Duality and Zero-Point Length of Spacetime

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The action for a relativistic free particle of mass m receives a contribution $-mds$ from a path of infinitesimal length ds. Using this action in a path integral, one can obtain the Feynman propagator for a spinless particle of mass m. Assuming that the path integral amplitude is invariant under the "duality" transformation $ds \to L_P^2/ds$, one can calculate the modified Feynman propagator. I show that this propagator is the same as the one obtained by assuming that quantum effects of gravity lead to modification of the spacetime interval $(x-y)^2$ to $(x-y)^2 + L_P^2$. The implications of this result are discussed. [S0031-9007(97)02531-3]

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From the fundamental constants $G, \hbar$, and c, one can form a quantity with dimensions of length, $L_P \equiv (G\hbar/c^3)^{1/2}$, which is expected to play a vital role in the "ultimate" theory of quantum gravity. Simple thought experiments indicate that it is not possible to devise experimental procedures which will measure lengths with an accuracy greater than about $\mathcal{O}(L_P)$ [1]. This result suggests that one could think of Planck length as some kind of "zero-point length" of spacetime. In some simple models of quantum gravity, L_P^2 does arise as a mean square fluctuation to spacetime intervals, due to quantum fluctuations of the metric [2]. In more sophisticated approaches, such as models based on string theory or Ashtekar variables, similar results arise in one guise or another (see, e.g., [3,4]).

The existence of a fundamental length implies that processes involving energies higher than Planck energies will be suppressed, and the ultraviolet behavior of the theory will be improved. All sensible models for quantum gravity provide a mechanism for good ultraviolet behavior, essentially through the existence of a fundamental length scale. One direct consequence of such an improved behavior will be that the Feynman propagator (in momentum space) will acquire a damping factor for energies larger than Planck energy.

If the ultimate theory of quantum gravity has a fundamental length scale built into it, then it seems worthwhile to formulate quantum field theory, using this principle as the starting point. This could, for example, help in understanding some of the effects of quantizing gravity on the matter fields. I will show in this Letter that such a procedure leads to some interesting results for a spin-zero (scalar) field.

To keep things well defined and general, I will work in a D-dimensional Euclidean space. Feynman propagator $G(\mathbf{x}, \mathbf{y})$ for a spin-zero, free particle of mass m in D dimension is

$$G_{\text{conv}}(\mathbf{x}, \mathbf{y}) = \int \frac{d^D\mathbf{p}}{(2\pi)^D} \frac{e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{(p^2 + m^2)}. \qquad (1)$$

This propagator—which arises in the standard formulation of quantum field theory—does not take into account the existence of any fundamental length in the spacetime. Let us ask how this propagation amplitude could be modified if there exists a fundamental zero-point length to the spacetime. This is best done using the path integral expression for the Feynman propagator

$$G_{\text{conv}}(\mathbf{x}, \mathbf{y}) = \sum_{\text{paths}} \exp[-ms(\mathbf{x}, \mathbf{y})], \qquad (2)$$

where $s(\mathbf{x}, \mathbf{y})$ is the length of any path connecting $\mathbf{x}$ and $\mathbf{y}$. To give meaning to the path integral we shall first introduce a cubic lattice with a lattice spacing ϵ in the D-dimensional Euclidean space. The propagator in the latticized spacetime is given by

$$\mathcal{G}_{\text{conv}}(\mathbf{R}, \epsilon) = \sum_{N=0}^{\infty} C(N, \mathbf{R}) \exp[-\mu(\epsilon)\epsilon N], \qquad (3)$$

where $C(N, \mathbf{R})$ is the number of paths of length $N\epsilon$ connecting the origin to the lattice point $\mathbf{R} = (n_1, n_2, \ldots, n_D)$ which is a D-dimensional vector with integer components. (The physical scale corresponding to $\mathbf{R}$ is $\mathbf{x} = \epsilon\mathbf{R}$.) The scaling factor $\mu(\epsilon)$ acts as the mass parameter on the lattice. The propagator for the continuum has to be obtained by multiplying (3) by a suitable measure $\mathcal{M}(\epsilon)$ and taking the limit $\epsilon \to 0$. Both the measure $\mathcal{M}(\epsilon)$ and the mass parameter on the lattice $\mu(\epsilon)$ should be chosen so as to ensure the finiteness of the limit. This procedure is straightforward to carry out (see, e.g., [5]) and one obtains the Feynman propagator given in Eq. (1).

