

NEWTONIAN UNIVERSES WITH SHEAR AND ROTATION

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Summary

Cosmological models involving shear and rotation are considered within the framework of Newtonian theory. It is shown that, under certain assumptions, a differential equation can be obtained for the volume expansion factor. The solutions of this equation show that it is not possible to get non-singular finitely oscillating cosmological models.

1. *Introduction.*—Attempts made in the nineteenth century to deal with the cosmological problem within the framework of Newtonian theory of dynamics and gravitation resulted in failure, mainly because the assumption was made that universe was static. This was interpreted as a failure of the Newtonian theory and no further progress in theoretical cosmology was made until the advent of general relativity. The success of relativistic cosmology completely overshadowed Newtonian cosmology until 1934 when Milne and McCrea (1) raised the Newtonian problem again. With suitable interpretation of Newtonian terms they were able to reproduce most of the results of relativistic cosmology.

One of the main difficulties of the relativistic cosmological models based on Einstein's field equations with zero cosmological constant and the Robertson-Walker line-element

$$ds^2 = c^2 dt^2 - R^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right\}, \quad k = 0, \pm 1, \quad (1)$$

has been the presence of a singularity in the solution. At time $t = t_0$ the function $R(t_0) = 0$ and the density of the universe is infinite. While this is interpreted by some authors as a singular event leading to the origin of the universe, others find the existence of such a phase very unsatisfactory. The introduction of the λ -term can avoid a singularity but does not produce finitely oscillating models. With a view to prevent such a singularity and to obtain oscillating solutions Heckmann and Schücking (2) proposed to modify the postulate of isotropy. This brings in rotation and shear. It was shown by Raychaudhuri (3) that if the motion of cosmic fluid is not chaotic so that one can associate a unique velocity vector with every point, then in the co-ordinate system in which the fluid is at rest one has

$$\frac{\dot{R}}{R} = \frac{1}{3}(\lambda - \kappa\rho c^4 - \Phi^2 + 2\Omega^2), \quad (2)$$

$$\rho R^3 = \text{constant}, \quad (3)$$

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where $R^6 = -\det. (g_{ik})$, g_{ik} being the metric tensor, ρ is the density of matter, λ the cosmical constant and $\kappa = 8\pi G/c^4$. The quantity Φ^2 vanishes if and only if the expansion is isotropic, and is positive otherwise. Ω can be identified with the local angular velocity.

This shows that while anisotropy in the expansion generally produces effects opposite to that required, rotation produces forces that oppose the force of gravitation. Thus the net outcome is left uncertain.

Relations similar to (2) and (3) were obtained subsequently by Heckmann and Schücking (4) for Newtonian cosmology by methods similar to those used by Milne and McCrea. This shows that the agreement in the results of relativistic and Newtonian cosmologies persists even in the case of anisotropic models. This is the reason for using the Newtonian framework in what follows. The equations in this case are easier to handle than in the case of general relativity. The question of legitimacy of the application of Newtonian methods will not be considered.

2. *The equations of Newtonian cosmology.*—Following the axioms of Newtonian mechanics we have a Euclidean space with rectangular co-ordinates x_μ ($\mu = 1, 2, 3$) and a uniform even-flowing time t . At any time t the universe presents the same large scale view to all observers of a special class. These observers constitute a substratum whose motion is idealized as the streaming of a uniform fluid. This is the homogeneity postulate contained in the cosmological principle.

Taking $x_\mu = 0$ for the co-ordinates of one such observer, the velocity distance relation can be written in the form

$$v_\mu = H_{\mu\nu} x_\nu, \quad (4)$$

where $H_{\mu\nu}$ is a tensor depending on t only. If the isotropy postulate of the cosmological principle is also assumed, $H_{\mu\nu}$ becomes an isotropic tensor. We will, however, keep $H_{\mu\nu}$ quite general to include the possibility of shear and rotation. The density ρ and the pressure tensor $p_{\mu\nu}$ are also functions of t only.

The relation (2) can be written explicitly in terms of time dependent quantities in the form

$$x_\mu = a_{\mu\nu} x_\nu^0, \quad v_\mu = \dot{a}_{\mu\nu} x_\nu^0, \quad (5)$$

where x_ν^0 are constants and $a_{\mu\nu}$ are related to $H_{\mu\nu}$ by

$$\dot{a}_{\mu\nu} = H_{\mu\lambda} a_{\lambda\nu}. \quad (6)$$

Let

$$\Delta = \det. (a_{\mu\nu}) = R^3. \quad (7)$$

Δ denotes the volume expansion factor. In the case of isotropic expansion R has its usual meaning.

