

The academy fellowship problem

The problem

A science academy elects new fellows every year. In order to improve the overall quality of its fellowship, it decides to impose a new criterion that every new fellow elected shall be better than the median level of the existing population of fellows*. How fast will the quality of the fellowship improve?

*This criterion was proposed for the Indian Academy of Sciences by Professor V. Radhakrishnan in the mid-seventies.

A toy model

The above problem needs to be recast in a quantitative form. It is difficult to quantify 'quality' of a scientist but suppose in a toy model we use a variable x to measure it on a scale ranging from 0 to 1, with 1 being the mark of perfection. Let us measure the time in years with $t=0$ denoting a starting year when this criterion is announced to be implemented in all subsequent years. Let $f_n(x)$ denote the cumulative quality distribution of fellowship in the year $t=n$, $n=0, 1, 2, \dots$. In the starting year the

total number of fellows was N_0 , say and suppose that by statutes, every year N fellows must be elected.

Denote by M_n the median of the distribution $f_n(x)$. Thus

$$f_n(M_n) = \frac{1}{2} f_n(1) = \frac{1}{2}(N_0 + nN), \quad (1)$$

and while electing in year n , care is taken that all new additions shall have the quality parameter $x > M_{n-1}$. Improvement in quality will then be indicated by how the median M_n steadily increases in value year by year.

We still need to specify the quality distribution of the pool of scientists

from which the new fellows are chosen. We denote by $g(x)$ the cumulative probability that a scientist selected at random from this pool will not have the quality parameter exceeding x . Thus

$$g(0) = 0, \quad g(1) = 1. \quad (2)$$

In this toy model we have ignored the reduction in the total population of fellows due to death or resignation (or even eviction).

With these specifications we can now pose the question more precisely: Given the functions $f_0(x)$ and $g(x)$, how does M_n increase with n ?

Solution

First note that for $t = n$, the median criterion implies

$$f_n(x) = f_{n-1}(x) \text{ for } x < M_{n-1}. \quad (3)$$

For $x \geq M_{n-1}$, there will be addition of N fellows whose distribution will be assumed to be proportional to the function $g(x)$ over $M_{n-1} \leq x \leq 1$. Therefore, we have for $x \geq M_{n-1}$

$$f_n(x) = f_{n-1}(x) + N \frac{g(x) - g(M_{n-1})}{1 - g(M_{n-1})}. \quad (4)$$

Since for the distribution $f_n(x)$ there are $\frac{1}{2}(N_0 + nN)$ members with $x < M_n$, these comprise of the $\frac{1}{2}(N_0 + (n-1)N)$ members with $x < M_{n-1}$ together with the number $f_{n-1}(M_n) - f_{n-1}(M_{n-1})$ from the earlier distribution and the new addition as per (4) above. Carrying out this book-keeping we have

$$f_{n-1}(M_n) - f_{n-1}(M_{n-1}) + N \frac{g(M_n) - g(M_{n-1})}{1 - g(M_{n-1})} = \frac{1}{2} N. \quad (5)$$

Next we consider the number $f_{n-1}(M_n) - f_{n-1}(M_{n-1})$ which has arisen from steady addition to $f_0(M_n) - f_0(M_{n-1})$ over all the preceding years following the rule (4). Therefore,

$$\begin{aligned} & f_{n-1}(M_n) - f_{n-1}(M_{n-1}) \\ &= N \sum_{r=1}^{n-1} \frac{g(M_n) - g(M_{r-1})}{1 - g(M_{r-1})} \\ &+ f_0(M_n) - f_0(M_{n-1}). \end{aligned} \quad (6)$$

From (5) and (6) we have the final relation

$$f_0(M_n) - f_0(M_{n-1}) + N \{g(M_n) - g(M_{n-1})\} \times \sum_{r=1}^n \frac{1}{1 - g(M_{r-1})} = \frac{1}{2} N. \quad (7)$$

This iterative relation determines the sequence $\{M_n\}$ in a step by step fashion starting with M_0 , since for determining M_n we have all other quantities known from the preceding application of this relation.

A simple example

We will illustrate the above solution with a simple example in which both the initial distribution and the general population distribution are uniform with respect to the attribute x . In this case

$$f_0(x) = N_0 x, \quad g(x) = x. \quad (8)$$

and the relation (7) gives

$$M_n = M_{n-1} + \frac{1}{2 \sum_{r=1}^n \frac{1}{1 - M_{r-1}} + \frac{2N_0}{N}}. \quad (9)$$

Even with this simple example we cannot find a solution in closed form, but Figure 1 illustrates the march of M_n for $0 \leq n \leq 25$, $N_0 = 500$ and $N = 20$ obtained by numerical methods. The median value rises from 0.5 to 0.75 over twenty-five years.

The continuum version

The solution, however, becomes tractable even for the general case if we modify the problem by having a continuous input of fellows at all times instead of the discrete annual input. A simple analysis shows that the relation (7) is changed to an integro-differential equation

$$\begin{aligned} \left(\frac{dM}{dt}\right)^{-1} &= \frac{2}{N} \frac{\partial f(x,t)}{\partial x} \Big|_{x=M(t), t_0} \\ &+ 2 \int_0^t \frac{g'(M(t_1)) dt_1}{1 - g(M(t_1))}, \end{aligned} \quad (10)$$

where the prime denotes derivative with respect to the argument. Here the rate of

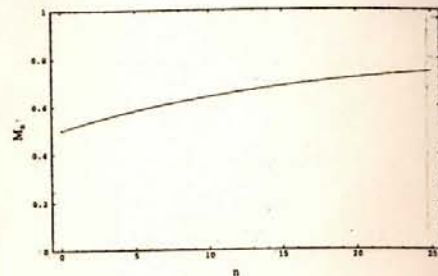


Figure 1. The figure shows how the median value of the fellowship quality improves with successive elections over 25 years in the simple toy model.

fellowship input is N , that is $N\delta t$ to be added in a time interval δt . Also, the fellowship distribution at time t is now denoted by $f(x, t)$ and its median by $M(t)$.

The equation (10) is solvable in closed form and in the general case we have

$$\begin{aligned} \frac{dt}{dM} &= \frac{2}{(1 - g(M))^2} \\ &\times \left[\int_{M(0)}^{M(t)} \alpha'(M) [1 - g(M)]^2 dM + A \right], \end{aligned} \quad (11)$$

$$\alpha(M) = \frac{1}{N} \frac{\partial f(x,t)}{\partial t} \Big|_{x=M, t=0} \quad (12)$$

where A is a constant of integration. For the simple case of uniform distributions discussed earlier the solution is very simple:

$$M(t) = \frac{1}{2} \frac{1 + (2tN / N_0)}{1 + (tN / N_0)}, \quad (13)$$

and it fits the curve of Figure 1 very well. In fact an approximate discrete solution based on equation (13), viz.

$$M_n = 1 - \frac{1}{2} (1 + nN / N_0)^{-1} \quad (14)$$

satisfies the iterative relation (7) with a good approximation.

JAYANT V. NARLIKAR

Inter-University Centre for Astronomy and Astrophysics, Post Bag 4, Ganeshkhind, Pune 411 007, India