Gravitational collapse of null strange quark fluid and cosmic censorship

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We study gravitational collapse of the general spherically symmetric null strange quark fluid having the equation of state, $p = (\rho - 4B)/n$, where B is the bag constant. An interesting feature that emerges is that the initial data set giving rise to naked singularity in the Vaidya collapse of null fluid gets covered due to the presence of strange quark matter component. Its implication to the Cosmic Censorship Conjecture is discussed.

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I. INTRODUCTION

A gravitating object when it undergoes indefinite collapse, the end product is singularity which is marked by divergence of physical parameters like energy density. As the singularity is approached, density diverges and it would therefore be of relevance to consider the state of matter at ultra high density beyond the nuclear matter. One of such possible states could be the strange quark matter which consists of u, d and s quarks. It is energetically most favored state of baryon matter. It could either be produced in quark-hadron phase transition in the early Universe or at ultra high energy neutron stars converting into strange (matter) stars [1]. In the context of gravitational collapse, which is our concern here, it is the latter process which would be pertinent.

The key question in the collapse process is whether the singularity so formed will be visible or will be covered by an event horizon prohibiting its visibility to an external observer. That the latter is the case goes by the name of Cosmic Censorship Conjecture (CCC) (see [2] for reviews of the CCC)), which remains one of the most important unresolved issues in classical general relativity. In the strong form of CCC, there could emerge no null rays from the singularity in a reasonable space-time, and hence it is invisible for all observers. That is, there occurs no naked singularity for any observer. On the other hand the weak form states that null rays can emerge from the singularity which is however covered by an event horizon and hence they cannot reach out the external observer. In the weak form singularity is locally naked, say for an observer sitting on the collapsing star, but it is globally not because it is safely hidden behind an event horizon.

There do however exist cases of regular initial data sets giving rise to possibility of null rays emanating from the singularity and reaching out to external observer [3]. That is, both the possibilities of singularity being naked and hidden inside a black hole can occur. The critical question is what decides between these two possibilities? In this context, it has recently been argued that it is the shearing effect in the collapsing inhomogeneous dust cloud that is responsible for the ultimate outcome [4]. This happens because shear produces distortion in the collapsing fluid congruence which could cause distortion in the geometry of the apparent horizon surface. Such a distortion of the apparent horizon could let null rays emanating from the singularity to escape to external observer. It turns out that if shear close to the center exceeds a threshold limit, it gives rise to a naked singularity and else a black hole.

Another critical question is, what is the state of matter as the singularity is approached? It is certainly a case of diverging density, and hence it would be appropriate to consider near the singularity matter in the highest known density form. That brings in the strange quark matter (SQM). Recently, collapse of charged strange quark fluid together with the Vaidya null radiation has been studied [5]. In this paper, we would further like to analyze this process from the point of view of formation of naked singularity and its strength, and more importantly to bring out the effect of SQM. It turns out that effect of SQM leads to covering up the region in the initial data set window for

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naked singularity. That is, it tends to favor black hole against naked singularity and consequently the CCC. This happens because SQM contributes an additional attractive potential.

The SQM fluid is characterized by the equation of state $p = (\rho - 4B)/3$ where B is the bag constant indicating the difference between the energy density of the perturbative and nonperturbative QCD vacuum, and ρ, p are the energy density and thermodynamic pressure of the quark matter [1, 6]. The fluid consists of zero mass particles with the QCD corrections for trace anomaly and perturbative interactions [5]. The boundary of a strange star is defined by $p \rightarrow 0$ which would imply $\rho \rightarrow B$. The typical value of the bag constant is of the order of $B \approx 10^{15} g/cm^3$ while the energy density, $\rho \approx 5 \times 10^{15} g/cm^3$ [1]. This shows that SQM will always satisfy the energy conditions because $\rho \geq p \geq 0$. We shall however consider the equation of state $p = (\rho - 4B)/n$ as a generalization of SQM fluid, and particularly the cases $n = 2, \rightarrow \infty$ correspond to known cases of the Vaidya - de Sitter and the Vaidya in constant potential bath collapse respectively.