Substituting (5) in the equation of continuity

$$\frac{\partial \rho}{\partial t} + \text{div} (\rho \mathbf{v}) = 0 \quad (8)$$

gives, after integration,

$$\Delta \rho = \text{constant} = \Delta_0 \rho_0 \text{ (say)}. \quad (9)$$

Euler's equation of motion can similarly be considered with an external force \mathbf{F} where

$$\text{div} \mathbf{F} = -4\pi G\rho. \quad (10)$$

Taking the divergence of the equation in the usual way and using (6) to eliminate $H_{\mu\nu}$ gives an equation for $a_{\mu\nu}$:

$$\ddot{a}_{\mu\nu} = -\frac{4\pi G\rho}{3} a_{\mu\nu}. \quad (11)$$

It is worth noting at this stage that in the corresponding relativistic case pressure terms are also present and may play an important role.

The equations (9) and (11) together with given initial conditions determine completely the subsequent behaviour of the system. It is easy to see that $\Delta = 0$ at some t is the necessary and sufficient condition for the existence of a singularity. For, if $\Delta = 0$ the equations (5), when consistent, do not have a unique solution. This means that given x_μ , provided they satisfy the consistency relation, there are a number of different possible solutions for x_μ^0 . If x_μ do not satisfy the consistency condition there is no solution. The converse follows by reversing the argument.

Thus if $\Delta = 0$ and $(a_{\mu\nu})$ has rank 2 the singularity is disk-shaped. If $a_{\mu\nu}$ has rank 1 it is line-shaped while if $a_{\mu\nu}$ has rank 0 it is a point singularity. The latter is the case in the isotropic expansion of the Einstein de Sitter model.

3. *The nature of solutions.*—In view of the physical significance of Δ , it is of interest to study its behaviour with t . Since $x_\mu = x_\mu^0$ gives $\Delta = 1$ we can take $\Delta_0 = 1$ and define

$$\tau = \sqrt{\frac{4\pi G\rho_0}{3}} t. \quad (12)$$

The equations (11) then become

$$\frac{d^2}{d\tau^2} (a_{\mu\nu}) = -\frac{a_{\mu\nu}}{\Delta}. \quad (13)$$

From these an equation can be obtained for Δ in the following way. Differentiate Δ with respect to τ successively, using the method of differentiating a determinant. Substitute for $d^2 a_{\mu\nu}/d\tau^2$ at each stage from (13). It turns out that all $a_{\mu\nu}$ and $da_{\mu\nu}/d\tau$ disappear after the fourth differentiation. A differential equation for Δ is accordingly obtained in the form

$$\Delta^2 \frac{d^4 \Delta}{d\tau^4} + 7\Delta \frac{d^2 \Delta}{d\tau^2} - 4 \left(\frac{d\Delta}{d\tau} \right)^2 + 9\Delta = 0. \quad (14)$$

One solution of (14) is $\Delta = 9/2 \tau^2$ —this being the Einstein de Sitter solution. In this case $\Delta \rightarrow 0$ as $\tau \rightarrow 0$. Other solutions correspond to various anisotropic models. It does not seem possible to solve (14) explicitly in terms of elementary functions. The following property of the equation is, however, useful in narrowing the class of solutions.

(P): "If $\Delta = f(\tau)$ is a solution so is $A^{-2}f(A\tau + B)$ where A, B are arbitrary constants".

If Δ is plotted against τ the above property means that all solutions of the form $A^{-2}f(A\tau + B)$ are obtained from $f(\tau)$ by change of τ , Δ -scales by suitable factors and shift of origin along the τ -axis. The physical properties of all such solutions are similar and one solution is enough to represent the class.

It should therefore be possible to reduce the order of (14) by two by suitable transformations. This can be accomplished in the following way:

Let

$$\Delta = F^2, \quad \left(\frac{dF}{d\tau} \right)^2 = X(F), \quad F = e^U, \quad \frac{dX}{dU} = Y(X). \quad (15)$$

The resulting equation for $Y(X)$ is a second order one:

$$XY^2 \frac{d^2Y}{dX^2} + XY \left(\frac{dY}{dX} \right)^2 + (X + \frac{1}{2}Y) Y \frac{dY}{dX} + Y^2 - 2XY + 7Y - 2X + 9 = 0. \quad (16)$$

Any particular solution of (16) leads, on further integration, to all solutions of the same class. It is sufficient therefore to consider only the equation (16). We note that as a result of the transformations (15) the quantities X , Y are related to F by the relations

$$X = \left(\frac{dF}{d\tau} \right)^2, \quad Y = 2F \frac{d^2F}{d\tau^2}. \quad (17)$$

As it did not seem possible to integrate (16) analytically, numerical integrations were carried out on EDSAC-2.

