A class of solutions for a rotating perfect fluid in general relativity

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Abstract

A class of perfect fluid solutions of the Einstein's field equations are presented. The Weyl tensor of the space-time is of Petrov type D and the fluid obeys an equation of state, namely $\epsilon + 3\Sigma =$ constant. The fluid rotates rigidly. The solution may be interpreted either as interior fluid of a finite body or a cosmological distribution. Particular cases are the NUT family of vacuum solutions, Schwazschild interior solution, static homogeneous cosmological space -times, Goedel universe and generalisations of these space -times with mass and other inhomoginities.

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1 INTROUCTION

We investigate here a space - time metric derived from Misra metric [1] by the procedure of contraction [2]. Our metric in general represents algebraically general gravitational field with perfect fluid as the source. By imposing conditions on Weyl scalars, we obtain type D fields with geodesic, shearfree and diverging null rays, having common complex expansion. The solutions represent stationary axisymmetric perfect fluid in rigid rotation with velocity vector lying in the 2-space defined by the principal null directions of Weyl tensor (so that they belong to class I of Wainwright's classification [3]). The paper is organised as follows: In section 2, the metric form and its geometrical properties are given. Field equations and solutions are given in section 3. Some particular solutions are discussed in section 4. We follow the notation of Debney et. al. [4].

2 METRIC AND PROPERTIES

The Misra metric [1] in coordinates x^i(p,σ,q,τ) is given by

$$ds^2 = -\frac{p^2+q^2}{PX}dp^2 - \frac{P}{p^2+q^2}(d\tau + q^2 d\sigma)^2 - \frac{p^2+q^2}{QY}dq^2 - \frac{Q}{p^2+q^2}(d\tau - p^2 d\sigma)^2$$
$$P = P(p), X = X(p), Q = Q(q), Y = Y(q).$$

It represents stationary axysymmetric rigidly rotating perfect fluid with or without electromagnetic field [5], [6] type D gravitational field of class II of Wainwright's classification [3].
By contracting 'p' coordinate [2], we obtain the metric in the form

$$ds^2 = -(p_0^2+q^2)(\frac{dp^2}{PX} + Pd\sigma^2) - \frac{dq^2}{QY} + Q(d\tau - 2p_0 p d\sigma)^2 \qquad (1)$$

where p_0 is a constant and

$$P = P(p), \qquad X = X(p),$$

$$Q = Q(q), \qquad Y = Y(q).$$

The signature of the metric is assumed to be (- - - +). The line-element (1) induces the natural choice of real orthonormal basis 1-forms $e^{a'}$ as:

$$e^{1'} = (\frac{p_0^2 + q^2}{PX})^{1/2} dp,$$

$$e^{2'} = ((p_0^2 + q^2)P)^{1/2} d\sigma, \qquad (2a)$$

$$e^{3'} = (QY)^{-1/2} dq,$$

$$e^{4'} = \sqrt{Q}(d\tau - 2p_0 p d\sigma).$$

The null basis 1-forms obtained from (2) is then

$$\begin{cases} e^1 \\ e^2 \end{cases} := \frac{(p_0^2 + q^2)^{1/2}}{(\sqrt{2})} \{(PX)^{-1/2} dp \pm i\sqrt{P} d\sigma\}, \qquad (2b)$$

$$\begin{cases} e^3 \\ e^4 \end{cases} := \frac{1}{\sqrt{2}} \{(QY)^{-1/2} dq \pm \sqrt{Q}(d\tau - 2p_0 p d\sigma)\}.$$

