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Inflationary Hubble Parameter from the Gravitational Wave Spectrum in the General Slow-roll Approximation

Minu Joy[1*] and Ewan D. Stewart[2]

[1] *Inter-University Centre for Astronomy and Astrophysics, Pune, India*

[2] *Department of Physics, KAIST, Daejeon, Republic of Korea*

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Abstract

We derive first and second order general slow-roll inverse formulae giving the Hubble parameter H in terms of the gravitational wave spectrum. We also derive improved forward general slow-roll formulae giving the gravitational wave spectrum in terms of the Hubble parameter H. Moreover, the general slow-roll consistency condition relating the scalar and tensor spectra is obtained.

*minu@iucaa.ernet.in

1 Introduction

Inflation inevitably leads to the spectra of scalar density perturbations and gravitational waves caused by the tensor perturbations to the spatial metric [1]. The relative contribution of scalar and tensor fluctuations to the Cosmic Microwave background (CMB) anisotropy depends upon the details of the infaltionary potential. Also, the scalar and tensor fluctuations, generate different patterns of polarisation and contribute independently to the observed value [2] of the microwave multipoles, $\mathcal{C}_l$. The exploration of gravitational wave might be possible from the detection of the B-mode of polarisation in the CMB anisotropy [3] and it is hopeful to make more progress in that direction once we get the more precise data from *Planck* [4] and *Big Bang Observer* [5].

The standard slow-roll approximation used for inflationary scenarios makes some strong assumptions about the properties of inflation, which have not yet been confirmed observationally. Hence, a more general slow-roll approximation has been put forward [6] which lifts the extra, unjustified, assumptions of the standard slow-roll approximation. The advantages of the general slow-roll approximation, compared with the standard approximation, are clearly discussed in Ref. [7].

The reconstruction of the inflationary potential from the observations of the scalar density perturbation spectrum [8] and also from the gravitational wave spectrum have been discussed by many authors [9, 10]. The nearly scale-invariant spectrum, $\mathcal{P}_\psi$ of gravitational wave traces the evolution of the Hubble parameter during inflation and it has been shown that one can reconstruct the time dependence of the very early Hubble parameter and matter energy density from the relic garvitational wave spectrum [11]. Recently, we proposed a general inverse formula [12] for extracting inflationary parameters from the scalar power spectrum. There, we invert the single field, general slow-roll formula for the curvature perturbation spectrum to obtain a formula for inflationary parameters. Here we apply our inverse formalism to the gravitational wave spectrum, so as to estimate the Hubble parameter H.

2 First order general slow-roll formulae

2.1 Gravitational wave spectrum

The formalism of general slow-roll approximation and the power spectrum calculation are clearly described in [6]. WIth this general formalism, the first order gravitational wave spectrum for a single field inflation model can be given by [13]

$$\ln \mathcal{P}_\psi(\ln k) = \int_0^\infty \frac{d\xi}{\xi} \left[-k\xi\, W'(k\xi)\right] \left[\ln \frac{1}{p^2} + \frac{2}{3}\frac{p'}{p}\right] \tag{1}$$

where $p = 2\pi a\xi$ and $\xi = -\int \frac{dt}{a} = \frac{1}{aH}\left[1 + \mathcal{O}\left(\frac{\dot{H}}{H^2}\right)\right]$ is minus the conformal time. The window function $-x\, W'(x)$ is given by,

$$W(x) = \frac{3\sin(2x)}{2x^3} - \frac{3\cos(2x)}{x^2} - \frac{3\sin(2x)}{2x} - 1 \tag{2}$$

It has the asymptotic behavior

$$\lim_{x\to 0} W(x) = \frac{2}{5}x^2 + \mathcal{O}(x^4) \tag{3}$$

and the window property,

$$\int_0^\infty \frac{dx}{x}\left[-x\,W'(x)\right] = 1 \tag{4}$$

An alternative form for the gravitational wave spectrum with a particularly simple window function is

$$\ln \mathcal{P}_\psi(\ln k) = \int_0^\infty \frac{d\xi}{\xi}\left[-k\xi\, v'(k\xi)\right]\left[\ln\frac{1}{p^2} - 2\frac{p'}{p}\right] \tag{5}$$

where

$$v(x) = \frac{\sin(2x)}{2x} - 1 \tag{6}$$

$v(x)$ has the asymptotic behavior

$$\lim_{x\to 0} v(x) = -\frac{2}{3}x^2 + \mathcal{O}(x^4) \tag{7}$$

and window property

$$\int_0^\infty \frac{dx}{x}\left[-x\,v'(x)\right] = 1 \tag{8}$$

Here, p is taken as a function of $\ln\xi$ so that

$$p' \equiv \frac{dp}{d\ln\xi} = \xi\frac{dp}{d\xi} = -\frac{p}{2\pi}\frac{dp}{dt} \tag{9}$$

