

Heat Flow in General Relativity

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Abstract :

A few models of bounded spheres containing heat flux in the radial direction are constructed. These models satisfy the conditions of fit at their boundaries with Vaidya's radiating metric. Further the causal heat transport equation is considered in the background of one of the above metrics and it is found to be consistent with the inflationary expansion. Our case is more general than that discussed previously by Maartens et al.

1. Introduction :

One of the interesting problems of gravitational collapse of stars is to study the solutions of various spherically symmetric fluid distributions with radial heat flux within the interior. Such study gives more general picture of the dynamics of gravitating objects than what is described in a simple adiabatic fluid.

It was first shown by Glass [1] that such exact solutions for heat conducting shearfree fluid distribution could be obtained by solving the differential equation arising out of the isotropy of pressure which of course, would hold irrespective of the presence of heat flux or not. Many such solutions were previously obtained by Bergmann [2], Modak [3], Maiti [4], Banerjee et. al [5], Dadhich and Patel [6]. The last one is, however, a nonsingular solution with the suitable choices of an arbitrary function of time. Some of the solutions mentioned above reduce to the Robertson – Walker cosmological solutions when the heat flux vanishes. Recently Tikekar [7] found another example of a nonsingular solution with heat flux.

It is now interesting to construct suitable models for bounded spheres which contain radial heat flux inside and at the same time satisfy the conditions of fit at the boundary with the radiating vaidya metric in the exterior. This is done in the section 2 of the present paper. The boundary conditions were first discussed by Santos [8] for a nonviscous fluid sphere conducting heat in the radial direction with the use of the continuity of the first and the second fundamental forms on the surface. It was pointed out that the pressure would not vanish at the boundary, it would rather be related with the heat flux in the form $p = (q, q^{-1})^{1/2}$. We have used this matching conditions to find explicit solutions for the time dependent parameters appearing in the solutions of Modak, Maiti and Tikekar mentioned earlier to give a few examples of heat conducting nonviscous fluid spheres.

In the section 3 the causal heat flow equation is considered, for simplicity we have used the truncated form of this equation (Israel [9], Hiscock and

Lindholm [10]) in Maiti's form of metric and have noted that the scale factor of this inhomogeneous model here admits inflationary expansion. It is a kind of generalization of the work of Maartens et al [11], who used one of the simplest solutions given by Modak. Our model is more general in the sense that it reduces to the Robertson-Walker metric with nonzero spatial curvature for vanishing heat flux, while that of Maartens et al reduces to the Robertson-Walker space time having zero spatial curvature in the same situation.

2. Models matching with the Vaidya metric at the boundary [3,4,5,7]

The energy momentum tensor for a heat conducting fluid is given by

$$T^{\mu}_{\gamma} = (\rho + p) v^{\mu} v_{\gamma} + p \delta^{\mu}_{\gamma} + q^{\mu} v_{\gamma} + v^{\mu} q_{\gamma}$$

q^{μ} is the heat flux vector which is spacelike and $q_{\mu} v^{\mu} = 0$.

We use comoving coordinates for the fluid that is the $v^{\mu} = \left(-\frac{1}{g_{00}}\right)^{1/2} \delta_0^{\mu}$

Example 1 : $ds^2 = - (1 + \xi(t) r^2)^2 + R^2(t) (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2)$ (2.1)

The above solution previously obtained by Modak is further specialised by assuming $\xi = \xi_0 = \text{constant}$. The matching condition at the boundary surface $r = r_0$ is given by Santos

$$p_{\Sigma} = (g_{11}^{-1/2} q^1)_{\Sigma}, \quad (2.2)$$

which yields a differential equation

$$2 R \ddot{R} + \dot{R}^2 + m \dot{R} = n \quad (2.3)$$

where $m = 4\xi_0 r_0$ and $n = 4\xi_0 (1 + \xi_0 r_0^2)$

The general solution of (2.3) is not available in closed form. We can however, consider a very simple solution for instance $R(b) = Ct$, C being an arbitrary constant. By suitable choices of constant parameters the density ρ and the

pressure p are found to be positive, which are infinitely large as $t \rightarrow 0$ and decrease indefinitely as $t \rightarrow \infty$. Here the constant C is given by

$$C = \frac{1}{2} [-m \pm (m^2 + 4n)^{1/2}] \quad (2.4)$$

Example – 2 :

$$ds^2 = - (1 + \xi_0 r^2)^2 dt^2 + \frac{R^2(t)}{(1 + \xi_0 r^2)^6} (dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) \quad (2.5)$$

The boundary conditions in this case lead to the same differential equation (2.3) with different values of the constant parameters and the density, pressure have similar behaviour as seen in the previous case.