In this paper, we shall first obtain the general solution for SQM null fluid with the generalized equation of state in the general spherically symmetric metric in the Bondi ingoing coordinates. We shall then bring out explicitly the effect of SQM on gravitational collapse in terms of covering of spectrum in the initial data set for naked singularity by finding the threshold values for the parameters involved in the mass function.

The paper is organized as follows. In the next section, we obtain the general solution and analyze the collapse to show the effect of SQM in shrinking the parameter window in the initial data set giving rise to naked singularity. In the following section we discuss the strength of the singularity and we conclude with a discussion.

II. STRANGE QUARK NULL FLUID COLLAPSE

Expressed in terms of Eddington advanced time coordinate (ingoing coordinate) v, the metric of general spherically symmetric space-time [7]

$$ds^2 = -A(v,r)^2 f(v,r)\, dv^2 + 2A(v,r)\, dv\, dr + r^2 d\Omega^2 \tag{1}$$

$d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2$. Here A is an arbitrary function. It is useful to introduce a local mass function $m(v,r)$ defined by $f = 1 - 2m(v,r)/r$. For $m = m(v)$ and $f = 1$, the metric (1) reduces to the standard Vaidya metric. We wish to find the general solution of the Einstein equation for the matter field given by Eq.(3) for the metric (1), which contains two arbitrary functions. It is the field equation $G^0_1 = 0$ that leads to $A(v,r) = g(v)$. This could be absorbed by writing $d\tilde{v} = g(v)dv$. Hence, without loss of generality, the metric (2) takes the form,

$$ds^2 = -\left[1 - \frac{2m(v,r)}{r}\right] dv^2 + 2dvdr + r^2 d\Omega^2 \tag{2}$$

The energy momentum tensor for the null fluid together with SQM can be written in the form [5, 8, 9]

$$T_{ab} = \mu l_a l_b + (\rho + p)(l_a n_b + l_b n_a) + p g_{ab} \tag{3}$$

Here ρ and p are functions of v and r and the two null vectors l_a and n_a

$$\begin{aligned} l_a = \delta^0_a,\ n_a &= \frac{1}{2}\left[1 - \frac{2m(v,r)}{r}\right]\delta^0_a - \delta^1_a \\ l^a = \delta^a_1,\ n^a &= -\delta^a_0 - \frac{1}{2}\left[1 - \frac{2m(v,r)}{r}\right]\delta^a_1 \\ l_a l^a &= n_a n^a = 0 \quad l_a n^a = -1. \end{aligned} \tag{4}$$

Substituting this in Eq. (3), we find $T^0_0 = T^1_1 = -\rho$, $T^2_2 = T^3_3 = p$ and $T^1_0 = -\mu$, and the trace $T = -2(\rho - p)$. Here $\rho,\ p$ are the strange quark matter energy density and thermodynamic pressure while μ is the energy density of the Vaidya null radiation. The Einstein field equations now take the form

$$4\pi\mu = \frac{\dot{m}}{r^2}, \tag{5a}$$

$$4\pi\rho = \frac{m'}{r^2}, \tag{5b}$$

$$8\pi p = -\frac{m''}{r}. \tag{5c}$$

At higher density, the equation of state becomes uncertain as is the case for nuclear matter and the strange quark matter would be no exception to it. It is therefore appropriate to keep the coefficient n free in the equation of state,

$$p = \frac{1}{n}(\rho - 4B) \tag{6}$$

for SQM. We should however be open to the possibility that in the unknown new super dense matter state, there could be altogether a different kind of contribution which could entirely change the situation. So far SQM is the most dense state of matter considered.