and since $\frac{d\xi}{dt} = -\frac{1}{a}$ and $H = \frac{\dot{a}}{a}$ we get,

$$\frac{p'}{p} = 1 - \left(\frac{H}{2\pi}\right)p \tag{10}$$

Therefore,

$$\frac{H}{2\pi} = \frac{1}{p}\left(1 - \frac{p'}{p}\right), \tag{11}$$

$$\ln\left(\frac{H}{2\pi}\right)^2 = \ln\left(\frac{1}{p^2}\right) - 2\left(\frac{p'}{p}\right) - \left(\frac{p'}{p}\right)^2 - \cdots \tag{12}$$

Substituting the above form in equation (5) we get the general slow-roll formula for gravitational wave spectrum in terms of the Hubble parameter H, up to the first order correction terms, as

$$\ln \mathcal{P}_\psi(\ln k) = \int_0^\infty \frac{d(aH)}{aH}\left[-\frac{k}{aH}\,v'\left(\frac{k}{aH}\right)\right]\ln\left(\frac{H}{2\pi}\right)^2 \tag{13}$$

2.2 Inverse

Using the inverse identity,

$$\int_0^\infty \frac{dk}{k}\, m(k\zeta)\, v(k\xi) = \frac{1}{2\zeta}\left[(\zeta-\xi)\,\mathrm{sgn}(\zeta-\xi) + (\zeta+\xi)\,\mathrm{sgn}(\zeta+\xi)\right] - \frac{1}{\zeta}\left[\zeta\,\mathrm{sgn}(\zeta) + \xi\,\mathrm{sgn}(\xi)\right] \tag{14}$$

and its derivative with respect to ξ,

$$\int_0^\infty \frac{dk}{k}\, m(k\zeta)\left[-k\xi\, v'(k\xi)\right] = \frac{\xi}{2\zeta}\left[\mathrm{sgn}(\zeta-\xi) - \mathrm{sgn}(\zeta+\xi)\right] + \frac{\xi}{\zeta}\mathrm{sgn}(\xi) \tag{15}$$

where $\mathrm{sgn}(x) = -1$ for $x < 0$ and $\mathrm{sgn}(x) = 1$ for $x > 0$ and also where

$$m(x) = \frac{2}{\pi}\left[\frac{1}{x} - \frac{\cos(2x)}{x} - \sin(2x)\right] \tag{16}$$

with the asymptotic behaviour

$$\lim_{x\to 0} m(x) = \frac{4}{3\pi}x^3 + \mathcal{O}(x^5) \tag{17}$$

and the window properties

$$\int_0^\infty \frac{dx}{x}\, m(x) = 1\,, \tag{18}$$

$$\int_0^\infty \frac{dx}{x}\,\frac{1}{x}\, m(x) = \frac{2}{\pi} \tag{19}$$

we can write the first order inverse expression for the gravitational wave spectrum as

$$\ln\left(\frac{H}{2\pi}\right)^2 = \int_0^\infty \frac{dk}{k}\, m\left(\frac{k}{aH}\right)\left[\ln\mathcal{P}_\psi - \frac{\mathcal{P}_\psi{}'}{\mathcal{P}_\psi}\right] \tag{20}$$

The above equation gives an explicit formula for H in terms of the gravitational wave spectrum $\mathcal{P}_\psi$.

3 Second order general slow-roll formulae

Under the general slow-roll formalism [6], the second order spectrum for the scalar [14] and tensor perturbations [13] have been calculated. Here we present an improved second order general slow-roll formula for the gravitational wave spectrum in terms of the Hubble parameter H.

3.1 Gravitational wave spectrum

The general slow-roll gravitational wave spectrum can be derived up to second order terms as [13]

$$\begin{aligned}\ln\mathcal{P}_\psi(\ln k) \;=\; & \int_0^\infty \frac{d\xi}{\xi}\left[-k\xi\, W'(k\xi)\right]\left[\ln\frac{1}{p^2} + \frac{2}{3}\frac{p'}{p}\right] + \frac{\pi^2}{2}\left[\int_0^\infty \frac{d\xi}{\xi} m(k\xi)\frac{p'}{p}\right]^2 \\ & - 2\pi\int_0^\infty \frac{d\xi}{\xi} m(k\xi)\frac{p'}{p}\int_\xi^\infty \frac{d\zeta}{\zeta}\frac{1}{k\zeta}\frac{p'}{p}\end{aligned} \tag{21}$$

Using equation (11) and its derivative, we get

$$-\frac{\dot{H}}{H^2} = -\frac{\frac{p''}{p} + \frac{p'}{p}\left(1 - 2\frac{p'}{p}\right)}{\left(1 - \frac{p'}{p}\right)^2} \tag{22}$$