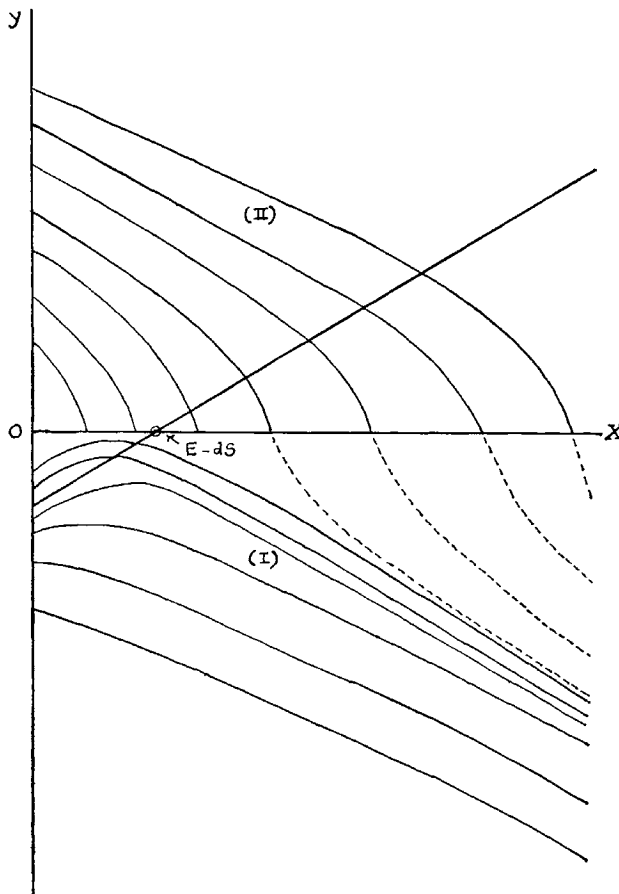


FIG. 1.

First it was investigated whether any solution of (16) leads to a finitely oscillating universe. One can start integrating from a point $X=0$, $Y < 0$ into the region $X > 0$. If an oscillating solution with no singularity is possible, this curve should join continuously on to a similar curve obtained by integrating from a point $X=0$, $Y > 0$. The two types of curves actually obtained are labelled I and II in Fig. 1. Clearly it is not possible to join any of the curves I on to any of the curves II. Curves I in fact show that Y eventually decreases with X . This corresponds to universes where the rotation has been ineffective in preventing a singular state.

There is one exceptional curve of type I, namely the straight line

$$Y = \frac{2}{3}X - 3. \quad (18)$$

In terms of $F = \Delta^{1/2}$ this gives

$$\frac{2}{3} \left(\frac{dF}{d\tau} \right)^2 = 3 + CF^{2/3} \quad (19)$$

where C is constant. If $C < 0$, F oscillates between 0 and $(-3/C)^{3/2}$ and the singularity persists. If $C > 0$ there is no oscillating solution and $F \rightarrow \infty$ as $\tau \rightarrow \infty$.

The Einstein de Sitter solution is represented by the point $Y = 0$, $X = 9/2$. This is, of course, a singular solution. There do not appear to be other points representing singular solutions in the X - Y plane.

The integration along the curves II can be continued into the region $Y < 0$, as shown by dotted lines in the figure. These do not lead to any non-singular solutions.

The ratio by which the universe has changed its size between any two points (X_1, Y_1) , (X_2, Y_2) of any curve can be calculated without it being necessary to know $\Delta(t)$ explicitly. (This curious situation is due to the property (P) mentioned before). Using (17) we get, along any curve $Y(X)$

$$\begin{aligned} \int_{(X_1, Y_1)}^{(X_2, Y_2)} \frac{dX}{Y} &= \int_{(X_1, Y_1)}^{(X_2, Y_2)} \frac{d[(dF/d\tau)^2]}{2F[d^2F/d\tau^2]} \\ &= \int_{F_1}^{F_2} \frac{dF}{F} \\ &= \ln \frac{F_2}{F_1} \end{aligned}$$

where F_1, F_2 are the values of F at (X_1, Y_1) , (X_2, Y_2) respectively. The corresponding results for volumes can be written as

$$\frac{\Delta_2}{\Delta_1} = \exp \left[2 \int_{X_1}^{X_2} \frac{dX}{Y} \right]. \quad (20)$$

All the curves (I) have slope -2 asymptotically so that $Y = -2X$. The integral in (20) therefore diverges as $X_2 \rightarrow \infty$ and $\Delta_2/\Delta_1 \rightarrow 0$. This is precisely the case of singularity.

Many solutions have to be rejected because they do not satisfy (i) the expansion factor is always positive (ii) the solution should contain a phase similar to the one observed now i.e., one of isotropic expansion, with negligible rotation.

4. *Conclusion.*—The above calculations show that, in Newtonian cosmology any rate, the singularity is not prevented simply by the introduction of anisotropy in the form of shear and rotation. The rotation may be able to prevent gravitational collapse into the axis of rotation, but it is unable to do so along the axis of rotation which may itself be changing its direction with time. In the above calculations the λ -term has been ignored, since it does not arise in a theory of

gravitation that is purely Newtonian. In any case the λ -term, if it gives a force of repulsion, is unimportant when the universe is in a state of high density and is therefore not able to offer significant resistance to the force of gravitation.

Raychaudhuri has pointed out that in the Newtonian cosmology there are fewer equations to be satisfied than in the relativistic cosmology and this implies that whereas a relativistic model has a Newtonian analogue the converse may not hold. In this sense the above discussion should be more general than in the relativistic case which is certainly more difficult to handle.

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