

# Generating $\Lambda$ from the vacuum

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**Abstract:** The close relationship between the cosmological constant and the vacuum has been emphasised in the past by Zeldovich amongst others. We briefly discuss different approaches to the cosmological constant issue including the possibility that  $\Lambda$  could be generated by vacuum polarization in a static Universe. Fresh possibilities occur in an expanding Universe. An Inflationary Universe generically leads to particle creation from the vacuum, the nature and extent of particle production depending upon the mass of the field and its coupling to gravity. For ultra-light, non-minimally coupled scalar fields, particle production can be large and the resulting vacuum energy-momentum tensor will have the form of an effective cosmological constant. The Inflationary scenario therefore, could give rise to a Universe that is *both flat and*  $\Lambda$ -dominated, in agreement with observations.<sup>1</sup>

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In 1998, in what could be a milestone discovery for Cosmology, the team from the ‘Supernova Cosmology Project’ published their results demonstrating the need for a cosmological constant on the basis of a redshift- magnitude analysis of 7 Type 1a high redshift supernovae [1]. This result has subsequently been strongly supported by the analysis of a larger number of high redshift supernovae both by the Supernova Cosmology Project [3] and by the High- $z$  Supernova search team [2]. In all, at the time of writing, over 50 type 1a supernovae have been analysed and the results, when combined with cosmic microwave background constraints, strongly favour a spatially flat Universe  $\Omega_m + \Omega_\Lambda \simeq 1$ , with a substantial component of the energy density in the form of a cosmological constant  $\Omega_\Lambda \simeq 0.6 - 0.7$  [4].

The history of the cosmological constant is interesting. It was introduced by Einstein in 1917 to make General Relativity compatible with Mach’s principle which Einstein felt, favoured a static and spatially closed Universe. Discarded after the discovery by Friedmann of expanding cosmological solutions,  $\Lambda$  was reinvented in 1968 to help explain the ‘discovery’ of an excess of QSOs at a redshift 1.98 which could be easily accommodated in the quasi-static cosmology of Lemaitre (1927). Intrigued by this debate over  $\Lambda$ , Zeldovich set forward the first physical model of a cosmological constant by showing that one-loop zero-point vacuum fluctuations had the Lorenz invariant equation of state  $p_{vac} = -\rho_{vac}$  which gave rise to the vacuum expectation value of the energy momentum tensor having the form  $\langle T_{ik} \rangle_{vac} = \Lambda g_{ik}$ , where  $\rho_{vac} = \Lambda/8\pi G$ . This meant that the vacuum within the quantum framework has properties identical to those of a cosmological constant.

Unfortunately, one-loop quantum effects contribute an infinite zero-point energy which for bosons takes the form

$$\langle 0|H_b|0\rangle = \frac{1}{2} \sum_{\mathbf{k}} \omega_{\mathbf{k}}. \quad (1)$$

The fermionic zero point energy is also infinite, but comes with an opposite sign

$$\langle 0|H_f|0\rangle = -\frac{1}{2} \sum_{\mathbf{k}} \omega_{\mathbf{k}}. \quad (2)$$

The advent of supersymmetric models incorporating a fundamental symmetry between bosons and fermions led to the hope that the mutual cancellation of infinities in SUSY models might rid us of the cosmological constant problem. However this view was shown to be mistaken since, at low temperatures of  $T \simeq 2.7^\circ\text{K}$ , supersymmetry would be broken, giving rise to a large (formally infinite) value of  $\Lambda$  today. The present thinking on  $\Lambda$  is that it must be set either to zero or else to a small value, either by normal ordering or by some other regularisation procedure.<sup>1</sup>

Zeldovich suggested that after the removal of one-loop divergences, the 'regularised' vacuum polarization contributed by a fundamental particle of mass  $m$  would be described by the self-interaction between particle & anti-particle participating in a virtual loop. This would give rise to the following vacuum density at the *two-loop* level

$$\rho_\Lambda \sim \frac{Gm^2}{\hbar c} m \left(\frac{mc}{\hbar}\right)^3. \quad (3)$$

One can arrive at this result from the following argument: the vacuum consists of virtual particle-antiparticle pairs of mass  $m$  and separation  $\lambda = \hbar/mc$ . Although the regularised self-energy of these pairs is zero, their gravitational interaction is finite and results in the vacuum energy density  $\epsilon_{vac} \equiv \rho_{vac}c^2 \sim \frac{Gm^2}{\lambda}/\lambda^3 = Gm^6c^4/\hbar^4$ , corresponding to (3). Unfortunately the electron mass  $m_e$  when substituted in (3) gives too small a value of  $\rho_\Lambda$ , whereas the proton mass  $m_p$  gives a vacuum density which exceeds observational estimates by several orders of magnitude. (The pion mass  $m_\pi$  incidentally, gives just the right value  $\rho_{vac} \sim 10^{-29}\text{g cm}^{-3}$  !)