Example – 3 :

$$ds^2 = - (1 + \alpha r^2) dt^2 + R^2(t) \left(\frac{1 + 2\alpha r^2}{1 + \alpha r^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right) \quad (2.6)$$

This is a spherically symmetric model with heat flow obtained by Tikekar in the context of singularity free cosmology. It has regular density and pressure and other kinematic scalars for a particular choice of the function $R(t)$. Our aim is to find a suitable expression for $R(t)$ which represents a bounded fluid distribution along with radial heat flux. The matching condition at the boundary again give the same differential equation (2.3) with

$$m = - \frac{2\alpha r_0}{(1+2\alpha r_0^2)^{1/2} (1 + \alpha r_0^2)^{1/2}} \quad \text{and} \quad n = \frac{\alpha (1 + \alpha r_0^2)}{(1 + 2\alpha r_0^2)}$$

and we have the solution in a special case $R(t) = Ct$.

The density, pressure may be expressed as

$$8\pi\rho = \frac{\alpha (3 + 2\alpha r^2)}{(1+2\alpha r^2)^2} - \frac{1}{R^2} + \frac{3}{(1+\alpha r^2)} - \frac{\dot{R}^2}{R^2} \quad (2.7)$$

$$8\pi\rho = \frac{\alpha}{(1+2\alpha r^2)^2} \frac{1}{R^2} + \frac{1}{(1+\alpha r^2)} \left(\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} \right) \quad (2.8)$$

It is evident from (2.7) & (2.8) that the density and pressure diverge as $t \rightarrow 0$. The density ρ is always positive and the pressure remains positive if we choose $\alpha \geq 2C^2$.

Example – 4 :

The metric form we consider in this example is a special case of Maiti's metric mentioned earlier. It is as follows :-

$$ds^2 = \left(1 + \frac{a}{1+\xi_0 r^2}\right)^2 dt^2 - \frac{R^2(t)}{(1+\xi_0 r^2)^2} (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) \quad (2.9)$$

Here 'a' and ξ_0 both are constants.

This case is discussed in more details because we consider it in the background of causal heat flow equation in the next section. The density ρ , the pressure p , the heat flow vector q^1 and the expansion scalar θ are given by

$$8\pi\rho = \frac{12\xi_0}{R^2} + \frac{3\ddot{R}^2}{R^2} \left(1 + \frac{a}{1+\xi_0 r^2}\right)^{-2} \quad (2.10)$$

$$8\pi p = \frac{-4\xi_0}{R^2} - \frac{4a\xi_0}{R^2} \frac{(1-\xi_0 r^2)}{(1+\xi_0 r^2)} \left(1 + \frac{a}{1+\xi_0 r^2}\right)^{-1} - \left(\frac{2R}{R} + \frac{\dot{R}^2}{R^2}\right) \left(1 + \frac{a}{1+\xi_0 r^2}\right)^{-2} \quad (2.11)$$

$$8\pi q^1 = \frac{4a\xi_0 \dot{R}r}{R^3} \left(1 + \frac{a}{1+\xi_0 r^2}\right)^{-2} \quad (2.12)$$

$$\theta = v_{;\alpha}^{\alpha} = \frac{3\dot{R}}{R} \left(\frac{1+a+\xi_0 r^2}{1+\xi_0 r^2} \right) \quad (2.13)$$

The condition of fit at the boundary as mentioned earlier yields again the same differential equation.

$$2R\ddot{R} + \dot{R}^2 + m\dot{R} = n$$

$$\text{where } m = \frac{4a\xi_0 r_0}{(1 + \xi_0 r_0^2)} \quad \text{and} \quad n = - \frac{4\xi_0 (1 + a + \xi_0 r_0^2)}{(1 + \xi_0 r_0^2)} [1 + 2a + \xi_0 r_0^2 (1-a)]$$

.....(2.14)

In principle here also one simple solution will be $R(t) = ct$. It is possible to make $n = 0$ and assume the constant 'a' negative and find a suitable solution for $R(t)$, which is different from that found in each previous case and at the same time yield positive density and pressure for the fluid.

