

The Vanishing Likelihood of Space-Time Singularity in Quantum Conformal Cosmology

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A general formalism is developed for studying the behavior of quantized conformal fluctuations near the space-time singularity of classical relativistic cosmology. It is shown that if the material contents of space-time are made of massive particles which obey the principle of asymptotic freedom and interact only gravitationally, then it is possible to estimate the quantum mechanical probability that, of the various possible conformal transforms of the classical Einstein solution, the actual model had a singularity in the past. This probability turns out to be vanishingly small, thus indicating that within the regime of quantum conformal cosmology it is extremely unlikely that the universe originated out of a space-time singularity.

1. INTRODUCTION

This work is a sequel to that presented in an earlier paper,⁽¹⁾ hereafter referred to as I. In Paper I a formalism was developed for studying the quantum gravitodynamical behavior of the conformal fluctuations of Einstein's classical equations of general relativity:

$$R_{ik} - \frac{1}{2} g_{ik} R = -8\pi T_{ik} \quad (1)$$

(We have taken the gravitational constant $G = 1$, the speed of light $c = 1$, and we will also take the crossed Planck constant $\hbar = 1$.) A conformal fluctuation ϕ is a C^2 function of space-time coordinates x^i ($i = 0, 1, 2, 3$, x^0 timelike) defined by the transformation

$$\tilde{g}_{ik} = (1 + \phi)^2 g_{ik} \quad (2)$$

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Since Einstein's equations are not conformally invariant, \tilde{g}_{ik} will not, in general, satisfy Eq. (1). The space-time geometry described by $\{\tilde{g}_{ik}\}$ will be termed nonclassical geometry. The classical and nonclassical manifolds will be denoted by \mathcal{M} and $\tilde{\mathcal{M}}$ and their geometries by ${}^{(4)}\mathcal{G}$ and ${}^{(4)}\tilde{\mathcal{G}}$, respectively.

In quantum gravity the notion of a clearly defined space-time geometry has to be replaced by the probabilistic view in which we talk of a whole range of geometries describing the space-time, each having a certain probability. The Feynman path integral technique can be used to give a formal meaning to this view. The path integral approach was outlined in Paper I.

It is clear at the outset that the full range of nonclassical geometries ${}^{(4)}\tilde{\mathcal{G}}$ is far wider than that limited by the conformal transformations (2). A theory which limits itself to conformal fluctuations alone cannot therefore claim to be a full theory of quantum gravity.

Nevertheless, while treading new territory it always helps to set oneself limited goals in the beginning, if only to get familiar with the difficulties of the problem. Simplifications dictated by various symmetry arguments have invariably helped such attempts in the past. It is in the same spirit that we limit ourselves to conformal fluctuations on the grounds that global causality is preserved under these transformations. That is, under conformal transformations the global light cone structure remains unchanged. Hence the fact that two space-time events A and B are (or are not) causally connected remains unchanged under conformal fluctuations. The normal quantum mechanical propagators describing the behavior of various fields in such conformally related geometries are not therefore drastically altered and it becomes possible to attach unambiguous meanings to such causal notions as "in" and "out" states.

This simplification helps to understand answers to certain clearly posed questions in quantum cosmology. It does *not* provide answers to *all* deep questions of quantum gravity, and to clearly indicate this limitation we will henceforth call our subject *quantum conformal cosmology* (QCC in brief).

In Paper I an important question was posed and answered within the framework of QCC. There it was shown that the quantum conformal fluctuations diverge at the epoch where the classical geometry becomes singular. That is, $\langle \phi^2 \rangle^{1/2}$, the root-mean-square fluctuation tends to infinity at the classical singularity, thereby rendering the classical solution of dubious value. The investigation in Paper I, however, raises another important question which can be formulated thus. Given the full range of nonclassical geometries in QCC, what fraction of these are nonsingular at the classical singular epoch? If this fraction is close to 1 we can assert that even in QCC the problem of space-time singularity (which is considered inevitable in classical relativistic cosmology) remains. If, on the other hand the fraction is

close to zero, we can conclude that it is extremely unlikely that the universe had a singular origin. The present paper is devoted to answering this question.