The above analysis was done in flat space. With the arrival of the Inflationary

<sup>1</sup> In curved space-time a single regularisation is not enough to rid  $\langle T_{ik} \rangle$  of all its divergences. Three remaining 'infinities' must be regularised, leading to the renormalisation of additional terms in the one-loop effective Lagrangian for the gravitational field, which, in an FRW Universe has the form:  $\mathcal{L}_{\text{eff}} = \sqrt{-g}[\Lambda_\infty + R/16\pi G_\infty + \alpha_\infty R^2 + \beta_\infty R_{ij}R^{ij}]$ . Renormalisation of the first term  $\Lambda_\infty \rightarrow 0$  corresponds to normal ordering. The presence of the second term  $R/16\pi G_\infty$  led Sakharov to postulate that the gravitational field might be 'induced' by one-loop quantum effects in a curved background geometry, since one could recover the ordinary Einstein action by renormalising the 'bare' value  $G_\infty$  to its observed value:  $G_\infty \rightarrow G_{\text{obs}}$  [7]. The remaining two terms in  $\mathcal{L}_{\text{eff}}$  give rise to vacuum polarization effects and have been extensively discussed in the literature [8].

scenario it was discovered that zero-point vacuum fluctuations could be ‘super-adiabatically amplified’ giving rise, in the case of tensor modes, to a stochastic background of relic gravity waves. These results, together with previous work on particle production and vacuum polarization in the vicinity of black holes, emphasised the close symbiotic ties between general relativity and quantum field theory in curved space-time. In this essay I present further evidence of the importance of the vacuum in general relativity by showing that the amplification of zero-point vacuum fluctuations of a *non-minimally coupled field* during an early Inflationary epoch can give rise to a small cosmological constant today, in agreement with observations.

Consider the the wave equation satisfied by a massive scalar field in curved space-time

$$[\square + \xi R + m^2]\Phi = 0 \quad (4)$$

where  $R = 6\ddot{a}/a^3$  is the Ricci scalar in a FRW Universe, and  $\xi$  parametrises the coupling to gravity. Since field variables separate, one can write

$$\Phi_k = (2\pi)^{-3/2} \phi_k(\eta) e^{-i\mathbf{k}\cdot\mathbf{x}}$$

for each wave mode, after the substitution  $\chi_k = a\phi_k$  this gives rise to

$$\ddot{\chi}_k + [k^2 + m^2 a^2 - (1 - 6\xi)\ddot{a}/a]\chi_k = 0, \quad (5)$$

where differentiation is carried out with respect to the conformal time  $\eta = \int dt/a$ . Equation (5) strongly resembles the one dimensional Schrödinger equation in quantum mechanics [9]

$$\frac{\hbar^2}{2m} \frac{d^2\Psi}{dx^2} + [E - V(x)]\Psi = 0. \quad (6)$$

Comparing (5) and (6) we find that the role of the potential barrier in space  $V(x)$  is now played by the “potential barrier in time”

$$V(\eta) = -m^2 a^2 + (1 - 6\xi)\ddot{a}/a. \quad (7)$$

(The form of the barrier is shown in Fig. 1 for an Inflationary epoch succeeded by radiative and matter dominated eras.) The presence of a barrier

in quantum mechanics leads to particles being reflected and transmitted, so that  $\Psi_{in}(x) = \exp(ikx) + R(k)\exp(-ikx)$  in the incoming region, and  $\Psi_{out}(x) = T(k)\exp(ikx)$  in the outgoing region. Similarly, the time-like barrier  $V(\eta)$  will lead to particles moving both forwards and backwards in time. The scalar field at late times will therefore not be in its vacuum state  $\phi_k^+$  but in a superposition of positive and negative frequency states

$$\phi_{out}(k, \eta) = \alpha\phi_k^+ + \beta\phi_k^-. \quad (8)$$

Within the framework of field theory this result signifies quantum particle production.

The role of reflection and transmission coefficients  $R, T$  is now played by the Bogoliubov coefficients  $\alpha, \beta$  which determine the extent of particle production. <sup>2</sup> From (5) it is easy to see that as  $k \rightarrow \infty$  the amplitude of scalar field modes decreases conformally with the expansion of the Universe  $\phi^+ = \frac{1}{\sqrt{2ka}} \exp(-ik\eta)$ . However large wavelength modes (small  $k$ ) can have their amplitudes ‘superadiabatically amplified’ upon reentering the horizon after Inflation, as shown in figure 1.

