

Quantum conformal fluctuations in a singular space-time

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The cosmological solutions of Einstein's general relativistic equations lead inevitably to space-time singularities¹. However, general relativity is only an approximation to a fully quantized theory of gravity and we need to consider whether singularity persists in the quantum domain. Although a full quantum theory of gravity has not yet been developed, we show here that the above question can be tackled in a simplified model where only the conformal degree of freedom is quantized. Previous applications of this technique had shown that in specific cases the quantum conformal fluctuations (QCF) from the classical solutions diverge at the classical singularity, thus rendering the classical solution physically meaningless^{2,3}. Recently one of us (J.V.N. ref. 4) has generalized this result to cover all dust cosmologies. Here we show that this conclusion is applicable to even more general types of cosmological singularities.

To arrive at this result we draw on the mathematical details of previous work; in particular, we make extensive use of the formalism developed in ref. 4.

Classical equations of general relativity are derived from the variation of the action S given by

$$S = \frac{1}{16\pi} \int R \sqrt{-g} d^4x + \int \mathcal{L}_m \sqrt{-g} d^4x \quad (1)$$

We have taken $c = 1$ and $G = 1$. As we are discussing quantum phenomena we will also take $\hbar = 1$, thus making the Planck length the basic unit. \mathcal{L}_m is the lagrangian density for matter.

In the thin sandwich problem⁵ of classical gravity, if data are given on initial and final hypersurfaces Σ_1 and Σ_2 , $\delta S = 0$ determine the geometry of the space-time sandwiched between Σ_1 and Σ_2 . The initial and final data are specified by 3-geometries $(^3)\mathcal{G}_1$ and $(^3)\mathcal{G}_2$. In the quantum problem there is no unique sequence of 3-geometries leading from Σ_1 to Σ_2 . Instead we have a probability amplitude $K[(^3)\mathcal{G}_2, \Sigma_2; (^3)\mathcal{G}_1, \Sigma_1]$ for the system to have the specified initial and final data. The kernel K is formally written as the Feynman functional integral

$$K[(^3)\mathcal{G}_2, \Sigma_2; (^3)\mathcal{G}_1, \Sigma_1] = \int \exp iS[\mathcal{G}] \mathcal{D}\mathcal{G} \quad (2)$$

where $S[\mathcal{G}]$ is the action for an arbitrary geometry \mathcal{G} which satisfies the given boundary conditions. If we are considering only the conformal degrees of freedom we have to take all geometries given by the metrics

$$g_{ik} = \Omega^2 \bar{g}_{ik} \quad (3)$$

where Ω is a function of space and time ($i = 0, 1, 2, 3$), while \bar{g}_{ik} is the metric tensor describing the solution of the classical problem (all physical quantities pertaining to the classical solution carry an overbar).

Using a time coordinate t to label the spacelike hypersurfaces Σ_1, Σ_2 as t_1, t_2 we may rewrite equation (2) as

$$K[\Omega_2, t_2; \Omega_1, t_1] = \int \exp iS[\Omega] \mathcal{D}\Omega \quad (4)$$

with

$$S[\Omega] = \frac{1}{16\pi} \int (\bar{R}\Omega^2 - 6\Omega_t \Omega') \sqrt{-\bar{g}} d^4x + \int \mathcal{L}_m \sqrt{-g} d^4x \quad (5)$$

The lack of a unique space-time geometry in the quantum regime may also be expressed through a wavefunction Ψ which 'propagates' with the kernel K :

$$\psi(\Omega_2, t_2) = \int K[\Omega_2, t_2; \Omega_1, t_1] \psi(\Omega_1, t_1) \mathcal{D}\Omega_1 \quad (6)$$

A physical interpretation can be given to the above relation by

choosing a Gaussian wavepacket form for $\psi(\Omega_1, t_1)$ and then studying its dispersion as the final hypersurface Σ_2 approaches the singularity. For the examples discussed in refs 2 and 3, it was shown that while the 'mean' of the wavepacket stays on the classical solution $\Omega = 1$, its dispersion diverges as the classical singularity is approached.

It was shown in ref. 4 that the above results were in fact part of a general result which was valid for all singular space-times containing only dust. By a similar analysis it can be shown that whenever $S[\Omega]$ is a quadratic functional in Ω and its derivatives, the kernel can be written as

$$K[\Omega_2, t_2; \Omega_1, t_1] = F(t_2, t_1) \exp [i\bar{S}] \quad (7)$$

where $F(t_2, t_1)$ depends only on t_2 and t_1 , and

$$\bar{S} = \sum_{P=1}^2 \sum_{Q=1}^2 \frac{3}{8\pi} \iint A_{PQ}(\mathbf{x}_P, t_P; \mathbf{x}'_Q, t'_Q) \times \Omega_P(\mathbf{x}_P) \Omega_Q(\mathbf{x}'_Q) d^3\mathbf{x}_P d^3\mathbf{x}'_Q \quad (8)$$

The coefficients A_{11} and A_{22} are given by certain integrals involving the retarded Green's function $G(\mathbf{x}_2, t_2; \mathbf{x}_1, t_1)$ of a scalar wave equation determined by the background classical geometry and the source. In the dust case discussed in ref. 4, the scalar wave operator is simply $\square + \bar{R}/6$.

