obtained by correlating two neighboring normalized filters in the usual way. This work is now in progress.

IV. CONCLUSION

We have given here a formulation to optimally detect the Newtonian chirp with a network of detectors in which the noise is additive, Gaussian, and uncorrelated between detectors. We have shown how this can be done efficiently by analytically maximizing the LR over four parameters and using FFT to maximize over the time-of-arrival. In a future work we hope to address key issues such as required computational power for such a search and also estimate errors in parameter values.

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