<sup>2</sup> The Bogoliubov coefficients are determined by matching ‘in’ and ‘out’ modes, in analogy with quantum mechanics. The ‘in’ modes are defined by the adiabatic vacuum during Inflation and the ‘out’ modes are defined during the radiative or matter dominated epochs. In all cases mode functions can be written in terms of Hankel functions.

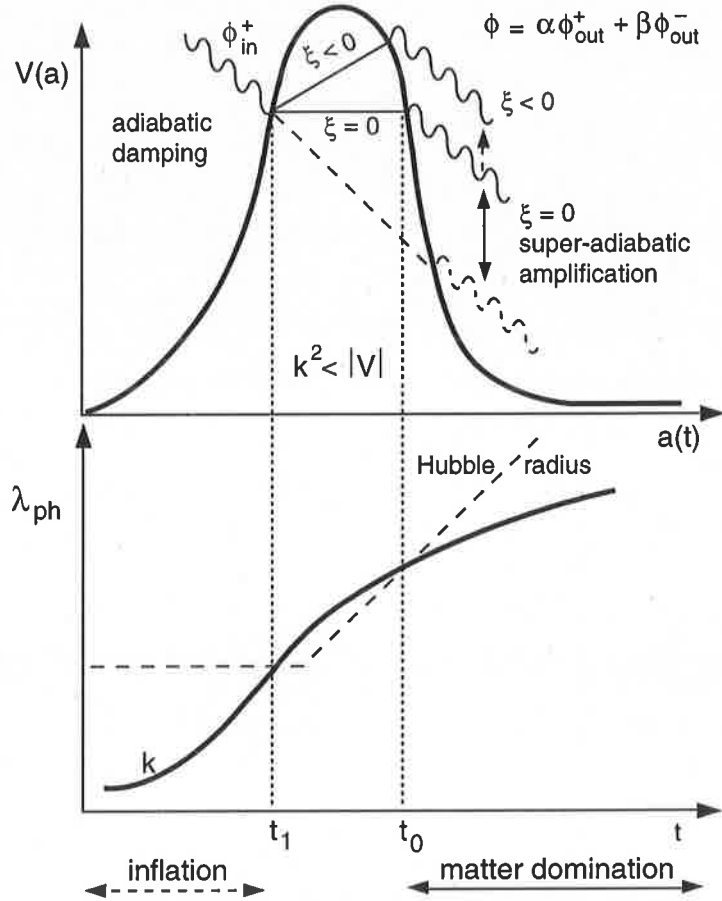


Fig. 1. The time-like potential  $V(\eta)$  of (7) is illustrated for a Universe that underwent an early Inflationary epoch. The amplitude of modes having wavelengths larger than the Hubble radius either increases ( $\xi < 0$ ) or remains frozen to a fixed value ( $\xi = 0$ ) as the Universe expands. The amplitude of smaller wavelength modes, on the other hand decreases conformally with the expansion of the Universe. As a result, modes with  $\xi \leq 0$  have their amplitude super-adiabatically amplified on re-entering the Hubble radius after inflation signifying particle production. Clearly more particles are created for  $\xi < 0$  than for  $\xi = 0$  [15]. (The case  $\xi = 0$  also describes the quantum mechanical production of gravity waves during Inflation [9].)

An exact calculation of the quantum creation of gravitons by Allen (1988) led to the surprising result that the ratio of the energy density of gravity waves to the matter density remains unchanged if the Hubble parameter did not evolve during Inflation (*i.e.*  $a \propto \exp Ht$ ). (Gravity waves behave exactly like massless minimally coupled scalars, *i.e.*  $m = \xi = 0$  in (5).) This treatment was soon generalised to power law Inflation ( $a(t) \propto t^p$ ) for which the ratio  $\rho_g/\rho_m$  was shown to grow with time [12]! Although the density in gravity waves is

probably too small to play an important role in the expansion dynamics of the Universe today, these early results of graviton production served to highlight some important features of the particle production process, namely: (i) the equation of state of created particles need not be  $w_g = p_g/\rho_g = 1/3$ , but could have the ‘chameleon-like’ property  $w_g \simeq w_m$ , which allows the gravity wave density to scale exactly as the matter density during an extended expansion period (here  $w_g, w_m$  denote the equation of state of relic gravity waves and matter respectively); (ii) in many cases  $w_g \geq w_m$  which permits the relic graviton density to fall off at a slower rate than the matter density, and hence to influence the expansion of the Universe at very late times (since  $\rho \propto a^{-3(1+w)}$ ). Similar properties are exhibited by scalar fields exhibiting ‘quintessence’, which are currently in vogue to explain a dynamical  $\Lambda$ -term (Ratra & Peebles 1988, Zlatev, Wang & Steinhardt 1999).