To proceed further we first formulate the basic structure of QCC along the lines of Paper I.

2. THE BASIC FORMALISM OF QUANTUM CONFORMAL COSMOLOGY

Following the notation of Paper I we consider the part of space-time sandwiched between two spacelike hypersurfaces Σ_1 and Σ_2 . Choosing a coordinate system such that the space-time is foliated by spacelike hypersurfaces $\{\Sigma\}$, given by $x^0 \equiv t = \text{const}$ with Σ_1 and Σ_2 given by $t = t_1$ and $t = t_2$, respectively, we look upon classical general relativity as defining a succession of 3-geometries $\{^{(3)}\mathcal{G}(t)\}$ for $t_1 < t < t_2$. This succession of 3-geometries is obtained by solving the Einstein equations together with the specified boundary conditions on Σ_1 and Σ_2 . The Einstein equations themselves are obtained from the Hilbert variational principle

$$\delta J = 0 \quad (3)$$

We will shortly discuss the form of the action J .

The specification of boundary conditions in a consistent manner has been discussed by Isenberg and Wheeler⁽²⁾ who argued that instead of the 3-geometry it is more appropriate to specify on Σ_1 or Σ_2 the conformal part of the 3-geometry together with the extrinsic curvature of the three-dimensional hypersurface. The classical gravitodynamics is then posed as the solution of (3) together with such boundary conditions on Σ_1 and Σ_2 .

For quantum gravitodynamics we introduce the concept of "paths," with each path I describing a succession of 3-geometries from Σ_1 to Σ_2 consistent with the end point conditions. For each path compute $J[I]$ and evaluate the sum

$$K = \sum_I \exp iJ[I] \quad (4)$$

over all permissible paths. As Feynman⁽³⁾ showed, the above sum is in general a complicated path integral which defines the probability amplitude for the system to start with the specified condition on Σ_1 and end with the specified conditions on Σ_2 . We may therefore rewrite (4) as a path integral

$$K[2, 1] = \int \exp iJ[I] \mathcal{D}I \quad (5)$$

The formidable problems posed by (5) in general become simplified in QCC, as was shown in Paper I. Taking the classical Einstein solution for the background metric, it is possible to write (5) as a path integral over ϕ . To see the explicit form of (5), we proceed as follows.

The classical Hilbert action is given by

$$J = \frac{1}{16\pi} \int_{\nu} R \sqrt{-g} d^4x + J_s + J_m \quad (6)$$

Here ν is the four-dimensional space-time region (e.g., the thin sandwich mentioned before) under consideration. In Paper I we had commented on the "surface term" J_s defined over the boundary $\partial\nu$ of ν . The necessity of this term and its explicit form in asymptotically flat space-time have been discussed in detail by Gibbons and Hawking.⁽⁴⁾ It is needed in order to eliminate the variations of metric derivatives, $\delta g_{ik,l}$ on $\partial\nu$. Although the asymptotic form of J_s at infinity is not yet known in general space-times, we will assume that whatever it is, it is well defined.

For J_m , the matter action, we will take the simple, yet fairly general form

$$J_m = \sum_a \int m_a ds_a \quad (7)$$

describing a system of particles a, b, \dots of masses m_a, m_b, \dots and proper time elements ds_a, ds_b, \dots . These particles do not interact except through gravity. If it is assumed that close to the classical singularity the particles have very high energies, then the principle of asymptotic freedom (assumed in most grand unified theories) leads us to the above conclusions.

Given these details, the analysis of Paper I leads us, for the space-time with the metric given by (2), to

$$\tilde{J} = \frac{1}{16\pi} \int (R\phi^2 - 6\phi_i\phi^i) \sqrt{-g} d^4x + J_E \quad (8)$$

where J_E has been computed for the classical solution and does not contain ϕ or its derivatives $\phi_i \equiv \partial\phi/\partial x^i$. Accordingly (5) becomes

$$K[\phi_2, t_2; \phi_1, t_1] = \int \exp i \left\{ \frac{1}{16\pi} \int (R\phi^2 - 6\phi_i\phi^i) \sqrt{-g} d^4x \right\} \mathcal{D}\phi \quad (9)$$

Any weight factors arising from measure of the path integral over ϕ are absorbed in $\mathcal{D}\phi$.