In single field general slow-roll, $-\frac{\dot{H}}{H^2} \simeq$ constant and also, $\xi \simeq \frac{1}{aH}\left[1 - \frac{\dot{H}}{H^2} + \cdots\right]$. But this is not the case for the general slow-roll inflation with multi-component fields [15] and one has to treat it separately.
Now, we can rewrite the equation (21) to get the second order general slow-roll gravitational wave spectrum in terms H as,

$$\begin{aligned}\ln \mathcal{P}_\psi(\ln k) &= \int_0^\infty \frac{d\xi}{\xi}\left[-k\xi\, v'(k\xi)\right] \ln\left(\frac{H}{2\pi}\right)^2 + \frac{\pi^2}{2}\left[\int_0^\infty \frac{d\xi}{\xi} n(k\xi)\frac{\dot{H}}{H^2}\right]^2 \\ &\quad - 2\pi \int_0^\infty \frac{d\xi}{\xi} n(k\xi)\frac{\dot{H}}{H^2}\left\{\frac{1}{k\xi}\frac{\dot{H}}{H^2} - \int_\xi^\infty \frac{d\zeta}{\zeta}\frac{1}{k\zeta}\frac{\dot{H}}{H^2}\right\}\end{aligned} \tag{23}$$

where

$$n(x) = \frac{1}{\pi}\left[\frac{1}{x} - \frac{\cos(2x)}{x}\right] \tag{24}$$

$n(x)$ has the asymptotic behavior

$$\lim_{x\to 0} n(x) = -\frac{2}{\pi}x + \mathcal{O}(x^3)\,, \tag{25}$$

window property

$$\int_0^\infty \frac{dx}{x}\, n(x) = 1\,, \tag{26}$$

and is related to $m(x)$ by

$$m(x) = n(x) - x\, n'(x)\,. \tag{27}$$

3.2 Inverse

Substituting first order inverse expression, equation (20), into equation (23) and following the same formalism of Ref. [12] we get the second order inverse formula for gravitational wave spectrum as,

$$\begin{aligned}\ln\left(\frac{H}{2\pi}\right)^2 &= \int_0^\infty \frac{dk}{k}\, m(k\xi)\left[\ln \mathcal{P}_\psi - \frac{\mathcal{P}_\psi{}'}{\mathcal{P}_\psi}\right] \\ &\quad - \frac{1}{2\pi^2}\int_0^\infty \frac{dk}{k}\, m(k\xi)\int_0^\infty \frac{dl}{l}\ln\left|\frac{k+l}{k-l}\right| \frac{\mathcal{P}_\psi{}'(\ln l)}{\mathcal{P}_\psi(\ln l)}\int_0^\infty \frac{dq}{q}\left[\ln\left|\frac{k+q}{k-q}\right| - \frac{2qk}{q^2-k^2}\right]\frac{\mathcal{P}_\psi{}'(\ln q)}{\mathcal{P}_\psi(\ln q)} \\ &\quad + \int_0^\infty \frac{dl}{l}\int_0^\infty \frac{dq}{q}\, N(l\xi, q\xi)\,\frac{\mathcal{P}_\psi{}'(\ln l)}{\mathcal{P}_\psi(\ln l)}\frac{\mathcal{P}_\psi{}'(\ln q)}{\mathcal{P}_\psi(\ln q)}\end{aligned} \tag{28}$$

where $N(x,y) = M_1(x,y) - M_2(x,y)$ with

$$\frac{M_j(x,y)}{j} = \frac{2}{\pi^2 xy}\left[g_j(x) + g_j(y) - \frac{1}{2}\,g_j(x-y) - \frac{1}{2}\,g_j(x+y)\right] \tag{29}$$

for which the index j takes values 1 and 2.

$$g_j(x) = x\,\mathrm{Si}(2x) + \frac{j}{2}\,(\cos(2x) - 1) \tag{30}$$

where we denote

$$\mathrm{Si}(x) \equiv \int_0^x \frac{\sin t}{t}\,dt \tag{31}$$

$M_j(x,y)$ has the window property

$$\int_0^\infty \frac{dx}{x}\int_0^\infty \frac{dy}{y}\,\frac{M_j(x,y)}{j} = 1 \tag{32}$$

and the asymptotic behaviour

$$\lim_{x,y\to 0}\frac{M_j(x,y)}{j} = \frac{4xy}{3\pi^2}\left[1 + \mathcal{O}\left(x^2+y^2\right)\right] \tag{33}$$