Imposing the equation of state and combining Eqs. (5b) and (5c), we obtain the following differential equation

$$m''(v,r) = -\frac{2}{nr}m'(v,r) + \frac{32\pi B}{n}r \tag{7}$$

From this equation it is clear that the term involving the bag constant B makes the contribution similar to the cosmological constant. We shall thus seek the solution in the form,

$$m(v,r) = m_0(v,r) + \frac{\Lambda}{3}r^3 \tag{8}$$

which would lead to

$$B = \frac{(n+1)\Lambda}{16\pi} \tag{9}$$

and the differential equation

$$m_0''(v,r) + \frac{2}{nr}m_0'(v,r) = 0 \tag{10}$$

This has the general solution

$$m_0(v,r) = S(v)r^{(n-2)/n} + M(v) \tag{11}$$

The two arbitrary functions $M(v)$ and $S(v)$ would be restricted only by the energy conditions. Here Λ is not the cosmological constant but instead is related to the bag constant B via Eq.(9). Thus the metric describing the radial collapse of null SQM in (v,r,θ,ϕ) coordinates reads as:

$$ds^2 = -(1 - \frac{2M(v)}{r} - \frac{2S(v)}{r^{2/n}} - \frac{\Lambda r^2}{3})dv^2 + 2dvdr + r^2 d\Omega^2 \tag{12}$$

This metric represents a solution to the Einstein equations for a collapsing null SQM. The physical quantities for this metric as in [5, 8] are given by

$$\mu = \frac{1}{4\pi r^2}\left[\dot{M}(v) + \dot{S}(v)r^{(n-2)/n}\right] \tag{13}$$

$$\rho = \frac{1}{4\pi r^2}\left[\frac{n-2}{n}S(v)r^{-2/n} + \Lambda r^2\right] \tag{14}$$

$$p = \frac{1}{4n\pi r^2}\left[\frac{n-2}{n}S(v)r^{-2/n} - \Lambda n r^2\right] \tag{15}$$

Clearly, all the energy conditions would be satisfied for $n \geq 2$ because it would ensure $\rho \geq 0$ and $p \geq 0$, while $\mu \geq 0$ would be taken care of when we choose the mass functions for both null radiation and SQM. The initial radius of the star from which the collapse begins would be given by $p = 0$ which would also relate the bag constant with the mass function $S(v)$.

We are studying the collapse of SQM null fluid on a non-flat but empty cavity. The first shell arrives at $r = 0$ at time $v = 0$ and the final at $v = T$. A central singularity of growing mass would develop at $r = 0$. For $v < 0$, $M(v) = S(v) = 0$, i.e., we have

$$ds^2 = -(1 - \frac{\Lambda r^2}{3})dv^2 + 2dvdr + r^2 d\Omega^2 \tag{16}$$

and for $v > T$, $\dot{M}(v) = \dot{S}(v) = 0$, $M = M_0 > 0$. The space-time for $v = 0$ to $v = T$ is given by the generalized Vaidya metric (12), and for $v > T$ we have the generalized Schwarzschild metric:

$$ds^2 = -(1 - \frac{2M_0}{r} - \frac{\Lambda r^2}{3})dv^2 + 2dvdr + r^2 d\Omega^2 \tag{17}$$

A. Occurrence of naked singularities

In this section, we adapt above solution to study the existence of a naked singularity. Let $K^a = dx^a/dk$ be the tangent vector to the null geodesic, where k is an affine parameter. The geodesic equations, on using the null condition $K^a K_a = 0$, take the simple form

$$\frac{d^2v}{dk^2} + \frac{1}{r}\left[\frac{M(v)}{r} + \frac{2S(v)}{r^{2/n}} - \frac{\Lambda}{3}r^2\right]\left(\frac{dv}{dk}\right)^2 = 0 \tag{18}$$

$$\frac{d^2r}{dk^2} + \left[\frac{\dot{M}(v)}{r} + \frac{\dot{S}(v)}{r^{2/n}}\right]\left(\frac{dv}{dk}\right)^2 = 0 \tag{19}$$

Radial (θ and $\phi = const.$) null geodesics of the metric (11) must satisfy the null condition

$$\frac{dr}{dv} = \frac{1}{2}\left[1 - \frac{2M(v)}{r} - \frac{2S(v)}{r^{2/n}} - \Lambda\frac{r^2}{3}\right] \tag{20}$$