We write $a = -A^2$ and assume $A^2 < 1$ in order to avoid the divergence of density and pressure within the sphere $0 \leq r \leq r_0$.

$$\text{We now write } -m = \frac{4A^2 \xi_0 r_0}{(1 + \xi_0 r_0^2)} = b^2 \quad \text{and put } n = 0, \text{ the}$$

boundary condition reduces to the equation.

$$2R\ddot{R} + \dot{R}^2 - b^2\dot{R} = 0$$

The assumption $n = 0$ requires

$$\xi_0 r_0^2 = \frac{2A^2 - 1}{A^2 + 1} \quad (2.16)$$

which again imposes another restriction on A^2 for example $A^2 > \frac{1}{2}$. So we have both upper and lower limits to A^2 expressed in the form

$$\frac{1}{2} < A^2 < 1 \quad (2.17)$$

The solution of the equation (2.15) may be obtained in the form

$$\frac{2}{b^6} \ln(1-b^2 R^{1/2}) + \frac{2}{b^4} R^{1/2} + \frac{R}{b^2} = t \quad (2.18)$$

$$\text{so that } \dot{R} = \left(b^2 - \frac{1}{R^{1/2}} \right) \quad (2.19)$$

At $b^2 R^{1/2} = 1$ we have $\dot{R} = 0$ and this occurs at $t \rightarrow -\infty$. If now R is allowed to decrease it goes on decreasing in course of time ($\dot{R} < 0$) till $R(t)$ falls to zero at $t \rightarrow 0$. This is no doubt a collapsing situation, where finally we have $|\dot{R}|$ exploding as $R \rightarrow 0$.

The matter density ρ is evidently positive using the relation (2.19) the pressure is explicitly written as

$$8\pi p = \frac{4\xi_0}{R^2} \left[\frac{(2A^2-1) - \xi_0 r^2 (A^2+1)}{1-A^2 + \xi_0 r^2} \right] + \frac{b^2}{R^{5/2}} (1-b^2 R^{1/2}) \frac{(1+\xi_0 r^2)}{(1-A^2 + \xi_0 r^2)}$$

In view of (2.16) the above expression for pressure may also be written as

$$8\pi p = \frac{4\xi_0}{R^2} \frac{(A^2+1)(r_0^2 - r^2)}{(1-A^2 + \xi_0 r^2)} + \frac{b^2 (1-b^2 R^{1/2})}{R^{5/2}} \frac{(1+\xi_0 r^2)}{(1-A^2 + \xi_0 r^2)} \quad (2.20)$$

It is clear from (2.20) that since $r_0 > r$ and $b^2 R^{1/2} < 1$ the pressure 'p' is positive. At the moment $b^2 R^{1/2} = 1$ that is when $\dot{R} = 0$ the pressure vanishes at the boundary in view of the fact that at this instant the heat flux vanishes.

In fact there may be another solution which shows expansion. The is given by

$$\frac{2}{b^6} \ln(b^2 R^{1/2} - 1) + \frac{2}{b^4} R^{1/2} + \frac{R}{b^2} = t \quad (2.21)$$

and as in (2.19) $\dot{R} = (b^2 - \frac{1}{R^{1/2}})$

The solution (2.21) requires $b^2 R^{1/2} \geq 1$, which means we have in this expansion only $\dot{R} > 0$. So the function $R(t)$ increases from a finite magnitude as $t \rightarrow -\infty$ to an indefinitely large value at $t \rightarrow \infty$. At the final stage $R \rightarrow \infty$, \dot{R} approaches a finite positive magnitude. One drawback of this solution is that one cannot ensure positivity of pressure everywhere within the heat conducting sphere.

3. The Causal heat transport equation

As is mentioned in the introduction that this section discusses the truncated heat transport equation of Israel and Stewart and also Hiscock and Lindholm. This equation may be written as

$$\tau h^{\alpha\beta} q_{\beta;\gamma} u^\gamma + q^\alpha = -\lambda (h^{\alpha\beta} T_{;\beta} + \tau \dot{u}^\alpha) \quad (3.1)$$

τ in the above equation is called the relaxation time, T is the temperature and $h^{\alpha\beta} = (g^{\alpha\beta} + u^\alpha u^\beta)$ – the so-called projection tensor and the constant λ represents the thermal conductivity of the fluid. The equation (3.1) is discussed for the metric (2.9). the metric (2.1) is already discussed by Maartens et al. Their conclusion is that the simple Modak's solution (2.1) is constant with the inflationary expansion and at the same time it satisfied the equation (3.1).