3.3 Standard slow-roll approximation

In the context of the standard approximation for slow-roll, the gravitational wave spectrum has the form

$$\ln \mathcal{P}_\psi = \ln \mathcal{P}_{\psi_\diamond} + n_{\psi_\diamond}\ln\left(\frac{k}{k_\diamond}\right) + \frac{1}{2}n_{\psi_\diamond}'\ln^2\left(\frac{k}{k_\diamond}\right) + \cdots \tag{34}$$

where $n_\psi \equiv \frac{d\ln\mathcal{P}_\psi}{dlnk}$, is the gravitational wave spectral index and $k_\diamond$ is some reference wavenumber. Applying our inverse formula equation (28), using the window properties Eqs. (18) and (32), and the results

$$\int_0^\infty \frac{dx}{x}\,m(x)\,\ln(x) = \alpha \tag{35}$$

$$\int_0^\infty \frac{dx}{x}\,m(x)\,\ln^2(x) = \alpha^2 + \frac{\pi^2}{12} \tag{36}$$

$$\int_0^\infty \frac{dy}{y}\ln\left|\frac{x+y}{x-y}\right| = \frac{\pi^2}{2}\,, \quad \texttt{for}\ \ k>0 \tag{37}$$

where $\alpha = 2 - \ln 2 - \gamma \simeq 0.7296$, we get

$$\ln\left(\frac{H}{2\pi}\right)^2 = \ln\mathcal{P}_{\psi_\diamond} + (\alpha_\diamond - 1)\,n_{\psi_\diamond} + \frac{1}{2}\left(\alpha_\diamond^2 - 2\alpha_\diamond + \frac{\pi^2}{12}\right)n_{\psi_\diamond}' + \left(1 - \frac{\pi^2}{8}\right)n_{\psi_\diamond}{}^2 + \cdots \tag{38}$$

where $\alpha_\diamond = \alpha - \ln(k_\diamond\xi)$. Equation (38) reproduces the standard slow-roll inverse, which is trivially obtained from the standard slow-roll formula,

$$\ln\mathcal{P}_\psi = \ln\frac{1}{p_\star^2} - 2\alpha_\star\frac{p_\star'}{p_\star} - \left(\alpha_\star^2 - \frac{\pi^2}{12}\right)\frac{p_\star''}{p_\star} + \left(\alpha_\star^2 - 4 + \frac{5\pi^2}{12}\right)\left(\frac{p_\star'}{p_\star}\right)^2 + \cdots \tag{39}$$

where $\alpha_\star = \alpha - \ln(k\xi_\star)$ and $\xi_\star$ is an arbitrary evaluation point usually taken to be around horizon crossing.

4 General slow-roll consistency condition

The standard slow-roll consistency condition relating the scalar and tensor spectra is detailed in [9] for the single field case and [16] discusses the same for the multi-field case. In this section we do the general slow-roll generalisation of the constraint on the spectra.

The inverse formula for the scalar power spectrum, $\mathcal{P}_s$ [12] is given by,

$$\ln \frac{1}{f^2} = \int_0^\infty \frac{dk}{k} m(k\xi) \ln \mathcal{P}_s \tag{40}$$

where $f = \frac{2\pi a \xi \dot{\phi}}{H}$. Also, for the tensor spectrum we have,

$$\ln \frac{1}{p^2} = \int_0^\infty \frac{dk}{k}\, m(k\xi) \ln \mathcal{P}_\psi \tag{41}$$

Combining the above two equations,

$$\ln \left(\frac{f}{p}\right)^2 = \int_0^\infty \frac{dk}{k}\, m(k\xi) \ln \left(\frac{\mathcal{P}_\psi}{\mathcal{P}_s}\right) \tag{42}$$

Thus we can write

$$\ln \left(\frac{\dot{\phi}}{H}\right)^2 = \int_0^\infty \frac{dk}{k}\, m(k\xi) \ln \left(\frac{\mathcal{P}_\psi}{\mathcal{P}_s}\right) \tag{43}$$

Now, taking the derivative of equation (20),

$$2\frac{\dot{H}}{H^2} = \int_0^\infty \frac{dk}{k}\, m(k\xi) \left[\frac{\mathcal{P}_\psi{}'}{\mathcal{P}_\psi} - \left(\frac{\mathcal{P}_\psi{}'}{\mathcal{P}_\psi}\right)'\right] \tag{44}$$

The slow-roll parameter ϵ is defined as

$$\epsilon \equiv \frac{1}{2}\left(\frac{\dot{\phi}}{H}\right)^2 = -\frac{\dot{H}}{H^2} \tag{45}$$

Therefore we can equate the equations (43) and (44) to get the general slow-roll constraint on the spectra as,

$$\ln \frac{\mathcal{P}_\psi}{\mathcal{P}_s} = -n_\psi \tag{46}$$

where we neglect the n'_ψ term which is small in single field.

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