Most of Paper I was devoted to evaluating $K[\phi_2, t_2; \phi_1, t_1]$. We

therefore quote the result without proof. However, since we will be interested in applications of $K[\phi_2, t_2; \phi_1, t_1]$, we will use a compact notation which plays the same role for integration over continuous variables as the summation convention for discrete tensor indices. This notation and the form of K are described next.

3. THE MASTER PROPAGATOR IN QCC

In Paper I, Section 5.2 we had replaced integrations by summations over discrete sets of values of space variables $\mathbf{x}_1, \mathbf{x}_2$, etc. This is unnecessary if we adopt a compact notation illustrated below. In analogy with the tensor relation

$$A_i = \delta_i^k A_k \tag{10}$$

for the Kronecker delta, we write the continuum version

$$A(x) = \int \delta(x - y) A(y) dy \tag{11}$$

as simply

$$A(x) = \delta(x - y) A(y) \tag{12}$$

In (12) it is understood that the repeated variable y on the right-hand side implies integration over the entire range.

Using this notation we can write, for example, the propagation of a wave functional $\psi_1(\phi_1)$ at $t = t_1$ to $\psi_2(\phi_2)$ at $t = t_2$ in the form

$$\psi_2(\phi_2) = K[\phi_2, t_2; \phi_1, t_1] \psi_1(\phi_1) \tag{13}$$

On the right-hand side an integral over ϕ_1 is implied. In the same way we now express the form of $K[\phi_2, t_2; \phi_1, t_1]$ obtained in Paper I as

$$\begin{aligned} K[\phi_2, t_2; \phi_1, t_1] = & F(t_1, t_2) \exp i\{A_{11}(\mathbf{x}_1, \mathbf{x}'_1) \phi_1(\mathbf{x}_1) \phi_1(\mathbf{x}'_1) \\ & + A_{22}(\mathbf{x}_2, \mathbf{x}'_2) \phi_2(\mathbf{x}_2) \phi_2(\mathbf{x}'_2) \\ & + 2A_{12}(\mathbf{x}_1, \mathbf{x}_2) \phi_1(\mathbf{x}_1) \phi_2(\mathbf{x}_2)\} \end{aligned} \tag{14}$$

In the above expression we have taken $\mathbf{x} = (x^\mu)$, $\mu = 1, 2, 3$, and $F(t_1, t_2)$ is a known function of t_1, t_2 . Following the work of DeWitt,⁽⁵⁾ we may formally express $F(t_1, t_2)$ as a van Vleck determinant. However, its

form does not concern us here. The coefficients A_{11} , A_{22} , A_{12} are related to Green's functions of the wave operator $\square + \frac{1}{6}R$, defined over the background geometry ${}^{(4)}\mathcal{G}$. As in I we will write the Green's functions as

$$G^A(x_1^\mu, t_1; x_2^\mu, t_2) = G^R(x_2^\mu, t_2; x_1^\mu, t_1) \equiv G(\mathbf{x}_2, \mathbf{x}_1) \quad (15)$$

where the superscripts A and R denote the advanced and retarded property. Thus in (15) G^A has support only over the past light cone of (x_2^μ, t_2) , while G^R has support only over the future light cone of (x_1^μ, t_1) . The time variables are not explicitly indicated in $G(\mathbf{x}_2, \mathbf{x}_1)$. However, they are supposed to be present, and we denote by G_{t_1} the time derivative of G with respect to t_1 .