The interesting part of \bar{S} in equation (8) is, however, the cross term containing A_{12} , which retains the memory of the initial wavefunction at $t = t_1$. The coefficient A_{12} is given by

$$A_{21}(\mathbf{x}_2, t_2; \mathbf{x}_1, t_1) = A_{12}(\mathbf{x}, t_1; \mathbf{x}_2, t_2) = [G(\mathbf{x}_2, t_2; \mathbf{x}_1, t_1)]^{-1} \quad (9)$$

Suppose now that G diverges as the space-time singularity is approached. We then have $A_{12} \rightarrow 0$ and a decoupling of the final wavefunction from the initial one. This is precisely the state of complete uncertainty of the final state represented in the wavepacket case by infinite dispersion.

In ref. 4 the divergence of G at the singularity was deduced from the conformal invariance of the operator $\square + \bar{R}/6$. However, the result also seems to be valid for other cases. For example, it was shown recently⁶ that the QCF diverge for a Robertson-Walker universe containing a massless scalar field. The QCF also diverge at the singularity of the empty Kasner universe⁷. These results suggest that we should look for a proof of the divergence of G in a fairly general scenario as follows.

Belinskii *et al.*⁸ argued that a general approach to the space-time singularity can be described by a succession of Kasner type solutions, in which the Universe contracts along two axes and expands along the third. (We have reversed the direction of time so that $t \leq 0$ and the final singular state is described by $t = 0$.) Without going into detail we will simply accept the claim that number, 8, of arbitrary functions in the Belinskii *et al.*⁸ solution makes it a general solution of Einstein's equations.

Belinskii *et al.*⁸ have argued that near the space-time singularity the contributions from \mathcal{L}_m can be neglected and the wave operator for our Green's function then becomes simply \square . We therefore have to show that in the space-time of their model the Green's function of the wave equation

$$\square \bar{G}(\mathbf{x}_2, t_2; \mathbf{x}_1, t_1) = [-\bar{g}(\mathbf{x}_1, t_1)]^{-1/2} \delta_4(\mathbf{x}_2, t_2; \mathbf{x}_1, t_1) \quad (10)$$

diverges as $t_2 \rightarrow 0$.

The line element of the Belinskii *et al.* model is given by

$$ds^2 = dt^2 - \gamma_{\alpha\beta} dx^\alpha dx^\beta, \quad \alpha, \beta = 1, 2, 3 \quad (11)$$

$$\gamma_{\alpha\beta} = \sum_{\mu=1}^3 |t|^{2p_\mu} l_{\mu\alpha} l_{\mu\beta} \quad (12)$$

where $p_\mu, l_{\mu\alpha}$ are functions of space coordinates. As in the Kasner universe, the p_μ satisfy

$$\sum_{\mu=1}^3 p_\mu = \sum_{\mu=1}^3 p_\mu^2 = 1 \quad (13)$$

Simple algebra shows that close to the singularity the Green's function is given by

$$G(\mathbf{x}_2, t_2; \mathbf{x}_1, t_1) \equiv \frac{\delta_3(\mathbf{x}_2, \mathbf{x}_1)}{f(\mathbf{x}_2)} \ln \left| \frac{t_1}{t_2} \right| \quad 0 > t_2 > t_1 \quad (14)$$

while $G=0$ otherwise. The function f depends on the space coordinates. We are interested in the behaviour of the Green's function as $t_2 \rightarrow 0$, the singular epoch.

From equation (14) we have $G \rightarrow \infty$ logarithmically as $t_2 \rightarrow 0$, leading to divergence of QCF. Matter terms are unimportant in the final stages of the Belinskii *et al.* solution so our conclusion should hold for general distributions of matter and energy.

Although we have quantized only the conformal degree of freedom, the fact that the QCF diverge at the classical singularity is sufficient to demonstrate that quantum uncertainty becomes so large near the classical singular epoch that the classical singularity ceases to have any significance. In analogy with the example of the hydrogen atom can we therefore expect explicit examples of quantum behaviour in the very early Universe? Recent investigations^{9,10} have shown that closed Friedmann models have stationary states and that the Planck length sets the lower limit to the 'size' of the Universe. Whether this is a general feature of quantum cosmology remains to be seen.

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Individual electrons detected after the interaction of ionizing radiation with gases

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The interaction of ionizing radiation (X rays or high-energy particles) with gases results in the creation of electron-ion pairs. For low-energy X rays a small cluster of primary ionization is produced close to the point of the initial photoionization. High-energy particles, however, lose a small fraction of their energy in each of a number of ionizing collisions resulting in a long track composed of clusters of ionization. We report here that the light signals emitted by a parallel-plate proportional counter are fast enough to allow the individual electrons in the ionization clusters to be resolved. By counting the number of electrons for low energy X ray events we demonstrate an improvement in energy resolution of more than a factor of two over conventional proportional counters.