Clearly, the above differential equation has a singularity at $r = 0$, $v = 0$. If the singularity is naked, there must exist null ray emanating from it. By investigating the behavior of radial null geodesics near the singularity, it is therefore possible to determine whether outgoing null curves meet the singularity in the past. To go any further we would require specific form of functions $M(v)$ and $S(v)$, which we choose as follows:

$$2M(v) = \quad \alpha v \ (\alpha > 0), \tag{21a}$$

$$2S(v) = \quad \beta v^{2/n} \ (\beta > 0), \tag{21b}$$

Let $X \equiv v/r$ be the tangent to a possible outgoing geodesic from the singularity. In order to determine the nature of the limiting value of X at $r = 0$, $v = 0$ on a singular geodesic, we let $X_0 = \lim_{r\to 0\ v\to 0} X = \lim_{r\to 0\ v\to 0} \frac{v}{r}$. Using (20), (21) and L'Hôpital's rule we get

$$X_0 = \lim_{r\to 0\ v\to 0} X = \lim_{r\to 0\ v\to 0} \frac{v}{r} = \lim_{r\to 0\ v\to 0} \frac{dv}{dr} = \frac{2}{1 - \alpha X_0 - \beta X_0^{2/n} - \Lambda r^2/3} \tag{22}$$

which implies,

$$\alpha X_0^2 + \beta X_0^{2/n+1} - X_0 + 2 = 0 \tag{23}$$

This is the equation which would ultimately decide the end state of collapse: a black hole or a naked singularity.

Thus by analyzing this algebraic equation, the nature of the singularity can be determined. The central shell focusing singularity would atleast be locally naked (for brevity we have addressed it as naked throughout this paper), if Eq. (23) admits one or more positive real roots [3]. The values of the roots give the tangents of the escaping geodesics near the singularity. When there are no positive real roots to Eq. (23), there are no out going future directed null geodesics emanating from the singularity. Thus, the occurrence of positive roots would imply the violation of the strong CCC, though not necessarily of the weak form. Hence in the absence of positive real roots, the collapse will always lead to a black hole. The positive roots would define the range for the tangent slopes from which the null geodesics can escape to infinity. The critical slope would be given by the double root, marking the threshold between black hole and naked singularity.

1. *Case $n = 3$*

We now examine the condition for the occurrence of a naked singularity for $n = 3$. First note that the Eq. (23) is free of Λ and hence it has no effect on the question under study. However its presence makes the background space-time asymptotically non flat. This happens because when $r \to 0$ the term $\Lambda r^2/3$ in Eq.(22) tends to zero. For $\beta = 0$, the allowed range for α is given by (0, 1/8] as obtained earlier [10] for the Vaidya null radiation collapse. In this case, it would be black hole for $\alpha > 1/8$.

The numerical computation reveals that Eq. (23) would always admit two positive roots for $\alpha \leq \alpha_C$. Tangent to all outgoing radial null geodesics would lie in the range $X_2 < X < X_1$, where X_1 and X_2 are the two roots. Table I

TABLE I: Variation of α_C and X_0 for various β ($n=3$)

β	Critical Valueα_C	Equal Roots X_0
0	1/8	3.9999
0.05	0.093728958525	4.18083
0.1	0.06294108366	4.39273
0.15	0.032689168719	4.64701
0.2	0.0030431168445	4.9625

TABLE II: Values of Roots X_0 for $\alpha < \alpha_C$ for different β ($n=3$)

β	$\alpha < \alpha_C$	Roots (X_0)
0.0	0.1	1.40338, 1.9342
0.05	0.09	1.52319, 1.7237
0.1	0.06	1.55259, 1.74615
0.15	0.03	1.57943, 1.78409
0.2	0.003	1.69165, 1.72026

shows the critical values of α for various values of β. The window for naked singularity is defined by (0,α_C], and it is black hole for $\alpha > \alpha_C$. Table II indicates the slope range, given by the two roots, for the null geodesics to escape. It is seen that α_C decreases with increase in β, i.e., initial data set (0,1/8] for a naked singularity of the Vaidya collapse shrinks by the introduction of SQM. There exists a threshold value $\beta_T = 0.205198$ such that for $\beta \geq \beta_T$, gravitational collapse of strange quark null fluid would always end into a black hole for all α.

Note that α refers to rate of collapse of the null radiation while β would refer to that of SQM. The β-threshold would therefore define a critical rate of collapse for SQM required for collapse to end in a black hole. Then it would fully respect CCC. Thus introduction of quark matter favors formation of black hole.