In Paper I it was shown that

$$A_{11}(\mathbf{x}_1, \mathbf{x}'_1) = (3/8\pi) G(\mathbf{x}_2, \mathbf{x}_1)^{-1} G_{t_1}(\mathbf{x}_2, \mathbf{x}'_1) \sqrt{-g'_1} \quad (16)$$

$$A_{22}(\mathbf{x}_2, \mathbf{x}'_2) = -(3/8\pi) G(\mathbf{x}_2, \mathbf{x}_1)^{-1} G_{t_2}(\mathbf{x}'_2, \mathbf{x}_1) \sqrt{-g'_2} \quad (17)$$

$$A_{12}(\mathbf{x}_1, \mathbf{x}_2) = (3/8\pi) G(\mathbf{x}_2, \mathbf{x}_1)^{-1} \quad (18)$$

The inverse of the Green's function is to be interpreted in an operator sense (see Appendix of Paper I and also Morse and Feshbach⁽⁶⁾).

In the expression (14) the exponent has the first two terms as quadratic forms in ϕ_1 and ϕ_2 , respectively. Linear transformations of the kind

$$\phi_1(\mathbf{x}_1) = \alpha_1(\mathbf{x}_1, \tilde{\mathbf{x}}_1) \tilde{\phi}_1(\tilde{\mathbf{x}}_1) \quad (19)$$

$$\phi_2(\mathbf{x}_2) = \alpha_2(\mathbf{x}_2, \tilde{\mathbf{x}}_2) \tilde{\phi}_2(\tilde{\mathbf{x}}_2) \quad (20)$$

can be used to diagonalize these quadratic forms. For reasons that will be clear later we will diagonalize only A_{22} . Then we get

$$\tilde{A}_{12}(\mathbf{x}_1, \tilde{\mathbf{x}}_2) = \alpha_2(\mathbf{x}_2, \tilde{\mathbf{x}}_2) A_{12}(\mathbf{x}_1, \mathbf{x}_2) \quad (21)$$

In the discussion that follows we will be concerned with the entire range of conformal fluctuations ϕ . Hence transformations of the type (19) and (20) do not introduce any new inputs but are useful as simplifying devices. We will therefore assume without loss of generality that the continuum matrix $A_{22}(\mathbf{x}_2, \mathbf{x}'_2)$ defined in (17) is diagonal and write

$$A_{22}(\mathbf{x}_2, \mathbf{x}'_2) = Q_2(\mathbf{x}_2) \delta(\mathbf{x}_2 - \mathbf{x}'_2) \quad (22)$$

where Q_2 is a known function. We will refer to K expressed in this form as the master propagator of QCC.

4. THE AVOIDANCE OF SINGULARITY BY CONFORMAL TRANSFORMATIONS

Before proceeding with the applications of the master propagator, we look at some results of classical nature concerning conformal transformations, starting with the simple example of Einstein–de Sitter cosmology.

The Einstein–de Sitter model is the simplest of the Friedmann cosmological models and is described by the line element

$$ds_2 = dt^2 - (t/t_0)^{4/3} [dr^2 + r^2(d\phi^2 + \sin^2 \theta d\theta^2)] \quad (23)$$

where $t_0 = \text{const}$. This model has space-time singularity at $t = 0$ and hence the space-time manifold \mathcal{M} is restricted to $t \geq 0$.

The manifold $\tilde{\mathcal{M}}$ obtained by the conformal transformation

$$d\tilde{s} = (t_0/t)^{2/3} ds \quad (24)$$

is, however, singularity-free at $t = 0$. In fact the coordinate transformation

$$\tau = 3t_0^{2/3} t^{1/3} \quad (25)$$

expresses ds in the manifestly Minkowskian form

$$d\tilde{s}^2 = d\tau^2 - [dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)] \quad (26)$$

It is also noticed that the conformal function

$$\Omega_c = (t_0/t)^{2/3} \quad (27)$$

diverges at the singular hypersurface $t = 0$ of the original space-time \mathcal{M} . Moreover, if we choose another conformal function $\Omega(t)$ instead of Ω_c in (24) we would find that the corresponding $\tilde{\mathcal{M}}$ is singular at $t = 0$ if

$$\Omega/\Omega_c \rightarrow 0 \quad \text{as} \quad t \rightarrow 0 \quad (28)$$

For $\tilde{\mathcal{M}}$ to be nonsingular, Ω must tend to infinity as $t \rightarrow 0$ at least as fast as Ω_c .