The mean number of electrons, \bar{n} , created by the photoelectric absorption of X rays in a gas is a function of the X-ray energy, E , and the mean energy required to create an ion pair, W , such that $\bar{n} = E/W$ (where $W \sim 26$ eV in Ar). In conventional gas proportional counters the electrons produced are accelerated by a strong electric field such that a Townsend charge avalanche results. At a constant mean charge gain, \bar{A} , the charge signal from the proportional counter is directly proportional to the X-ray energy. The charge signal pulse amplitude distribution which is observed results from the variation in n and in A . Fano¹ pointed out that the standard deviation in n is $\sigma_n = (F\bar{n})^{1/2}$, where F (the Fano factor) is ~ 0.17 for Ar. The total fractional deviation in the charge pulse amplitude, σ_N/\bar{N} , is given by,

$$\frac{\sigma_N}{\bar{N}} = \left[\left(\frac{\sigma_n}{\bar{n}} \right)^2 + \frac{1}{\bar{n}} \left(\frac{\sigma_A}{\bar{A}} \right)^2 \right]^{1/2} \quad (1)$$

where \bar{N} is the mean number of final electron-ion pairs, and

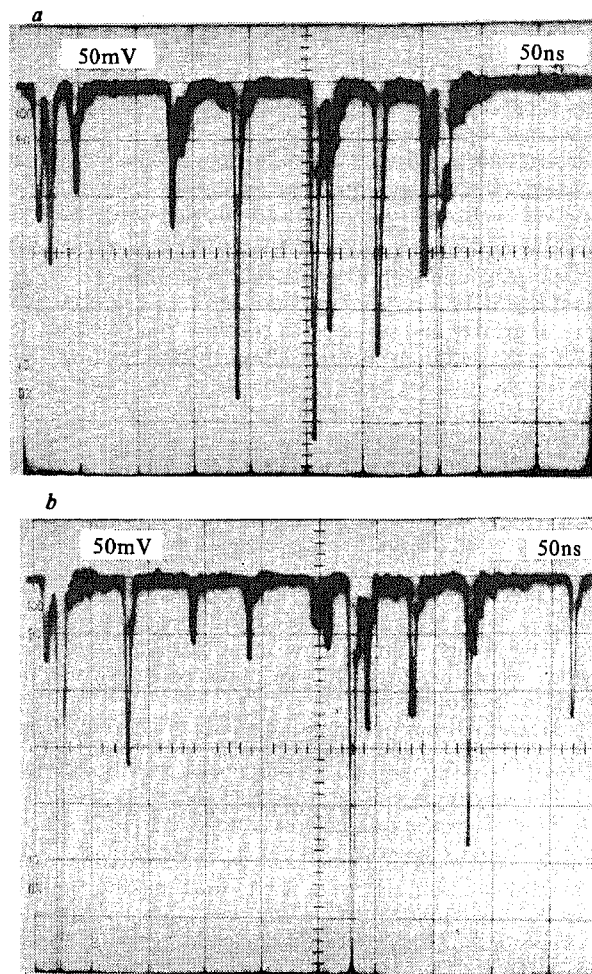


Fig. 1 Traces of the light pulses for two C-K X-ray events. The individual peaks correspond to avalanches initiated by single electrons. The gas used was Ar-CH₄(5%)-CO₂(5%)-N₂(7.5%), with a drift region field strength of ~ 10 V cm⁻¹ and a gas gain of 10^6 .

σ_A/\bar{A} is the fractional deviation in the charge gain. We may write $f = (\sigma_A/\bar{A})^2$ for the variance in A (f is typically² 0.6–0.7 when $A = 10^4$ – 10^5). Rewriting equation (1) in the form $\sigma_N/\bar{N} = [(F+f)W/E]^{1/2}$, it is clear that the elimination of f would result in a factor of more than two improvement in energy resolution. This is accomplished in the gas scintillation proportional counter^{3,4}. The same result may be achieved by counting the individual initial electrons.

In the approach described here the light pulses emitted by Townsend avalanches in a parallel plate proportional counter⁵ are detected by a fast photomultiplier tube (PMT, EMI 9818 QBM, ~ 2 ns rise time). The PMT views the 1 mm-deep avalanche region through an optically transparent anode deposited on a fibre optic substrate^{5,6}. The gas mixture used was Ar-CH₄(5%)-CO₂(5%)-N₂(7.5%) at a pressure of 760 torr. Ar and N₂ in the mixture are responsible for the light emission. Ar emits in the range 7,000–8,000 Å and N₂ emits in the range 3,000–4,000 Å (from the second positive band of N₂), with an optimum light emission attained for of the order of 10% N₂ (ref. 6). CH₄ and CO₂ are used in the gas mixture to inhibit secondary electron emission processes, thus enabling high charge gains to be achieved, and to reduce the lateral diffusion of the electron cluster so that acceptable position resolution may be obtained in applications requiring imaging^{6,7}. Addition of CO₂ also reduces the electron drift velocity⁸ (A. J. F. Den Boggende and C. J. Schrijver, personal communication). Fairly low concentrations (5%) of CH₄ and CO₂ were used as these gases also suppress the light emission⁶.

For an X-ray event the overall duration of the light signal is set by the time taken for the initial electron cluster to cross the grid