Now adopting the standard procedure one may compute the Ricci

rotation coefficients. In Newman - Penrose (NP) notation one finds that

$$\chi = \sigma = 0 = \nu = \lambda \tag{3}$$

identically. Therefore the null congruences e_3 and e_4 are both geodesic and shearfree. Further, both the congruences have a common complex expansion given by

$$\rho := -(\theta + iw) = \frac{\sqrt{Q}(q\sqrt{Y} + ip_0\sqrt{X})}{\sqrt{2}(p_0^2 + q^2)} \tag{4}$$

where θ and w are the expansion and the twist respectively for the null congruences. From the above expression it is evident that in the limit of $p_0 \to 0$,the congruences will be twistfree. The geodesic and nonshearing null vectors may be taken as

$$K_\mu^{(\pm)} dx^\mu = d(\tau \pm \int \frac{dq}{Q\sqrt{Y}}) - 2p_0 p d\sigma. \tag{5}$$

The Weyl scalars $C^{(A)}$ are obtained as

$$C^{(1)} = C^{(5)} = 0,$$

$$C^{(2)} = -C^{(4)} = -\frac{ip_0}{4}\left(\frac{PQ}{(p_0^2 + q^2)^3}\right)^{1/2}\dot{X} \tag{6}$$

where an overhead dot denotes derivatives with respect to the argument. Equation (6) reveals that our choice of the null frames makes $C^{(1)}$ and $C^{(5)}$ to vanish identically i.e. e_3 and e_4 are indeed the eigen- directions of the Weyl tensor [7]. Further to obtain type D gravitational fields one may either take $p_0 = 0$ or $\dot{X} = 0 \Rightarrow X = a$ constant, otherwise the gravitational field would be algebraically general. However, if $p_0 \to 0$ the fluid distribution

is found to be nonrotating. Therefore, for type D fields and a rotating fluid distribution we choose

X= an arbitrary constant. (7)

3 FIELD EQUATIONS AND SOLUTIONS

On computing the Einstein tensor in the orthonormal frame one finds that

$$G^{1'}_{1'} = G^{2'}_{2'},$$

(8)

$$G^{1'}_{2'} = G^{1'}_{3'} = G^{1'}_{4'} = G^{2'}_{3'} = G^{2'}_{4'} = G^{3'}_{4'} = 0.$$

identically. Now the field equations $G^{a'}_{b'} = kT^{a'}_{b'}$ where

$$T^{a'}_{b'} = (\epsilon + \Sigma)U^{a'}U_{b'} - \Sigma\delta^{a'}_{b'}$$

(ϵ and Σ are the energy density and the pressure of the distribution respectively) and the structure of the Einstein tensor as given in (8) reveal that $U^{1'}$ and $U^{2'}$ must be zero. Consequently the velocity four- vector of the fluid lies in the 2-space spanned by the real null vectors e_3 and e_4. Thus the solutions discussed here belong to class I of Wainwright classification of type D fields [3]. To set up the field equations it is preferable to work in a comoving frame. We take τ as the time-like coordinate. σ is a cyclic coordinate so that this subclass of solutions will be stationary in general. The choice gives $U^{3'}$=0. In this case the field equations are seperable into ordinary differential equations for the variables p and q.
One obtains

$$\ddot{P} = -2\gamma_0, \tag{9}$$

$$\frac{2Q}{p_0^2+q^2}(q^2Y-p_0^2X)-Q(Y+1/2q\dot{Y})-1/4(p_0^2+q^2)(2Y\ddot{Q}+\dot{Y}\dot{Q}) = \gamma_0 X \tag{10}$$

where γ_0 is the constant of seperation. Equation (9) determines P as a second order polonomial in p. To determine Q and Y we have only one differential equation namely (10). Thus one of these functions may be chosen suitably to render Eq. (10) integrable. Thus in this manner different class of solutions may be obtained. In what follows we discuss some particular solutions that may be obtained from (7)-(10).
We choose X and Y in the following manner:

$$X = \mu + \lambda p_0^2 \tag{11a}$$

$$Y = \mu - \lambda q^2 \tag{11b}$$

and also introduce the function S as

$$S := Q(p_0^2 + q^2) \tag{12}$$

where μ and λ are arbitary constants.
Now equation (10) becomes

$$2Y\ddot{S} + \dot{Y}\dot{S} + 8\lambda S = 4(2Aq^2 + B) \tag{13}$$

where B is a constant of integration and A is given as

$$2Ap_0^2 = B - \gamma_0 X_0. \tag{14}$$

Now the solution of (13) is easily found to be

$$S = \frac{q_0(\mu - 2\lambda q^2)}{\mu} - q\sqrt{Y}(\frac{2m}{\mu} + \frac{A}{\lambda\sqrt{\lambda}} sin^{-1}\sqrt{(\frac{\lambda}{\mu})}q) + (\frac{B}{\mu} + \frac{A}{\lambda})q^2 \tag{15}$$

Further the pressure and density of the distribution are obtained as

$$k\Sigma = A - \lambda Q \,, \tag{16}$$

$$k\epsilon = -A + 3\lambda Q \,,$$

so that the equation of the state of the fluid is found to be

$$k(\epsilon + 3\Sigma) = 2A. \tag{17}$$

Equations (7), (9), (11) - (15) describe the new class of perfect fluid solutions. One finds that the expansion and shear of the fluid vanish in general whereas the rotation is non-zero if $p_0 \neq 0$ Thus the fluid is rotating rigidly. Due to these properties the metric is L.R.S. and admits a G^4 [8].

4 PARTICULAR SOLUTIONS

Our metric incorporates a large number of well known space-time as particular cases. For examples, if we let $\lambda \to 0, \mu \neq 0$ and put $A = 0$ one obtains vaccum solutions belonging to the NUT family of space-times with p_0 as the NUT parameter and 'm' the mass parameter [9]. But if $\lambda \to 0$ and $A \neq 0$ one obtains NUT metric in the de Sitter or Anti-de Sitter background depending upon the sign of A.

In the case of λ different from zero and for vanishing p_0 we have static space-times. The conformally flat member is the Einstein static cosmos. If $p_0 \neq 0$ but $A = 0$, we have NUT metric in the background of Einstein space-time. However if $A \neq 0$, we have a generalisation of the space -time discussed above.
A different class of type D space-times are obtained if we introduce a new variable v as

$$dv = \frac{dq}{\sqrt{Y}}$$

and let $\mu \to 0$. If $m = 0$ we have the homogeneous but rotating Goedel cosmos [10]. If m is different from zero we have a mass inhomogenity with Goedel space-time as the background [11].
We have thus demonstrated that the particular choice of X and Y via Eqs. (7) and (11) leads to a very general space-time when all the parameters are different from zero. It is àlso obvious that the space - time characterised by (1), (7), (11)-(16) may be interpreted either as the interior field of a rigidly rotating fluid body ('A' is then to be interpreted as surface energy- density of the distribution) or a cosmological model with A playing a role similar to the cosmological term. Our analysis also demonstrates the intimate relationship between the NUT metric and Goedel space-time.
We would also like to emphasize that different functions Y will give rise to different class of space-times. For examples, if $p_0 = 0$

$$Y = \frac{1 - C_1 q^2}{Q},$$

then

$$Q^{1/2} = C_2 - D(1 - C_1 q^2)^{1/2}$$

where C_1, C_2, D are constants of integration. In this case one finds that the Weyl tensor vanishes and our metric (1) then becomes the well known Schwarzschild solution.
In our discussions so far we have taken τ as the time- like comov-

ing coordinate. However it is also possible to take q as time- like and we then get time dependent perfect fluid solutions of the field equations. Particular cases are the Friedman, Robertson- Walker (FRW) space-times. This class of solutions will be discussed elsewhere.

5 CONCLUSION

To best of our knowledge, we have obtained for the first time Wainwright class I perfect fluid type D fields from class II perfect fluid solutions. The solutions contain a wide class of vacuum as well as perfect fluid rotating and non rotating solutions as special cases.

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