Can these results be generalized? Suppose that the classical space-time \mathcal{M} has a singular hypersurface Σ . If the space-time singularity in \mathcal{M} is due to the divergence of one or more curvature invariants, we may envisage the circumstance that by choosing Ω suitably, the metric scales are stretched fast enough to make $\tilde{\mathcal{M}}$ nonsingular. This would require Ω to diverge sufficiently

rapidly as the singular hypersurface Σ is approached. We also expect the existence of a critical conformal function Ω_c such that if and only if $\Omega/\Omega_c \rightarrow 0$ at the singular hypersurface does the singularity in $\tilde{\mathcal{M}}$ persist.

A quantitative discussion of the above statement (but not a rigorous proof) is given in the Appendix. Explicit constructions of Ω_c in various singular cosmological space-time have been given by Kembhavi.⁽⁷⁾ These include the “most general” approach to cosmological singularity described by Belinskii *et al.*⁽⁸⁾

Another way to describe space-time singularity is through the notion of geodesic incompleteness.⁽⁹⁾ In this context Beem⁽¹¹⁾ has shown that if the original space-time \mathcal{M} is causal and if it satisfies the nonimprisonment condition [i.e., for each compact subset K of \mathcal{M} there is no future inextendible nonspacelike curve $x(t)$ such that $x(t) \in K$ for all $t \geq t_1$ for some t_1], then there is some conformal function Ω such that the conformally transformed manifold $\tilde{\mathcal{M}}$ is null and timelike complete. Since most physically meaningful solutions of Einstein’s equations are expected to satisfy the causality and nonimprisonment conditions, Beem’s construction provides the method of eliminating space-time singularity through conformal transformations.

Since Einstein’s equations are not conformally invariant, the manifolds $\tilde{\mathcal{M}}$, whether singular or not, do not have any physical significance in classical relativistic cosmology. In QCC, however, they have all got to be included in the study of the dynamical evolution of space-time geometry. As a consequence of the above discussion, we will make the following assumption.

Assumption A. Given a classical space-time manifold \mathcal{M} which is singular on a spacelike hypersurface Σ , there exists a conformal function Ω_c such that

- (i) the conformally transformed manifold $\tilde{\mathcal{M}}_c \equiv \Omega_c^2 \mathcal{M}$ is nonsingular on Σ ,
- (ii) all conformally transformed manifolds $\tilde{\mathcal{M}} \equiv \Omega^2 \mathcal{M}$ are singular on Σ for which $\Omega/\Omega_c \rightarrow 0$ on Σ_c (we denote the class of such manifolds by \mathcal{C}_s),
- (iii) all conformally transformed manifolds $\tilde{\mathcal{M}} \equiv \Omega^2 \mathcal{M}$ are nonsingular on Σ , for which $\Omega/\Omega_c \rightarrow 0$ on Σ_c (such manifolds fall in a class to be denoted by \mathcal{C}_{ns}).

It is of course understood that Ω_c itself must diverge on Σ (see Appendix).

5. THE QUANTUM EVOLUTION OF CONFORMAL MANIFOLDS

We now return to the master propagator obtained in Section 3 and apply it to study the evolution of quantum conformal cosmological models. We first tackle the following problem.

It is known that the present state of the universe is describable by classical gravity. Denoting the classical model by \mathcal{M} we therefore argue that at present ϕ is very nearly zero. The present state of the universe could therefore be described by a wave functional which is strongly peaked at $\phi = 0$. We regard the present state as the final state (at $t = t_2$) and write the wave functional as a wave packet:

$$\Psi_2[\phi_2(\mathbf{x})] = \frac{1}{[2\pi\sigma_2(\mathbf{x})]^{1/4}} \exp \left\{ -\frac{[\phi_2(\mathbf{x})]^2}{4[\sigma_2(\mathbf{x})]^2} \right\} \quad (29)$$

The result that the present state of the universe is almost classical means that $|\sigma_2| \ll 1$.