2. Other Cases

a. Case $n = 2$

Then we find that

$$\mu = \frac{1}{4\pi r^2}\left[\dot{M}(v) + \beta\right], \qquad \rho = -p = \frac{\Lambda}{4\pi} \tag{24}$$

and the algebraic equation takes the form

$$(\alpha + \beta)X_0^2 - X_0 + 2 = 0 \tag{25}$$

The metric in this case takes the form of the Vaidya - de Sitter metric. The singularity is visible for $(\alpha+\beta) < 1/8$. It is the null fluid collapse in the background of the de Sitter space, where Λ is generated by the bag constant.

b. Case $n \to \infty$.

We have

$$\mu = \frac{1}{4\pi r^2}\dot{M}(v), \qquad p = 0, \qquad \rho = \frac{B}{4\pi r^2} \tag{26}$$

TABLE III: Variation of β_T with n

$2/n$	β_T
0.9	0.144
0.8	0.167
0.75	0.18
0.5	0.2052
0.25	0.45
0.125	0.6194
0.0625	0.756

TABLE IV: Values of equal roots X_0 for different n and β

$2/n$	β	α_C	Equal Root (X_0)
0.90	0.14	0.00343411423125	4.21544
0.80	0.16	0.00502531901422	4.47484
0.75	0.17	0.00679411554354	4.61836
0.5	0.20	0.0306384999095	5.1784
0.25	0.44	0.00178952699837	9.51405
0.125	0.61	0.00077533872957	16.519
0.0625	0.74	0.00061702385153	27.1684

In this case we have the dual Vaidya metric or Vaidya metric with constant potential [11]. The algebraic equation: $\alpha X_0^2 + (\beta - 1)X_0 + 2 = 0$, would admit a positive root for $\alpha \leq 1/8(\beta-1)^2$, giving the range for naked singularity as obtained in [11]. This is simply the null fluid collapse in the background of constant potential which is characterized by $T_0^0 = T_1^1 = const./r^2$, as is the case in Eq.(26) above. Note that $\beta < 1$ else the metric signature would change.

c. Other n

We also note that as n increases, so does the threshold value β_T. This is shown in Tables III and IV.

III. CURVATURE STRENGTH OF SINGULARITIES

An important aspect of a singularity is its gravitational strength [12]. A singularity is gravitationally strong in the sense of Tipler, if it destroys any object which falls into it and weak otherwise. It is now widely believed that space-time does not admit an extension through a strong curvature singularity, i.e., space-time is geodesically incomplete. Through a weak singularity, space-time could be analytically extended to make it geodesically complete. There have been attempts to relate strength of a singularity to its stability [13]. Recently, Nolan [14] gave an alternative approach to check the nature of singularities without having to integrate the geodesics equations. It was shown [14] that a radial null geodesic which runs into $r = 0$ terminates in a gravitationally weak singularity if and only if dr/dk is finite in the limit as the singularity is approached (this occurs at $k = 0$, the over-dot here indicates differentiation along the geodesic). If the singularity is weak, we have

$$\frac{dr}{dk} \sim d_0 \qquad r \sim d_0 k \tag{27}$$

Using the asymptotic relationship ($dv/dk \sim d_0 X_0$ and $v \sim d_0 X_0 k$) and Eq. (21), the geodesic equation yields

$$\frac{d^2v}{dk^2} \sim -(\alpha X_0 d_0^{-1} k^{-1} + \beta X_0^{2/n} d_0^{-1} k^{-1} - \frac{\Lambda}{3} d_0 k) d_0^2 X_0^2 \tag{28}$$