Even more interesting effects arise for light, non-minimally coupled fields with  $\xi < 0, m/H \lesssim 1$ . In this case the leading order contribution to the vacuum expectation value of the energy-momentum tensor is given by

$$\langle T_{ik} \rangle \simeq -\xi(R_{ik} - \frac{1}{2}g_{ik}R)\langle \Phi^2 \rangle + \frac{1}{2}g_{ik}m^2\langle \Phi^2 \rangle + \dots \quad (9)$$

It is interesting that whereas the first term in the right hand side of (9) is simply proportional to the Einstein tensor, the second has the covariant form usually associated with a cosmological constant (*i.e.*  $T_{ik} = g_{ik}\Lambda$ ). The generalised Einstein equations

$$R_{ik} - \frac{1}{2}g_{ik}R = -8\pi G(T_{ik} + \langle T_{ik} \rangle), \quad (10)$$

reduce to

$$3H^2 = 8\pi G(\rho_m + \rho_{vac}), \quad (11)$$

where

$$\rho_{vac} \equiv \langle T_{00} \rangle \simeq 3\xi H^2 \langle \Phi^2 \rangle + \frac{1}{2}m^2 \langle \Phi^2 \rangle, \quad (12)$$

$$\langle \Phi^2 \rangle = \frac{1}{2\pi^2} \int dk k^2 |\phi_{out}(k, \eta)|^2. \quad (13)$$

The term proportional to  $H^2 \langle \Phi^2 \rangle$  in (12) may be absorbed into the left hand

side of (11) leading to

$$3H^2 \simeq 8\pi\bar{G}[\rho_m + \frac{1}{2}m^2\langle\Phi^2\rangle] \quad (14)$$

where  $\bar{G} \simeq G/(1 + 8\pi G|\xi|\langle\Phi^2\rangle)$  is the new, time dependent gravitational constant. (Observational bounds on the rate of change of  $\bar{G}$  set the constraint  $|\xi| \ll 1$ .) An important aspect of wave propagation in an expanding Universe is that the infrared (IR) structure of ultra-light modes is preserved during expansion (see [15] and references therein). Thus  $\phi_{out}$  inherits the infrared behaviour of  $\phi_{in}$  ensuring that  $\langle\Phi^2\rangle$  will be a very large quantity since it will share the IR behaviour of  $\langle\Phi_{in}^2\rangle$  evaluated during Inflation. (It is well known that  $\langle\Phi^2\rangle$  is very large during Inflation if  $\xi < 0$ ,  $m/H \lesssim 1$  [11].) As a result,  $\bar{G} \simeq 1/(8\pi|\xi|\langle\Phi^2\rangle)$  and

$$\begin{aligned} \Lambda_{eff} &\equiv 8\pi\bar{G}\langle T_{00}\rangle \simeq m^2/2|\xi|, \\ \Omega_\Lambda &\equiv \Lambda_{eff}/3H^2 \simeq \frac{1}{6|\xi|}(m/H)^2. \end{aligned} \quad (15)$$

The energy density of created particles thus defines an *effective cosmological constant* which is constructed from fundamental properties of the field such as its mass and coupling to gravity. It is easy to see that  $\Omega_\Lambda$  can contribute significantly to the total density of the Universe for a broad range of values of  $m/H$  and  $\xi$ . Although this calculation was performed for non-minimal scalars, the importance of particle production during the current epoch could extend to other fields as well. The similarity between the dilaton and non-minimally coupled scalar fields demonstrated by Damour & Vilenkin (1996) suggests that light scalar fields which abound in string theory (dilatons, moduli fields) could contribute substantially to the mass density of the Universe [16].

In this essay we have shown that particle production effects might cause the cosmological constant to ‘self-replicate’, in the sense that a large cosmological constant during an early Inflationary epoch could generate a small value of  $\Lambda$  during later epochs, including the present. The Inflationary scenario can therefore give rise to a Universe that is *both flat and*  $\Lambda$ -dominated:  $\Omega_m + \Omega_\Lambda \simeq 1$ , in agreement with current observations.

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