We wish to find out the possible range of initial states at $t = t_1 < t_2$ from which the present state could have emerged. In particular, we would be interested in finding out to what extent the initial state was close to the classical solution \mathcal{M} . To this end we apply the master propagator *backwards* in time:

$$\Psi_1[\phi_1(\mathbf{x})] = \int K[\phi_2, t_2; \phi_1, t_1] \Psi_2[\phi_2(\mathbf{x})] \mathcal{D}\phi_2 \quad (30)$$

From Section 3, we may express K in the form

$$K[\phi_2, t_2; \phi_1, t_1] = F(t_1, t_2) \exp \{ i[A_{11}(\mathbf{x}_1, \mathbf{x}'_1) \phi_1(\mathbf{x}_1) \phi_1(\mathbf{x}'_1) + Q_2(\mathbf{x}_2) \phi_2(\mathbf{x}_2)^2 + 2A_{12}(\mathbf{x}_1, \mathbf{x}_2) \phi_1(\mathbf{x}_1) \phi_2(\mathbf{x}_2)] \} \quad (31)$$

Using (29) and (31) to perform the functional integral (30), we get

$$\Psi_1[\phi_1(\mathbf{x})] \propto \exp \left[\frac{A_{12}^2 \phi_1(\mathbf{x})^2}{(iQ_2 - 1/4\sigma_2^2)} + iA_{11}(\mathbf{x}_1, \mathbf{x}'_1) \phi_1(\mathbf{x}_1) \phi_1(\mathbf{x}'_1) \right] \quad (32)$$

The constant of proportionality is a normalizing function independent of ϕ_1 .

Therefore the probability density is given by

$$|\Psi_1|^2 = f \exp(-\phi_1^2/2\sigma_1^2) \quad (33)$$

where f is a normalizing function independent of $\phi_1(\mathbf{x})$ and

$$\sigma_1^2 = \sigma_2^2 [A_{22}^2 + 1/16\sigma_2^4] A_{12}^{-2} \quad (34)$$

This is the generalized form of the relation (48) of Paper I for the homogeneous dust ball. There are two differences, however. First, in the case of the dust ball, we were interested in the final state so that in (48) of Paper I the subscripts 1 and 2 are interchanged. Here we are interested in speculating about the possible range of initial states from which the universe could have emerged to the present form. The second point of difference is that all the quantities in (34) are not constants or functions of time only; they depend on space coordinates also. Unlike the case of the homogeneous dust ball, we are dealing with a general inhomogeneous space-time, and the functionals Ψ_1, Ψ_2 serve to remind us of that.

Nevertheless, with the results obtained in the previous section, we are now able to use (34) to estimate the relative importance of singular states (belonging to class \mathcal{C}_s) and nonsingular states (belonging to class \mathcal{C}_{ns}).

To do so we first consider the manifold \mathcal{M}_c implied in assumption A. The Green's functions defined in relation (15) transform under conformal transformation as follows:

$$\mathcal{M} \rightarrow \tilde{\mathcal{M}}_c \tag{35}$$

$$G(\mathbf{x}_1, \mathbf{x}_2) \rightarrow \tilde{G}_c(\mathbf{x}_1, \mathbf{x}_2) = [\Omega_c(\mathbf{x}_1, t_1) \Omega_c(\mathbf{x}_2, t_2)]^{-1} G(\mathbf{x}_1, \mathbf{x}_2) \tag{36}$$

Since $\tilde{\mathcal{M}}_c$ is well behaved on Σ , $\tilde{G}_c(\mathbf{x}_1, \mathbf{x}_2)$ is well behaved as $(\mathbf{x}_1, t_1) \rightarrow \Sigma$. However, $\Omega_c(\mathbf{x}_1, t_1) \rightarrow \infty$ as $(\mathbf{x}_1, t_1) \rightarrow \Sigma$. Hence $G(\mathbf{x}_1, \mathbf{x}_2)$ diverges as $(\mathbf{x}_1, t_1) \rightarrow \Sigma$. Note also that because $\Omega_c(\mathbf{x}_1, t_1)$ cancels with its reciprocal in the two factors of A_{22} as defined by (17), A_{22} remains finite on Σ . Thus the function α_2 defined in (20) also remains finite on Σ . Hence from the transformation (21) we may safely assume that

$$\tilde{A}_{12} \sim [\phi_c(\mathbf{x}_1, t_1)]^{-1} \quad \text{as } (\mathbf{x}_1, t_1) \rightarrow \Sigma \tag{37}$$