In the above procedure, the weight given for a path of length l is $\exp(-ml)$ which is a monotonically decreasing function of l. The existence of a fundamental length L_P would suggest that paths with length $l \ll L_P$ should be suppressed in the path integral. This can, of course, be done in several different ways by arbitrarily modifying the expression in Eq. (3). In order to make a specific choice I shall invoke the following "principle of duality." I will postulate that the weight given for a path should

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be invariant under the transformation $l \to L_P^2/l$. Since the original path integral has the factor $\exp(-ml)$, we have to introduce the additional factor $\exp(-mL_P^2/l)$. We therefore modify Eq. (3) to

$$\mathcal{G}(\mathbf{R},\epsilon) = \sum_{N=0}^{\infty} C(N,\mathbf{R}) \exp\left[-\mu(\epsilon)\epsilon N - \frac{\lambda(\epsilon)}{\epsilon N}\right], \quad (4)$$

where $\lambda(\epsilon)$ is a lattice parameter which will play the role of (mL_P^2) in the continuum limit.

I will take this to be the basic postulate arising from the "correct" theory of quantum gravity. It may be noted that the principle of duality invoked here is similar to that which arises in string theories (though not identical). In fact, we may think of Eq. (4) as the simplest realization of duality for a free particle; we have demanded that the existence of a weight factor $\exp(-ml)$ necessarily requires the existence of another factor $\exp(-mL_P^2/l)$. We shall now study the consequences of the modifications we have introduced.

To evaluate this path integral on the lattice we begin by noticing that the generating function for $C(N,\mathbf{R})$ is given by [5]

$$\begin{aligned} F^N &\equiv \sum_{\mathbf{R}} C(N;\mathbf{R}) e^{i\mathbf{k}\cdot\mathbf{R}} \\ &= (e^{ik_1} + e^{ik_2} + \cdots + e^{ik_D} \\ &\quad + e^{-ik_1} + e^{-ik_2} + \cdots + e^{-ik_D})^N. \end{aligned} \quad (5)$$

Therefore we can write

$$\begin{aligned} \sum_{\mathbf{R}} e^{i\mathbf{k}\cdot\mathbf{R}} \mathcal{G}(\mathbf{R},\epsilon) &= \sum_{N=0}^{\infty} e^{-\mu\epsilon N - (\lambda/\epsilon N)} \sum_{\mathbf{R}} C(N,\mathbf{R}) e^{i\mathbf{k}\cdot\mathbf{R}} \\ &= \sum_{N=0}^{\infty} e^{-N(\mu\epsilon - \ln F) - (\lambda/\epsilon N)}. \end{aligned} \quad (6)$$

Thus, our problem reduces to evaluating the sum of the form

$$\begin{aligned} S(a,b) &\equiv \sum_{n=0}^{\infty} \exp\left(-a^2 n - \frac{b^2}{n}\right) \\ &= \sum_{n=1}^{\infty} \exp\left(-a^2 n - \frac{b^2}{n}\right). \end{aligned} \quad (7)$$

This expression can be evaluated by some algebraic tricks [6], and the answer is

$$\begin{aligned} S(a,b) &= \int_0^{\infty} \frac{kdk}{2b^2} \frac{J_0(k) e^{-(a^2+k^2/4b^2)}}{[1 - e^{-(a^2+k^2/4b^2)}]^2} \\ &= \frac{1}{(1 - e^{-a^2})} - \int_0^{\infty} dq \frac{J_1(q)}{[1 - e^{-(a^2+q^2/4b^2)}]}, \end{aligned} \quad (8)$$

where $J_\nu(x)$ is the Bessel function of order ν. The first form of the integral shows that the expression is well defined while the second form has the advantage of separating out the b-independent part as the first term. (Note that the two summations in (7) will differ by unity if $b = 0$; the results in (8) will go over to the second summation in (7) if the limit $b \to 0$ is taken.) Using the second form in Eq. (8) and introducing the continuum variables $\mathbf{x} = \epsilon\mathbf{R}$, $\mathbf{p} = \epsilon^{-1}\mathbf{k}$, we can write the propagator as the sum of two terms $\mathcal{G} = \mathcal{G}_0 + \mathcal{G}_c$, where

$$\mathcal{G}_0(\mathbf{R}) = \int \frac{d^D\mathbf{k}}{(2\pi)^D} \left\{\frac{e^{-i\mathbf{k}\cdot\mathbf{R}}}{1 - 2e^{-\mu\epsilon}\sum_{i=1}^{D}\cos k_i}\right\}, \quad (9)$$

$$\begin{aligned} \mathcal{G}_c(\mathbf{R}) &= -\int_0^{\infty} dq J_1(q) \int \frac{d^D\mathbf{k}}{(2\pi)^D} e^{-i\mathbf{k}\cdot\mathbf{R}} \\ &\quad \times \left\{\frac{1}{1 - 2e^{-\epsilon[\mu+(q^2/4\lambda)]}\sum_{i=1}^{D}\cos k_i}\right\}. \end{aligned} \quad (10)$$