But this gives

$$\frac{d^2v}{dk^2} \sim c k^{-1} \tag{29}$$

where $c = (\alpha X_0^{(n-2)/n} + \beta) X_0^{2(n+1)/n} d_0^{-1}$. This is inconsistent with $dv/dk \sim d_0 X_0$, which is finite. Since the coefficient c is non-zero, the naked singularity is gravitationally strong in the sense of Tipler [15]. Having seen that the naked singularity is a strong curvature singularity, we check it for scalar polynomial singularity. The Kretschmann scalar for the metric (12) with the prescriptions (21), takes the form

$$\begin{aligned} K = \; & \frac{4}{3n^4r^4}\Bigg[\alpha^2 n^4 X_0^2 + (4\beta\Lambda n^2 - 6\beta\Lambda n^3 + 2\beta\Lambda n^4) X_0^{2/n} + (12\alpha\beta\Lambda n^2 + \\ & 8\alpha\beta\Lambda n^3 + 6\alpha\beta\Lambda n^4) X_0^{(n+2)/2} \\ + \; & (12\beta^2 n + 15\beta^2 n^2 + 3\beta^2 n^4) X^{4/n}\Bigg] + \frac{8}{3}\Lambda^2 \end{aligned} \tag{30}$$

which diverges at the naked singularity and hence the singularity is also a scalar polynomial singularity. The Ricci scalar also diverges. It however vanishes for the Vaidya space-time [16]. Thus the naked singularities studied here are strong curvature singularity and hence are physically significant. Lastly, we shall calculate Weyl scalar

$$C = \frac{4}{3n^4r^4}\left[\alpha n^2 X + 2\beta X^{2/n} + 3\beta X^{2/n} + \beta n^2 X^{2/n}\right]^2 \tag{31}$$

which too would diverge. The Weyl curvature describes non local effects of gravitation produced by free part of the field. It is generated by inhomogeneity and anisotropy, particularly divergence of shear [17]. In the context of naked singularity, like shear and inhomogeneity the Weyl curvature would also play significant role.

IV. DISCUSSION

In this paper, we have obtained the general solution for null SQM fluid with the equation of state given by Eq. (6) for the general spherically symmetric metric (2) in the Eddington advanced time coordinate (ingoing coordinates). We have used the solution to study the end state of the collapse. The present case is an example of non self-similarity as well as non asymptotic flatness, and yet there does occur a regular initial data set which would lead to naked singularity.

The relevant question is what effect does the presence of the SQM have on formation or otherwise of a naked singularity. Our results imply that the presence of SQM leads to shrinking of the initial data space for naked singularity of the Vaidya null fluid collapse. That is, it tends to favor black hole. This tendency is caused by the additional attractive potential, varying as $r^{-2/n}$, produced by SQM which results in strengthening of gravity. There also exists a threshold value $\beta_T = 0.205198$ such that for $\beta \geq \beta_T$ the end state of collapse of null SQM is always a black hole for all α. This is the critical value of the rate of collapse of SQM for respecting CCC. The case $n = 3$ represents the zero mass particles with QCD corrections for trace anomaly and perturbative interactions [5]. The energy conditions require $n \geq 2$. The case of $n = 2$ corresponds to the null fluid collapse in the background of de Sitter space where the bag constant provides the Λ. The case $n \to \infty$, corresponds to the null fluid collapse in the background of constant potential space as studied in [11]. These are the two extreme limiting cases encompassing the physically allowed cases. Though there is no much physical motivation in the context of SQM for $n \neq 3$ cases, yet these two particular cases are interesting. That is at least these three cases could be considered in a unified equation of state given by Eq. (8).

As mentioned earlier the strong version of the CCC doesn't allow even locally naked singularity, i.e., the space-time should be globally hyperbolic. It turns out that necessary and sufficient condition for a singularity to be locally naked is that the algebraic Eq. (23) should have atleast one or more positive root [3]. Hence existence of the positive roots of Eq. (23) is a counter example to the strong version of the CCC. In the absence of the proof of any version of the CCC, such examples remain the only tool to study this important and unresolved problem.

Quark stars could be formed in the realistic astrophysical setting. The core collapse of a massive star after the supernova explosion sets in first and second order phase transitions which result into deconfined quark matter. The other possibility is that some neutron stars could accrete matter and undergo phase transition to turn into quark stars [18, 19, 20]. Thus study of gravitational collapse with quark matter component is quite in order because it is perhaps astrophysically more realistic. In the ultimate stage of collapse close to the singularity, density is diverging. The quark matter contribution would therefore perhaps be most significant in deciding the ultimate result of the collapse.

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