We will express this by writing in (34)

$$A_{12} = \lambda(\mathbf{x}_1, t_1) / \phi_c(\mathbf{x}_1, t_1) \tag{38}$$

where $\lambda(\mathbf{x}_1, t_1)$ is well behaved on Σ .

Using this result in (34) we note that since as $(\mathbf{x}_1, t_1) \rightarrow \Sigma$ all quantities on the right-hand side of (34) are well behaved except A_{12}^{-1} , the asymptotic expression for σ_1 is given by

$$\sigma_1 \sim \beta(\mathbf{x}_1, t_1) \phi_c(\mathbf{x}_1, t_1) \tag{39}$$

where $\beta(\mathbf{x}_1, t_1)$ is well behaved as $(\mathbf{x}_1, t_1) \rightarrow \Sigma$.

Now recall assumption A (ii) and suppose that, in (34), $\mathcal{M} \in \mathcal{C}_s$. Then $\gamma \equiv \phi_1 / \phi_c = (\Omega_1 - 1) / (\Omega_c - 1)$ tends to zero as both Ω_1 and Ω_c diverge on Σ .

The singular manifolds therefore tend to crowd around the classical manifold given by $\phi_1 = 0$ and are confined to domains which get narrower and narrower as Σ is approached. In other words, the measure of probability

$$P(\mathcal{M} \in \mathcal{C}_s) = \int_{|\phi_1| < \gamma \sigma_1 \beta^{-1}} f \exp\left(-\frac{\phi_1^2}{2\sigma_1^2}\right) \mathcal{D}\phi_1 \tag{40}$$

tends to zero as $\gamma \rightarrow 0$.

By contrast, all $\tilde{\mathcal{M}} \in \mathcal{C}_{ns}$ satisfy the condition that $\gamma \rightarrow 0$, that is, γ is either finite or tends to infinity. These models contribute overwhelmingly to the probability measure as we push back our initial instant towards Σ .

This completes our proof that the likelihood of the universe emerging from a singular state within the fully permissible range of QCC is zero.

6. DISCUSSION

The singularity theorems in classical general relativity have established the occurrence of space-time singularity as an unavoidable feature of cosmology.⁽⁹⁾ The situation is somewhat analogous to that of the hydrogen atom in classical electrodynamics where the full application of Maxwellian equations led to the conclusion that the electron should spiral inwards and drop onto the proton in a time scale of the order of $e^2/mc^3 \sim 10^{-23}$ s.

The manifestly stable hydrogen atom demonstrated the inadequacy of classical electrodynamics, and the application of quantum theory was necessary and sufficient to solve the problem. The situation as regards cosmology is somewhat different. Theoreticians are divided in their attitude toward the cosmological singularity. The normal physicists' attitude is the same as it is toward any other singularity elsewhere in physics, namely one of suspicion that the theory leading to a singularity is necessarily incomplete. Yet there are others who consider the cosmological singularity as something lying beyond the scope of physics and implying a deeply significant event: the origin of the universe.

This latter attitude undermines any attempt to look for a wider framework than classical general relativity. It might have been defensible if space-time singularity had occurred only once in the space-time (or twice in case the universe ended also). This is not the case. Singularity is known to occur as the end state of finite compact objects undergoing gravitational collapse. The "cosmic censorship" hypothesis, like any other censorship law, may suppress the catastrophe for outside observers, but it does not eliminate it.