We now have to take the $\epsilon \to 0$ limit. The propagator $\mathcal{G}_0$ becomes, in the limit of small ϵ,

$$\mathcal{G}_0(\mathbf{x}) = \int \frac{d^D\mathbf{p}}{(2\pi)^D} \frac{A_1(\epsilon) e^{-i\mathbf{p}\cdot\mathbf{x}}}{p^2 + B_1(\epsilon)}, \quad (11)$$

where

$$A_1(\epsilon) = \epsilon^{D-2} e^{\epsilon\mu(\epsilon)}, \quad B_1(\epsilon) = \epsilon^{-2}\{e^{\epsilon\mu(\epsilon)} - 2D\}. \quad (12)$$

Similarly, $\mathcal{G}_c$ becomes, in the same limit,

$$\mathcal{G}_c(\mathbf{x}) = \int_0^{\infty} dq\, J_1(q) \mathcal{H}(q,\mathbf{x}) \quad (13)$$

with

$$\mathcal{H}(q,\mathbf{x}) = \int \frac{d^D\mathbf{p}}{(2\pi)^D} \frac{A_2(\epsilon,q) e^{-i\mathbf{p}\cdot\mathbf{x}}}{p^2 + B_2(\epsilon,q)}, \quad (14)$$

where

$$A_2(\epsilon,q) = -\epsilon^{D-2}\, e^{\epsilon[\mu(\epsilon)+q^2/4\lambda(\epsilon)]}, \quad (15)$$

$$B_2(\epsilon,q) = \epsilon^{-2}\{e^{\epsilon[\mu(\epsilon)+q^2/4\lambda(\epsilon)]} - 2D\}. \quad (16)$$

The continuum propagator is defined as

$$G(\mathbf{x}) = \lim_{\epsilon\to 0}\{\mathcal{M}(\epsilon)\mathcal{G}(\mathbf{x};\epsilon)\}, \quad (17)$$

where the small ϵ behavior of $\mathcal{M}(\epsilon)$, $\lambda(\epsilon)$, and $\mu(\epsilon)$ have to be fixed in such a manner that this limit is finite. One can easily see that finiteness of A_1 and B_1 requires

$$\lim_{\epsilon\to 0}\{\mathcal{M}(\epsilon)\epsilon^{D-2}e^{\epsilon\mu(\epsilon)}\} = 1, \quad (18)$$

$$\lim_{\epsilon\to 0}\left\{\frac{1}{\epsilon^2}[e^{\epsilon\mu(\epsilon)} - 2D]\right\} = m^2, \quad (19)$$

which is a standard result leading to (1) (see, e.g., [5]). The finiteness of B_2 requires the quantity $\beta(\epsilon) \equiv [\epsilon/\lambda(\epsilon)]$ to scale as ϵ^2 for small ϵ. Writing $\beta(\epsilon) \simeq l_0^{-2}\epsilon^2 + \mathcal{O}(\epsilon^3)$ in this limit (where we expect $l_0 \propto L_P$ in the continuum limit), we find that the final result can be expressed as $G = G_0 + G_c$, with

$$G_0(\mathbf{x}) = \int \frac{d^D\mathbf{p}}{(2\pi)^D} \frac{e^{-i\mathbf{p}\cdot\mathbf{x}}}{p^2 + m^2}, \tag{20}$$

$$G_c(\mathbf{x}) = -\int_0^\infty dq J_1(q) \int \frac{d^D\mathbf{p}}{(2\pi)^D} e^{-i\mathbf{p}\cdot\mathbf{x}} \times \left\{\frac{1}{p^2 + (D/2l_0^2)q^2 + m^2}\right\}. \tag{21}$$

Integrating the second term by parts and combining with the first term, we can express the full momentum space propagator as

$$\hat{G}(\mathbf{p}) = 2\nu^2 \int_0^\infty dq \frac{qJ_0(q)}{[q^2 + \nu^2(p^2 + m^2)]^2}, \tag{22}$$

where $\nu^2 \equiv (2l_0^2/D)$. Using the identity

$$\int_0^\infty dx \frac{xJ_0(x)}{(x^2 + b^2)^2} = -\frac{1}{2b} K_0'(b) = \frac{K_1(b)}{2b}, \tag{23}$$

where $K_n(z)$ is the modified Bessel function of order n, we can write

$$\hat{G}(\mathbf{p}) = \frac{\nu}{\sqrt{p^2 + m^2}} K_1(\nu\sqrt{p^2 + m^2}). \tag{24}$$