The metric (2.9) reduces to Robertson-Walker spacetime with nonzero spatial curvature when the heat flux vanishes, whereas the metric (2.1) discussed by Maartens et al reduces to that of Robertson-Walker for the special case of zero spatial curvature in the same situation. In this sense the cosmological model we consider is more general.

We assume that the relaxation time scale τ is proportional to the inverse of the expansion scalar θ , so that $\tau = \beta/\theta$, β being the proportionality constant. This is not any unreasonable assumption (see Trigriner and Pavon [12], Maartens et al [11]). For heat flow in the radial direction and using the comoving coordinates the truncated heat transport equation (3.1) reduces to

$$\tau h^{11} q_{1,0\nu}^0 + q^1 = -\lambda (h^{11} T_{,1} + T \dot{\nu}^1) \quad (3.2)$$

which when written in the background of the metric (2.9) yields

$$\frac{4a\xi_0 r}{(1+a+\xi_0 r^2)^3} \left[-\frac{\beta}{3} \left(\frac{R}{\dot{R}} \right) (1+a+\xi_0 r^2) \left(\frac{2\dot{R}^2}{R^2} - \frac{\ddot{R}}{R} \right) + \frac{\dot{R}}{R} (1+a+\xi_0 r^2) \right] = -\lambda \left[T_{,1} + \frac{2a\xi_0 r T}{(1+\xi_0 r^2)(1+a+\xi_0 r^2)} \right] \quad (3.3)$$

The equation (3.3) may have a simple solution independent of the choice of λ - the thermal conductivity and it may be obtained by putting both sides zero. It follows then

$$\frac{\beta}{3} \left(\frac{2\dot{R}^2}{R^2} - \frac{\ddot{R}}{R} \right) = \left(\frac{\dot{R}}{R} \right)^2 \quad (3.4)$$

The inflationary solution $R(t) = R_0 e^{H_0 t}$ gives immediately the value of $\beta =$

3. Putting the right hand side of (3.3) zero in turn yield on integration

$$T = f(t) \left[1 + \frac{a}{1 + \xi_0 r^2} \right] \quad (3.5)$$

One can choose both 'a' and ' ξ_0 ' positive, so that the temperature T decreases with increasing radial coordinate. Then from (2.12) the heat flow is outwards when $\dot{R} > 0$ that is for an expanding model. In other words the heat flow is along the direction of the temperature gradient, whereas in the case discussed

by Maartens et al the heat flow is against the direction of the temperature gradient. By choosing $\dot{f} < 0$ one can enforce inflationary cooling. From (2.10) and (2.11) it is possible to calculate.

$$8\pi(\rho + p) = \frac{8\xi_0}{R_0^2} e^{-2H_0 t} \left(\frac{1 + a/2 + \xi_0 r^2 + \frac{1}{2} a \xi_0 r^2}{1 + a + \xi_0 r^2} \right) \quad (3.6)$$

It is positive when a and ξ are both positive and approaches zero as $p = -\rho$ when t approaches infinitely large value. In our case the density decreases with time during inflation, whereas in the model described by Maartens et al the density remains independent of time. As $t \rightarrow \infty$, not only the heat flux vanishes but also the ratio $q/\rho \rightarrow 0$. Again at a given instant the ratio q/ρ vanishes at both ends $r = 0$ and $r \rightarrow \infty$ indicating that at the centre as well as far away from the centre the fluid is close to equilibrium, whereas in Maartens et al model it is in equilibrium only at the centre. Further the invariant expression for the heat flow may be written as

$$q = (q_1 q^1)^{1/2} = \frac{4a\xi_0 r (1 + \xi_0 r^2)}{(1 + a + \xi_0 r^2)^2} \frac{H_0}{R_0} e^{-H_0 t} \quad (3.7)$$

The above quantity vanishes both at $r \rightarrow 0$ and $r \rightarrow \infty$. So it must be maximum at some finite distance away from the centre.

It should be noted that this model also like that of the model discussed by Maartens et al in their earlier paper lacks some of the physical properties that would be expected from more realistic model, but it needs mentioning because it is more general and is characterized by inflationary expansion of the scale factor.

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