Taking therefore the view that classical general relativity is incomplete,

it is possible to widen its scope in several ways. At the classical level one may have new theories of gravity, or one may look for unusual behavior of highly dense matter which violates the assumed energy conditions in the singularity theorems. The other way to proceed is to follow the example of electrodynamics and look to quantum gravity for resolving the singularity problem of classical gravity.

In this paper we have taken the latter path as it involves the least radical departure from conventional physics. We have used the path integral approach, which has been known to work in quantum theory, and applied it to general relativity which is currently the most well-established theory of gravity. Since the formal apparatus of quantum gravity so generated is quite complex, we have restricted it to conformal degrees of freedom only. The simplification yields immediate dividends since we are able to show that within the framework of quantum conformal cosmology the appearance of space-time singularity is not only avoidable but is extremely unlikely.

In earlier papers^(12,13) it was shown that within the range of homogeneous and isotropic (Robertson–Walker) models the quantum mechanical likelihood of a singular origin is vanishingly small. The result of the present paper is considerably more general since it is not limited to homogeneous and isotropic space-times.

The result derived here could nevertheless be criticized on the ground that restriction to conformal degrees of freedom is artificial. To counter it, one has to argue that the restriction preserves global causality and is therefore conceptually easy to understand. If, for example, the space-time geometry made a quantum transition by breaking conformal symmetry, two events which were previously causally disconnected may become connected, and vice versa. The full implications of such phenomena are far from clear, but they are avoided altogether in quantum conformal cosmology.

In any case, simplified treatments have always provided valuable insights into more general situations. For this reason alone the result obtained here may be of some value to more general investigations of quantum cosmology.

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APPENDIX

The reasoning which leads to assumption A in Section 4 goes along the following lines, for curvature singularities.

The space-time singularity of \mathcal{M} on Σ implies that one or more of the 14 curvature invariants⁽¹¹⁾ diverge on Σ . The curvature invariants involve metric derivatives of up to second order. Since the set of invariants is finite, given any point P on Σ we can pick out one (or more) from the set which tends to infinity most rapidly as we approach a small neighborhood N_p containing P . Let $L(x^i)$ be this diverging function.

Consider the corresponding $\tilde{L}(x^i)$ in $\tilde{\mathcal{M}}$ for a conformal function Ω expressed as

$$\Omega = e^{\zeta} \tag{A1}$$

Then $L(x^i)$ has the form

$$\tilde{L}(x^i) = e^{-n\zeta} [L(x^i) + q(\zeta_{,i}, \zeta_{,ik}, g_{ik}, g_{ik,l}, g_{ik,lm})] \tag{A2}$$

where $n > 0$ and the function q contains $\zeta_{,i}$ and $\zeta_{,ik}$ in polynomial form. Hence, whatever be the behavior of $L, g_{ik}, g_{ik,l}, g_{ik,lm}$ on Σ , we can always choose $\zeta = \zeta_c$ such that $\tilde{L}(x^i)$ is finite on N_p . Even though ζ_c diverges at N_p and hence $\zeta_{c,i}$ and $\zeta_{c,ik}$ may diverge, the exponential function $\exp(-n\zeta_c)$ can always be chosen to suppress this divergence.

The function Ω_c implied in assumption A is obtained by covering Σ with neighborhoods N_p and then constructing a composite ζ_c from all neighborhood ζ_c 's. The construction of ζ_c can evidently be given in a more rigorous fashion than above to satisfy the purist. Our aim here is to give an intuitive reasoning for the existence of Ω_c .

It is also clear that Ω_c is not unique. For the purposes of our arguments in Section 5, it does not have to be so. Indeed, as the hypersurface Σ is approached, the main contribution to the probability measure

$$P(\tilde{\mathcal{M}} \in \mathcal{E}_{ns}) = \int_{|\phi_1| > x_{1\beta}^{-1}} f \exp\left(-\frac{\phi_1^2}{2\sigma_1^2}\right) \phi_1 \tag{A3}$$

(which approaches unity) comes mainly from those values of ϕ_1 for which Ω/Ω_c is finite and bounded. Thus only such conformal manifolds $\tilde{\mathcal{M}}$ are most likely near Σ .

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