This is our final result with $\nu \propto L_P$ in the continuum limit. This equation represents the Feynman propagator for a "free" spin-zero particle when our prescription—that the weight for a path of length l should be invariant under the transformation $l \to L_P^2/l$—has been invoked. This postulate (which, in the present context, may be called "lattice duality"), and the form of the standard free particle propagator, uniquely leads to our final result. From the asymptotic forms of $K_1(z)$ it is easy to see that the propagator in (24) has the limiting expressions

$$\hat{G}(\mathbf{p}) \to \begin{cases} \frac{1}{p^2+m^2} & (\text{for } \nu\sqrt{p^2+m^2} \ll 1) \\ \exp(-\nu\sqrt{p^2+m^2}) & (\text{for } \nu\sqrt{p^2+m^2} \gg 1) \end{cases}. \tag{25}$$

When $\nu \propto L_P \to 0$, the propagator reduces to the standard form, while for energies larger than Planck energies it is exponentially damped.

I shall now show that the result in (24) has an extremely simple interpretation and an alternative derivation. The standard Feynman propagator in Eq. (1) can be equivalently represented as

$$G_{\text{conv}}(\mathbf{x}) = \int \frac{d^Dp}{(2\pi)^D} \frac{e^{-i\mathbf{p}\cdot\mathbf{x}}}{p^2 + m^2} = \int \frac{d^Dp}{(2\pi)^D} e^{-i\mathbf{p}\cdot\mathbf{x}} \int_0^\infty ds e^{-s(m^2+p^2)} = \int_0^\infty \frac{ds}{(4\pi s)^{D/2}} \exp\left(-\frac{x^2}{4s} - m^2 s\right). \tag{26}$$

The last expression, in fact, constitutes the Schwinger proper time version of the propagator. Suppose we now postulate that the net effect of quantum fluctuations is to add a zero-point length to the spacetime interval, i.e., to change the interval from $(x - y)^2$ to $(x - y)^2 + l_0^2$ where $l_0 \propto L_P$. (In [1,2], it was suggested that $l_0 = L_P/2\pi$.) Making this replacement and doing the inverse Fourier transform, we immediately see that the modified momentum space propagator becomes

$$\hat{G}_{\text{mod}}(\mathbf{p}) = \frac{l_0}{\sqrt{p^2 + m^2}} K_1(l_0\sqrt{p^2 + m^2}), \tag{27}$$

which is identical in form to Eq. (24). In other words, *the modification of the path integral based on the principle of duality leads to results which are identical to adding a zero-point length in the spacetime interval*.

I wish to argue that the connection shown above is nontrivial; I know of no simple way of guessing this result. The standard Feynman propagator of quantum field theory can be obtained either through a lattice regularization of a path integral or from Schwinger's proper time representation. By adding a zero-point length in the Schwinger representation we obtain a modified propagator. Alternatively, using the principle of duality, we could modify the expression for the path integral amplitude on the lattice and obtain—in the continuum limit—a modified propagator. Both these constructions are designed to suppress energies larger than Planck energies. *However, there is absolutely no reason for these two expressions to be identical.* The fact that they are identical suggests that the principle of duality is connected in some deep manner with the spacetime intervals having a zero-point length. Alternatively, one may conjecture that any approach which introduces a minimum length scale in spacetime (such as in string models) will lead to some kind of principle of duality. This conjecture seems to be true in conventional string theories, though it must be noted that the term "duality" is used in a somewhat different manner in string theories. [The concept of duality in string theory is reviewed in several articles; see, e.g., Refs. [7–12], and the references cited therein. The closest to our approach seems to be the T duality.]

I stress that the path integral amplitude is modified on the lattice *before* taking the continuum limit. This allows us to introduce a factor $\exp(-\lambda/N\epsilon)$ along with the original $\exp(-\mu\epsilon N)$. Loosely speaking, we are changing the infinitesimal action for the relativistic particle from ds to $(ds + L^2/ds)$. It is not easy to interpret this term directly in the continuum limit or even find a modified *continuum* action for the relativistic particle which will lead to the same final propagator.

More generally, it is possible to define the path integral measure in such a way that the modified Green's function can be expressed in the form

$$G_{\text{mod}}(\mathbf{x}) = \int_0^\infty ds K(\mathbf{x}, s) \exp(-l_0^2/4s), \qquad (28)$$

where $K(\mathbf{x}, s)$ is the conventional Schwinger kernel of a *free* quantum field [6]. In the case of spin-$(1/2)$ particles, this seems to be equivalent to starting with an action which is an integral of $\gamma_a dx^a$ and defining the path integral with suitable path ordering. This question and related issues are under investigation.

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